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FREDHOLM DETERMINANTS, ANOSOV MAPS AND RUELLE RESONANCES

Carlangelo Liverani

Dipartimento di Matematica
II Università di Roma (Tor Vergata)
Via della Ricerca Scientifica, 00133 Roma, Italy

Abstract. I show that the dynamical determinant, associated to an Anosov diffeomorphism, is the Fredholm determinant of the corresponding Ruelle-Perron-Frobenius transfer operator acting on appropriate Banach spaces. As a consequence it follows, for example, that the zeroes of the dynamical determinant describe the eigenvalues of the transfer operator and the Ruelle resonances and that, for $\mathcal{C}^\infty$ Anosov diffeomorphisms, the dynamical determinant is an entire function.

1. **Introduction.** In the last years there has been a considerable interest in the study of dynamical determinants and dynamical zeta functions (see [4, 14, 2, 7, 25, 9, 19, 18, 6, 20], just to mention a few, [25, 21] for brief reviews of the field, [1] for a general introduction and [10] for a detailed discussion of physics related issues). Here, I will focus on Anosov diffeomorphisms T and the associated Fredholm determinant $d^\flat$ (2.11) for the transfer operator $\mathcal{L}$ (see (2.1) for a precise definition).

The most satisfactory results have been obtained for analytic systems [23, 11, 13, 26] and $\mathcal{C}^{r+1}$ expanding maps [24, 12]. For axiom A analytic and $\mathcal{C}^\infty$-expanding maps the above mentioned papers prove that the dynamical determinant is an entire function and its zeroes are exactly the inverse of the eigenvalues of the associated transfer operator; that is, it can be interpreted as a Fredholm determinant.

On the contrary for $\mathcal{C}^{r+1}$ Axiom A maps or flows the situation is still unsatisfactory. The strongest result to date is [18] where it is showed that the dynamical determinant for a C^{r+1} Anosov map, with expansion and contraction estimated by λ, is analytic in the disk $\lambda^{\frac{r}{2}}$. Nevertheless, in [18] the relation between the dynamical determinant and the transfer operator is only a formal one, in particular no information is available concerning the relation between the zeroes of such a function and the spectrum of the transfer operator. It was therefore a bit arbitrary to call such a function a Fredholm determinant.

In the present paper the missing relation is derived at the price of establishing the result in a smaller disk. Building on the results in [15] I will show that it is possible to make sense of the naïve idea of smoothing the singular kernel of

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the transfer operator, [1, page 103]. This yields a strategy greatly simplified with respect to previous approaches. In fact, it essentially boils down to a couple of pages computation. As a consequence one establishes the complete description of the correlation spectra (Ruelle resonances) in terms of periodic orbits. Finally, let me remark that, most likely, the present approach can be extended to more general transfer operators (e.g., with smooth weights), systems (e.g., Axiom A) and to the study of dynamical zeta functions (since the latter can be expressed as ratios of dynamical determinants [23]).

The plan of the paper is as follows. Section 2 details and proves the main results of the paper. Given the existence of a scale of *adapted Banach spaces* (see Definition 1) and Lemma 2.9 the proofs are completely self-contained. Lemma 2.9 is proven in section 3 while Proposition 2.1, proven in section 4, states the existence of the adapted spaces. This last result relies on a scale of Banach spaces introduced in [15], yet it should be emphasized that other choices of adapted spaces are possible, e.g. V.Baladi [3] and V.Baladi with M.Tsujii [5] have recently introduced different possibilities that could yield sharper bounds. Finally, an appendix contains an hardly surprising technical result that, for lack of references, needed to be proven somewhere.

Remark 1.1. *In this paper C stands for a generic constant depending only on the dynamical system (X, T) under consideration. Its actual value can thus change from one occurrence to the next.*

2. **The results.** In the following, I will discuss only the case $X = \mathbb{T}^d$ with the Euclidean metric. This simplifies the presentation and the notations since one can avoid the need to introduce local charts. The general case can be investigated in complete analogy by using partitions of unity and local charts along the lines exploited in [15]. Also, I will discuss only the transfer operator associated to the SRB measure although I do not see any real obstacle in treating more general, smooth, potentials.[1]

Let $T \in \text{Diff}^{r+1}(X, X)$ and $\mathcal{D}'_r$ be the space of distributions of order r, the transfer operator $\mathcal{L} : \mathcal{D}'_r \to \mathcal{D}'_r$ is defined by[2]

$$(\mathcal{L}h, \phi) := (h, \phi \circ T), \quad \forall \phi \in \mathcal{C}^r(X, \mathbb{C}). \tag{2.1}$$

In addition, consider a convolution operator $\hat{Q}_\varepsilon : \mathcal{C}^\infty \to \mathcal{C}^\infty$:

$$\begin{aligned} &\hat{Q}_\varepsilon f(x) := \int_{\mathbb{R}^d} q_\varepsilon(x-y) f(y) dy \\ &\int_{\mathbb{R}^d} q_\varepsilon(x) = 1; \quad \int_{\mathbb{R}^d} x^\alpha q_\varepsilon(x) = 0. \end{aligned} \tag{2.2}$$

[1]The main problem is than one needs the extension of [15] to such a setting. This is rather straightforward but to include it here would substantially increase the length of the paper without adding much to the presentation of the basic idea.

[2]Usually, the transfer operator is defined as acting on function but, in the present contest, it turns out to be essential that the operator can be defined also on distributions. In fact, by using the standard identification between functions and distributions (see also footnote 4), one can restrict the operator to $\mathcal{C}^r$ obtaining, for each $h \in \mathcal{C}^r(X, \mathbb{R})$, the usual formula $\mathcal{L}h(x) := f \circ T^{-1} |\det(D_x T^{-1})|$ which describes the evolution, under the dynamics, of the density of the measures absolutely continuous with respect to the Lebesgue measure m.

for each multi-index α such that $0 < |\alpha| \leq r$, and where $q_\varepsilon(x) = \varepsilon^{-d}\bar{q}(\varepsilon^{-1}x)$, $\bar{q}(x) = \bar{q}(-x)$, $\operatorname{supp}\bar{q} \subset \{x \in \mathbb{R}^d \ : \ |x| \leq 1\}$, $\bar{q} \in \mathcal{C}^\infty$. [3]

By duality one can then define the operator $Q_\varepsilon := \hat{Q}'_\varepsilon : \mathcal{D}' \to \mathcal{D}'$ which can be easily seen to be an extension of $\hat{Q}_\varepsilon$ to the space of distributions.

It is well known that the spectral properties of the transfer operator depend drastically on the space on which it acts. The space of distributions turns out to be too large a space to be useful, yet $\mathcal{C}^\infty$ is by far too small and the spectra of $\mathcal{L}$ on such a space bears little relevance on the statistical properties of the system. Below we give an abstract characterization of some properties that *good* dynamical spaces should enjoy.

Definition 1. *Given $T \in \operatorname{Diff}^{r+1}(X,X)$, a scale of Banach spaces $\{\mathcal{B}^s\}_{s\in\mathbb{N}}$ is called* adapted *to T if there exists $s_r \in \{1,\dots,r\}$ such that, for $s \in \{1,\dots,s_r\}$, $\mathcal{D}'_r \supset \mathcal{B}^{s-1} \supset \mathcal{B}^s \supset \mathcal{C}^s$.[4] More precisely, $\|\cdot\|_{s-1} \leq \|\cdot\|_s \leq |\cdot|_{\mathcal{C}^s}$, and $\overline{\mathcal{C}^s}^{\|\cdot\|_s} = \mathcal{B}^s$. In addition, there exist $\theta \in (0,1)$ and $\varepsilon_1 > 0$ such that, for all $0 < s \leq s_r$, $\mathcal{L} \in L(\mathcal{B}^s,\mathcal{B}^s)$ and, for each $\varepsilon \in (0,\varepsilon_1)$, $0 \leq l < s$ and $n \in \mathbb{N}$,[5]*

$$\left|\int f\phi\right| \leq C\|f\|_{\mathcal{B}^s} \cdot \sup_{k\leq r-s} |D_s^k\phi|_\infty \text{ for each } f,\phi \in \mathcal{C}^\infty; \tag{2.3}$$

$$\mathcal{L} : \mathcal{B}^s \to \mathcal{B}^{s-1} \text{ is compact}; \tag{2.4}$$

$$\|\mathcal{L}^n h\|_{\mathcal{B}^s} \leq B\|h\|_{\mathcal{B}^s} \text{ for each } h \in \mathcal{B}^s; \tag{2.5}$$

$$\|\mathcal{L}^n h\|_{\mathcal{B}^s} \leq A\theta^{sn}\|h\|_{\mathcal{B}^s} + B\|h\|_{\mathcal{B}^{s-1}} \text{ for each } h \in \mathcal{B}^s; \tag{2.6}$$

$$Q_\varepsilon \in L(\mathcal{B}^s,\mathcal{C}^\infty) \text{ and } \|Q_\varepsilon - \mathbf{Id}\|_{\mathcal{B}^s\to\mathcal{B}^{s-l}} \leq D\varepsilon^l; \tag{2.7}$$

$$\|hf\|_{\mathcal{B}^s} \leq D'\|h\|_{\mathcal{B}^s}|f|_{\mathcal{C}^r} \text{ for each } h \in \mathcal{B}^s,\ f \in \mathcal{C}^r; \tag{2.8}$$

where A, B, D, D' do not depend on ε and n.[6] If $T \in \operatorname{Diff}^\infty(X,X)$ and we have an adapted scale for each r, with $\lim_{r\to\infty} s_r = \infty$, then we say that we have a complete series *of adapted Banach spaces.*

From simple arguments (see, e.g., [15]) follows that on such spaces the spectrum of $\mathcal{L}$ has a *physical* interpretation: it describes the rate of decay of correlations and it is stable with respect to a large family of perturbations. In addition, in section 4 I prove:

Proposition 2.1. *If $T \in \operatorname{Diff}^{r+1}(X,X)$ is Anosov,[7] then there exists a scale of Banach spaces adapted to T with $s_r = \lceil \frac{r}{2} \rceil$ and $\theta = \lambda^{-1}$.[8] If $T \in \operatorname{Diff}^\infty(X,X)$ then the latter constitutes a complete series of adapted spaces.*

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[3] For a general manifold X, one must introduce coordinates charts Ψ_i and a subordinate partition of unity $\{\phi_i\}$, then, for ε small enough, one can define

$$\hat{Q}_\varepsilon f(x) := \sum_i \int J\Psi_i(\Psi_i^{-1}(x)+\xi)q_\varepsilon \circ \Psi_i(\Psi_i^{-1}(x)+\xi)\phi_i \circ \Psi_i(\Psi_i^{-1}(x)+\xi) f \circ \Psi_i(\Psi_i^{-1}(x)+\xi)$$

and the following holds essentially unchanged.

[4] Of course, to make sense of such a scale it is necessary to slightly abuse notations and identify each functions $f \in \mathcal{C}^s(X,\mathbb{R})$ with a linear functional (distribution) via the standard duality relation $(f,\varphi) := \int_X f\varphi dm$.

[5] By D_s I mean the derivative in the stable direction.

[6] In fact, Property (2.8), is needed only in the proof of Proposition 2.6.

[7] Anosov means that there exists a continuous splitting $E^u \otimes E^s$, $\dim(E^s) = d_s$ and $\dim(E^u) = d_u$, of the tangent bundle and a constant $\lambda > 1$ such that $\|D_xT^n v\| < \lambda^{-n}\|v\|$ for all $v \in E^s(x)$, $x \in X$ and $\|D_xT^{-n}v\| < \lambda^{-n}\|v\|$ for all $v \in E^u(x)$, $x \in X$.

[8] Given $a \in \mathbb{R}$, $\lceil a \rceil$ stands for the largest integer $n \leq a$.

Remark 2.2. *In section 4 I will define the wanted spaces based on the Banach spaces introduced in* [15]*, yet the present results hold for any other choice of adapted Banach spaces satisfying* (2.3)-(2.8)*.*

Remark 2.3. *The existence of an adapted scale of Banach spaces implies (see* [1]*): For each* $1 < s \leq s_r$ *and* $\sigma \in (\theta, 1)$*, the operator* $\mathcal{L}$ *is quasicompact on* $\mathcal{B}^s$*, more precisely it can be decomposed as* $\mathcal{L} = P_{\sigma,s} + R_{\sigma,s}$ *where* $P_{\sigma,s}R_{\sigma,s} = R_{\sigma,s}P_{\sigma,s} = 0$*,* $P_{\sigma,s}$ *is of finite rank and*

$$\|R^n_{\sigma,s}\|_{\mathcal{B}^s} \leq C\sigma^{sn}. \tag{2.9}$$

For further use let us set

$$\Gamma_n := \sum_{x\in \mathrm{Fix}\ T^n} |\det(\mathbf{Id} - D_xT^n)|^{-1}, \tag{2.10}$$

The following estimate is more or less standard. The proof can be found at the end of the section and is enclosed only for completeness.

Lemma 2.4. *If* $T \in \mathrm{Diff}^{r+1}(X,X)$ *is Anosov, for all* $n \in \mathbb{N}$*,* $\Gamma_n \leq C$*.*

The main result of the paper is the following.

Theorem 1. *If* $T \in \mathrm{Diff}^{r+1}(X,X)$ *is Anosov, define, for* $|z| < 1$*,*

$$d^\flat(z) := \mathrm{Exp}\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{x\in Fix\ T^n}|\det(\mathbf{Id} - D_xT^n)|^{-1}\right), \tag{2.11}$$

and consider its analytic extension. Then, if $\bar{s}_r := \frac{s_r^2}{s_r+r+2d}$*,* $d^\flat(z)\det(\mathbf{Id} - zP_{\sigma,s_r})^{-1}$ *is holomorphic and never zero in the disk* $|z| < \theta^{-\bar{s}_r}$*. Thus in such a disk* $d^\flat$ *is holomorphic and its zeroes are in one one correspondence with the eigenvalues of the operator* $\mathcal{L}$*. In addition, the algebraic multiplicity of the zeroes equal the dimension of the associated eigenspaces. In particular, if* $T \in \mathcal{C}^\infty$*, then* $d^\flat$ *is an entire function.*

Remark 2.5. *Using the spaces in* [15]*, see section 4, the above Theorem gives the analyticity domain* $\lambda^{-\frac{r}{6+\frac{8d}{r}}}$*. This is certainly far from optimal and can be easily improved.*[9] *Yet, since I do not see how to obtain the expected Kitaev-like bound* $\lambda^{-\frac{r}{2}}$*, I will not strive for superficial improvements at the expenses of clarity, simplicity and brevity.*

In the present language the SRB measures are the eigenvectors associated to the eigenvalue one. Probably the most interesting physical consequence of Theorem 1 is the following.

Proposition 2.6. *Given a transitive (hence mixing) Anosov map* $T \in \mathrm{Diff}^\infty(X,X)$*, let* μ_{SRB} *be the SRB measure and let* $f, g \in \mathcal{C}^\infty(X,\mathbb{R})$*,* $\mu_{SRB}(f) = \mu_{SRB}(g) = 0$*, then the function (the* correlation spectra*)*

$$C_{f,g}(e^{i\omega}) = \sum_{n\in\mathbb{Z}} e^{i\omega n}\mu_{SRB}(fg\circ T^n)\ ; \quad \omega \in \mathbb{R}$$

extends to a meromorphic function on $\mathbb{C}\setminus\{0\}$ *and its poles (often called* Ruelle resonances*) are exactly described by the zeroes of* $d^\flat$*.*

[9]For example, in the known examples, d can probably be replaced by d_u.

Proof. Fix any $r \in \mathbb{N}$ and consider an associated scale of adapted spaces. Calling m the Lebesgue measure[10] the SRB measure can be defined as

$$\mu_{SRB}(\phi) = \lim_{n\to\infty} m(\phi \circ T^n) = \lim_{n\to\infty} \mathcal{L}^n 1(\phi).$$

Note that, since the map is mixing, then one must be a simple eigenvalue and no other eigenvalues can be present on the unit circle.[11] Consequently, remembering Remark 2.3, $\mathcal{L}$ has a spectral gap, hence there exists $\rho > 1$ such that, for each $h \in \mathcal{B}^s$

$$\|\mathcal{L}^n h - (h,1)\mu_{SRB}\|_s \leq C\|h\|_s \rho^{-n}.$$

Since, $\mu_{SRB}(fg \circ T^n) = \lim_{p\to\infty}(\mathcal{L}^p 1, fg \circ T^n)$ it is natural to define the measures $m_{p,f}(h) := (\mathcal{L}^p 1, fh)$. In fact, by (2.8), we have $m_{p,f} \in \mathcal{B}^s$, $s \leq s_r$ and

$$m_{p,f}(g \circ T^n) = \mathcal{L}^n m_{p,f}(g) = \mu_{SRB}(g) m_{p,f}(1) + \mathcal{O}(\rho^{-n}) = \mathcal{O}(\rho^{-n}).$$

The above means that, for each $|z| < \rho$,

$$\begin{aligned}
\sum_{n\in\mathbb{N}} z^n \mu_{SRB}(fg \circ T^n) &= \lim_{p\to\infty} \sum_{n\in\mathbb{N}} z^n \mathcal{L}^n m_{p,f}(g) \\
&= \lim_{p\to\infty} [(\mathbf{Id} - z\mathcal{L})^{-1} m_{p,f}](g) \\
&= [(\mathbf{Id} - z\mathcal{L})^{-1} \lim_{p\to\infty} m_{p,f}](g) \\
&= [(\mathbf{Id} - z\mathcal{L})^{-1} \mu_f](g) =: G_{f,g}(z),
\end{aligned} \tag{2.12}$$

where $\mu_f(h) := \mu_{SRB}(fh)$. Since $\mu_f \in \mathcal{B}^s$, for each $s \leq s_r$, by Remark 2.3 follows that the function $G_{f,g}$ can be extended to a meromorphic function on $\{z \in \mathbb{C} \ : \ |z| < \theta^{-s_r}\}$. On the other hand, if $|z| > 1$,

$$\sum_{n\in\mathbb{N}} z^{-n} \mu_{SRB}(fg \circ T^{-n}) = \sum_{n\in\mathbb{N}} z^{-n} \mu_{SRB}(gf \circ T^n) = G_{g,f}(z^{-1}).$$

Hence the formula

$$C_{f,g}(e^{i\omega}) = G_{f,g}(e^{i\omega}) + G_{g,f}(e^{-i\omega}) - \mu_{SRB}(fg)$$

together with (2.12), shows that $C_{f,g}$ is meromorphic in the annulus $\{z \in \mathbb{C} \ : \ \theta^{s_r} < |z| < \theta^{-s_r}\}$. By Theorem 1 its poles are the zeroes and the inverse of the zeroes of $d^\flat$ in the annulus $\{z \in \mathbb{C} \ : \ \theta^{\bar{s}_r} < |z| < \theta^{-\bar{s}_r}\}$. The Lemma easily follows since we have a complete series of spaces and r can be chosen arbitrarily. $\square$

Remark 2.7. *For $\mathcal{C}^{r+1}$ maps the above argument shows that the correlation function is meromorphic in the anulus $\{\lambda^{-\lceil \frac{r}{2}\rceil} < |z| < \lambda^{\lceil \frac{r}{2}\rceil}\}$, but the relations between the poles and the zeroes of the dynamical determinat can be established only in the smaller anulus $\{\lambda^{-\frac{r}{6+8d/r}} < |z| < \lambda^{\frac{r}{6+8d/r}}\}$.*

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[10] Note that, since $(1,\phi) = \int \phi dm$ (see footnote 4), then m can also be seen as the element 1 of $\mathcal{B}^s$ or $\mathcal{D}'_s$.

[11] Indeed, if $e^{i\theta} \in \sigma(\mathcal{L})$, then $\Pi_\theta := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-i\theta k}\mathcal{L}^k$ is well defined and is exactly the projector on the associated eigenspace. Moreover, from (2.5) and (2.6) follows that Range$(\Pi_\theta) \subset \mathcal{B}_0$. Hence, by (2.3),

$$|(\Pi_\theta h, \phi)| = |(\Pi_\theta^2 h, \phi)| \leq \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |(\Pi_\theta h, \phi \circ T^n) \leq C\|\Pi_\theta h\|_s |\phi|_\infty.$$

That is the eigenspace would consist of measures, whereby violating the mixing assumption.

Remark 2.8. *Since $C_{f,g}(e^{i\omega})$ is the Fourier transform of the correlation function, it is a physically accessible function. Its poles on the complex plane (the Ruelle resonances) can be computed, e.g. via Pade approximants, hence they are physically observable as well.*

The proof of Theorem 1 rests on the next basic estimate proven in section 3.

Lemma 2.9. *For each $n \in \mathbb{N}$, $\sigma \in (\theta, 1)$, holds true*[12]

$$|\Gamma_n - \operatorname{Tr} P^n_{\sigma,s_r}| \leq C_\sigma \sigma^{\frac{s_r^2}{s_r+r+2d}n}.$$

Proof of Theorem 1. For $|z| < 1$, let

$$g(z) := \det(\mathbf{Id} - zP_{\sigma,s_r})^{-1} d^\flat(z) = \operatorname{Exp}\left(-\sum_{n=1}^{\infty} \frac{z^n}{n}(\Gamma_n - \operatorname{Tr} P^n_{\sigma,s_r})\right).$$

By the estimate of Lemma 2.9 , g is analytic and different from zero, in the disk $|z| < \sigma^{\frac{s_r^2}{s_r+r+2d}n}$. Since

$$d^\flat(z) = \det(\mathbf{Id} - zP_{\sigma,s_r})g(z)$$

the theorem trivially follows from the arbitrariness of σ. □

Proof of Lemma 2.4. Clearly we must worry only about large n. Consider $x \in \operatorname{Fix} T^n$, choose a coordinate system (ξ, η) in a neighborhood of x such that $W^u(x) = \{(\xi, 0)\}$ and $W^s(x) = \{(0, \eta)\}$. In such coordinates

$$D_x T^n = \begin{pmatrix} A_n(x) & 0 \\ 0 & B_n(x) \end{pmatrix}$$

where $\|A_n(x)^{-1}\| \leq C\lambda^{-n}$ and $\|B_n(x)\| \leq C\lambda^{-n}$. Accordingly

$$\begin{aligned} |\det(\mathbf{Id} - D_x T^n)| &= |\det A_n(x) \det(\mathbf{Id} - A_n(x)^{-1}) \det(\mathbf{Id} - B_n(x))| \\ &\geq C^{-1} |\det A_n(x)| = C^{-1} |\det(D_x T^n\big|_{E^u})|. \end{aligned} \tag{2.13}$$

Let us now consider a small fixed $\rho > 0$ and let $W^{u,s}_\rho(z)$ be the unstable and stable manifolds of z of size ρ, respectively. By (2.13) and standard distortion arguments

$$|\det(\mathbf{Id} - D_x T^n)|^{-1} \leq C\rho^{-d_u} \int_{W^u_\rho(x)} |\det(DT^n\big|_{E^u})|^{-1} d\xi = C\rho^{-d_u} \int_{T^{-n}W^u_\rho(x)} d\xi.$$

Next, let us consider the sets $Z_\rho(x) := \cup_{y \in T^{-n}W^u_\rho(x)} W^s_\rho(y)$ and notice that $\operatorname{Fix} T^n \cap Z_{2\rho}(x) = \{x\}$. Indeed, let $z \in \operatorname{Fix} T^n \cap Z_{2\rho}(x)$, then, if ρ has been chosen small enough, $W^s_{2\rho}(z) \cap W^u_{2\rho}(x)$ contain only one point, let it be y. But, by construction $y \in W^s_{2\rho}(z) \cap T^{-n}W^u_{2\rho}(x)$, hence $T^n y \in W^s_{2\rho}(z) \cap W^u_{2\rho}(x)$, that is $T^n y = y$. But $y = \lim_{n\to\infty} T^{-n}y = x$ and $y = \lim_{n\to\infty} T^n y = z$, hence $x = y = z$.

The above discussion implies that if $x_1, x_2 \in \operatorname{Fix} T^n$, $x_1 \neq x_2$, then $Z_\rho(x_1) \cap Z_\rho(x_2) = \emptyset$. Hence

$$\sum_{x \in \operatorname{Fix} T^n} |\det(\mathbf{Id} - D_x T^n)|^{-1} \leq C\rho^{-d} \sum_{x \in \operatorname{Fix} T^n} m(Z_\rho(x)) \leq C\rho^{-d} m(X).$$

□

[12]Since $P_{\sigma,s}$ is a finite rank operator, the usual trace Tr and the determinant are well defined.

3. **proof of Lemma 2.9.** The first step in the proof of Lemma 2.9 is to define, given an integral operator $Kh(x) := \int \kappa(x,y)h(y)dy$, $\kappa \in \mathcal{C}^0(X^2)$,[13]

$$\mathbf{Tr}\, K := \int_X \kappa(x,x)dx. \tag{3.1}$$

The first key ingredient is a representation of such an integral trace for small ε.

Lemma 3.1. *For each ε small enough, holds true*

$$\mathbf{Tr}\, Q_\varepsilon \mathcal{L}^n = \Gamma_n + \mathcal{O}(\varepsilon^r).$$

Proof. Since $Q_\varepsilon \mathcal{L}^n h(x) = \int_X q_\varepsilon(x - T^n y)h(y)dy =: \int_X \kappa_{\varepsilon,n}(x,y)h(y)dy$, we have

$$\mathbf{Tr}\, Q_\varepsilon \mathcal{L}^n = \int_X \kappa_{\varepsilon,n}(x,x)dx = \int_X q_\varepsilon(x - T^n x)dx. \tag{3.2}$$

Next, we consider the change of variable $z = \Phi_n(x) := x - T^n x$, clearly

$$\det D_x\Phi_n = \det(\mathbf{Id} - D_xT^n) \tag{3.3}$$

Let $B(0,\varepsilon) \subset \mathbb{R}^d$ be the ball of radius ε and center zero. If $z \in B(0,\varepsilon)$, then it turns to be useful to define the map $F_z : \mathbb{T}^d \to \mathbb{T}^d$

$$F_z(x) := T^n(x) + z \mod 1. \tag{3.4}$$

For ε small enough, F_z is still hyperbolic hence for each $x \in \text{Fix}\,(T^n)$ we can consider the F_z ε-orbit $\{x, x, \dots\}$. By shadowing the exists a unique point x_z , in a neighborhood of x, such that $x_z = F_z(x_z) = T^n(x_z) + z$. The latter fact means that $\Phi_n(x_z) = z$, that is $B(0,\varepsilon) \subset \text{Range}\,\Phi_n$. On the other hand, If $x \in \Phi_n^{-1}(B(0,\varepsilon))$, then, by shadowing, it is associated to a unique periodic point of period n. Indeed, given $z \in B(0,\varepsilon)$ and x such that $x - T^n x = z$, then there exists a unique periodic orbit of period n in a neighborhood of the periodic ε-pseudo-orbit $\{x, Tx, \dots, T^{n-1}x\}$. We can then define the function $\Psi : \Phi_n^{-1}(B(0,\varepsilon)) \to \text{Fix}\, T^n$. For each $x \in \text{Fix}\, T^n$ we can then define the set $\Delta_x := \Psi^{-1}(x)$. Due to hyperbolicity of T it is easy to verify that $\Phi : \Delta_x \to B(0,\varepsilon)$ is one-to-one beside being onto. We can then label the inverse branches of Φ_n by the elements of Fix T^n.

Sub-lemma 3.2. *There exists a constant M such that, for each inverse branch,*

$$\|\Phi_n^{-1}\|_{\mathcal{C}^{s+1}} \leq M\ ; \quad \|\det(D\Phi_n^{-1})\|_{\mathcal{C}^s} \leq M|\det(D_{x_*}\Phi_n^{-1})| \quad \forall s \in \{0, \dots, r\},$$

where $x_* \in \text{Fix}\, T^n$ labels the inverse branch.

The above estimate, whose technical but straightforward proof is postponed to the appendix, together with Lemma 2.4, (2.2) and (3.2) yields

$$\begin{aligned}\mathbf{Tr}\, Q_\varepsilon \mathcal{L}^n &= \sum_{x \in \text{Fix}\, T^n} \int_{\mathbb{R}^d} q_\varepsilon(z) \left|\det(D_{\Phi|_{\Delta_x}^{-1}(z)}\Phi_n^{-1})\right| dz \\ &= \sum_{x \in \text{Fix}\, T^n} |\det(\mathbf{Id} - D_xT^n)|^{-1} + \mathcal{O}(\varepsilon^r).\end{aligned}$$

□

[13] Notice that the definition below may not coincide necessarily with the usual trace even when the latter is well defined, e.g. it does not necessarily correspond to the sum of the eigenvalues.

Notice that $\mathbf{Tr}\, Q_\varepsilon \mathcal{L}^n Q_\varepsilon = \mathbf{Tr}\, Q_\varepsilon^2 \mathcal{L}^n$ and that Q_ε^2 has exactly the same properties as Q_ε, thus Lemma 3.1 applies to $Q_\varepsilon^2\mathcal{L}^n$ as well.

Setting $\varphi_{\varepsilon,y}(x) := q_\varepsilon(x-y) \in \mathcal{C}^\infty$, for each $A \in L(\mathcal{B}^s, \mathcal{B}^s)$ and $f \in \mathcal{C}^s$ holds[14]

$$AQ_\varepsilon f = \int A\varphi_{\varepsilon,y} f(y)dy. \tag{3.5}$$

Using (3.5) and remembering (2.3), (2.7), (2.9) yields

$$\begin{aligned}\left|\mathbf{Tr}\,(Q_\varepsilon R^n_{\sigma,s} Q_\varepsilon)\right| &= \left|\int dy (R^n_{\sigma,s}\varphi_{\varepsilon,y}, \varphi_{\varepsilon,y})\right| \leq C\varepsilon^{-d-r+s} \sup_y \|R^n_{\sigma,s}\varphi_{\varepsilon,y})\|_{\mathcal{B}^s} \\ &\leq C\sigma^{sn} \sup_y \|\varphi_{\varepsilon,y}\|_{\mathcal{B}^s} \varepsilon^{-r+s-d} \leq C\sigma^{sn}\varepsilon^{-r-2d}.\end{aligned} \tag{3.6}$$

The last step is given by the following perturbation result.

Lemma 3.3. *There exists $\varepsilon_1 > 0$ such that, for each $n \in \mathbb{N}$ and $\varepsilon \in (\sigma^n, \varepsilon_1)$, holds true*

$$|\operatorname{Tr} P^n_{\sigma,s} - \mathbf{Tr}\, Q_\varepsilon P^n_{\sigma,s} Q_\varepsilon| \leq C\varepsilon^{s_r}.$$

Proof. Since $P_{\sigma,s}$ is finite dimensional the usual trace of $P^n_{\sigma,s}$, $\operatorname{Tr} P^n_{\sigma,s}$, is well defined. More precisely, $P_{\sigma,s}h = \sum_i w_i \ell_i(h)$, $w_i \in \mathcal{B}^{s_r}$, $\ell_i \in (\mathcal{B}^s)'$, and $\operatorname{Tr} P_{\sigma,s} = \sum_i \ell_i(w_i)$. Hence, for each $h \in \mathcal{C}^\infty$, by (3.5) and (2.7),

$$\begin{aligned}(Q_\varepsilon P^n_{\sigma,s} Q_\varepsilon h)(x) &= \int (Q_\varepsilon P^n_{\sigma,s}\varphi_{\varepsilon,y})(x) h(y)dy \\ &= \int_X \sum_i (Q_\varepsilon w_i)(x)\ell_i(P^{n-1}_{\sigma,s}\varphi_{\varepsilon,y})h(y)dy.\end{aligned}$$

Next, let $\Delta_{\sigma,\tau} := P_{\sigma,\tau+1} - P_{\sigma,\tau}$, $\Delta_{\sigma,0} := P_{\sigma,1}$, clearly we can assume, without loss of gnerality, that $\Delta_{\sigma,\tau}(h) = \sum_{i=j_\tau}^{j_{\tau+1}} w_i\ell_i(h)$, with $\ell_i \in (\mathcal{B}^\tau)'$. Moreover, $\Delta_{\sigma,\tau}\Delta_{\sigma,\tau'} = 0$ for each $\tau \neq \tau'$ and $\|\Delta^n_{\sigma,\tau}\|_{\mathcal{B}^\tau} \leq C\sigma^{\tau n}$. Thus, by (3.5) and (2.7),

$$\begin{aligned}\mathbf{Tr}\, Q_\varepsilon P^n_{\sigma,s} Q_\varepsilon - \operatorname{Tr} P^n_{\sigma,s} &= \sum_{\tau=0}^{s-1}\sum_{i=j_\tau}^{j_{\tau+1}} \ell_i(\Delta^{n-1}_{\sigma,\tau}(Q^2_\varepsilon - \mathbf{Id})w_i) \\ &= \sum_{\tau=0}^{s-1} \sum_{j_\tau \leq i \leq j_{\tau+1}} \mathcal{O}(\|\Delta^n_{\sigma,\tau}(Q^2_\varepsilon - \mathbf{Id})w_i\|_{\mathcal{B}^\tau}) \\ &= \sum_{\tau=0}^{s-1} \mathcal{O}(\sigma^{\tau n}\varepsilon^{s_r-\tau}) = \mathcal{O}(\varepsilon^{s_r}).\end{aligned}$$

□

Collecting (3.6) and Lemma 3.3 yields

$$\mathbf{Tr}\,(Q_\varepsilon \mathcal{L}^n Q_\varepsilon) = \operatorname{Tr}(P^n_{\sigma,s_r}) + \mathcal{O}(\sigma^{sn}\varepsilon^{-r-2d} + \varepsilon^{s_r}). \tag{3.7}$$

Proof of Lemma 2.9. Lemma 3.1 and (3.7) imply

$$|\Gamma_n - \operatorname{Tr} P^n_{\sigma,s_r}| \leq C\left(\varepsilon^r + \varepsilon^{s_r} + \sigma^{s_r n}\varepsilon^{-r-2d}\right).$$

Finally, choose $\varepsilon = \sigma^{\frac{s_r}{s_r+r+2d}n}$, hence the lemma. □

[14]This follows immediately from the fact that, if $f_n \to f$ in L^1, then $Q_\varepsilon f_n \to Q_\varepsilon f$ in $\mathcal{C}^s$, hence $\lim_{n\to\infty} AQ_\varepsilon f_n = AQ_\varepsilon f$. One can then approximate f by piecewise constant function and compute the corresponding Riemann sums. Taking the limit and since $y \mapsto \varphi_{\varepsilon,y} \in \mathcal{B}^s$ is continuous one recovers the integral on the right which is meant in Bochner sense.

4. **proof of proposition 2.1.** Let us start by recalling the scale of Banach spaces introduced in [15].

It is well known that being Anosov is equivalent to the existence of a continuous strictly invariant vector field $\mathcal{C}$. Let $\mathcal{C}'$ be another continuous cone field contained in $\mathrm{Int}(\mathcal{C})$. Consider $\delta > 0$ and a set of d_s-dimensional manifolds (with boundary) Ω such that, if $W \in \Omega$, then there exists $x_W \in W$ and a d_s dimensional hyperplane E_W contained in $\mathcal{C}'$ such that, making an isometric change of coordinates such that $E_W = \{(\xi, 0) \ : \ \xi \in \mathbb{R}^{d_s}\}$, $W = \{x_W + (\xi, \gamma_W(\xi)) \ : \ \xi \in \mathbb{R}^{d_s}, \|\xi\| \leq \delta\}$, $\mathcal{T}_x W \subset \mathcal{C}(x)$, for each $x \in W$, and $|\gamma_W|_{\mathcal{C}^{r+1}(\mathbb{R}^{d_s}, \mathbb{R}^{d_u})} \leq M$ for some fixed M large enough.

Given $W \in \Omega$, we will denote by $\mathcal{C}^q_0(W, \mathbb{R})$ the set of functions from W to $\mathbb{R}$ which are $\lceil q \rceil$ times continuously differentiable and such that the $\lceil q \rceil$ derivative is $q - \lceil q \rceil$ Hölder continuos on W and vanish on a neighborhood of the boundary of W. For each $h \in \mathcal{C}^\infty(X, \mathbb{R})$ and $q \in \mathbb{R}_+$, $p \in \mathbb{N}^*$, let

$$\|h\|_{p,q} := \sup_{|\alpha| \leq p} \sup_{W \in \Omega} \sup_{\substack{\varphi \in \mathcal{C}^{q+|\alpha|}_0(W,\mathbb{R}) \\ |\varphi|_{\mathcal{C}^{q+|\alpha|}} \leq 1}} \int_W \partial^\alpha h \cdot \varphi, \tag{4.1}$$

and define the Banach spaces $\mathcal{B}^{p,q} := \overline{\mathcal{C}^\infty}^{\|\cdot\|_{p,q}}$.

In [15] it is proven that, setting $\mathcal{B}^s := \mathcal{B}^{s, r-s}$, $\mathcal{L} \in L(\mathcal{B}^s, \mathcal{B}^s)$ and it satisfies (2.4), (2.5) and (2.6), provided $s \leq \lceil \frac{r}{2} \rceil =: s_r$.

To prove (2.3), choose a smooth partition of unity $\{\Theta_i\}$ with $\mathrm{supp}\Theta_i$ contained in balls of size $\delta/2$ and choose a smooth partition $\{W_i(\xi)\}$, of the neighborhood of each support, made of elements of Ω. Then

$$\begin{aligned}\int f\phi &= \int \phi \circ T^{-n} \mathcal{L}^n f = \sum_i \int \Theta_i \phi \circ T^{-n} \mathcal{L}^n f = \sum_i \int d\xi \int_{W_i(\xi)} \Theta_i \phi \circ T^{-n} \mathcal{L}^n f \\ &= \sum_i \int d\xi \int_{T^{-n} W_i(\xi)} \Theta_i \circ T^n \phi |\det DT^n|^{-1} J_{W_i(\xi)} T^n f \\ &\leq C \|f\|_{\mathcal{B}^s} \sum_{ij} \int d\xi \Big| \Theta_i \circ T^n \rho_{\xi,i,j} \phi |\det DT^n|^{-1} J_{W_i(\xi)} \Big|_{\mathcal{C}^{r-s}(W_{ij}(\xi))},\end{aligned}$$

where the $W_{ij}(\xi) \in \Omega$ are a covering of $W_i(\xi)$ and $\rho_{\xi,i,j}$ is a smooth partition of unity subordinate to such a covering. Since ϕ is $\mathcal{C}^r$ it follows that, for n large enough, $|\phi|_{\mathcal{C}^{r-s}(W_{ij}(\xi))} \leq 2 \sup_{k \leq r-s} |D^k_s \phi|_\infty$, choosing such an n, and using Lemma 6.2 of [15], (2.3) readily follows.

Next, let us prove (2.7).

Lemma 4.1. *For each $l < s \leq s_r$ holds $Q_\varepsilon \in L(\mathcal{B}^s, \mathcal{C}^\infty)$ and*

$$\|Q_\varepsilon - \mathbf{Id}\|_{\mathcal{B}^s \to \mathcal{B}^l} \leq C \varepsilon^{s-l}.$$

Proof. Consider $W \in \Omega$. Let $W_\rho := \{x_W + (\xi, (1-\rho)\gamma(\xi)) \in \mathbb{R}^d \ : \ \|\xi\| \leq \delta\}$. Clearly, provided δ has been chosen small enough, for each $\rho \in [0,1]$, $W_\rho \in \Omega$ with $x_{W_\rho} = x_W$, $E_{W_\rho} = E_W$. Then, for each multi-index α, $|\alpha| \leq l$,

$$\begin{aligned}
&\int_W \partial^\alpha h\varphi = \int_{\|\xi\|\leq\delta} \partial^\alpha h(\xi,\gamma(\xi))\varphi(\xi,\gamma(\xi))J_W(\xi)d\xi \\
&= \sum_{i=0}^{s-l-1} \frac{1}{i!}\int_{\|\xi\|\leq\delta} \partial^\alpha \frac{d^i}{d\rho^i} h(\xi,(1-\rho)\gamma(\xi))\big|_{\rho=\varepsilon}\varphi(\xi,\gamma(\xi))J_W(\xi)d\xi \\
&+\int_0^\varepsilon d\rho_1\cdots\int_0^{\rho_{s-l-1}} d\rho_{s-l}\int_{\|\xi\|\leq\delta}\partial^\alpha\frac{d^{s-l}}{d\rho^{s-l}}h(\xi,(1-\rho)\gamma(\xi))\big|_{\rho=\rho_{s-l}}\varphi(\xi,\gamma(\xi))J_W(\xi)d\xi \\
&= \sum_{\substack{i=0\\|\beta|=i}}^{s-l-1}\frac{(-\varepsilon)^i}{i!}\int_{W_\varepsilon}\gamma^\beta\partial^{\alpha+\beta}h\varphi_\varepsilon + \sum_{|\beta|=s-l}(-1)^{s-l}\int_0^\varepsilon d\rho_1\cdots\int_0^{\rho_{s-l-1}}d\rho_{s-l} \\
&\qquad\times\int_{W_{\rho_{s-l}}}\gamma^\beta\partial^{\alpha+\beta}h\varphi_{\rho_{s-l}} \\
&\quad= \sum_{\substack{i=0\\|\beta|=i}}^{s-l-1}\frac{(-\varepsilon)^i}{i!}\int_{W_\varepsilon}\gamma^\beta\partial^{\alpha+\beta}h\varphi_\varepsilon + \mathcal{O}(\|h\|_{s,q_r}\varepsilon^{s-l})
\end{aligned}$$

Thus, in order to estimate the norms, it suffices to consider $\Omega_\varepsilon := \{W_\varepsilon \;:\; W\in\Omega\}$, that is manifolds uniformly strictly inside the cone field. Let $W\in\Omega_\varepsilon$, then

$$\int_W \partial^\alpha Q_\varepsilon h\varphi - \int_W \partial^\alpha h\varphi = \int dz\, q_\varepsilon(z)\int_W dy\,[\partial^\alpha h(y+z)-\partial^\alpha h(y)]\varphi(y).$$

If $W=\{\xi,\gamma(\xi)\}$, then $W_z=\{(\xi,\gamma(\xi))+z\}\in\Omega$, provided z is small enough. Thus, for $|\alpha|=l$, remembering (2.2),

$$\begin{aligned}
\int_W \partial^\alpha Q_\varepsilon h\varphi - \int_W \partial^\alpha h\varphi &= \sum_{|\beta|=s-l}\int dz q_\varepsilon(z)\int_0^1 dt_1\cdots\int_0^{t_{s-l-1}}dt_{s-l} \\
&\times\int_{W_{zt_{s-l}}} dy\partial^{\alpha+\beta}h(y)z^\beta\varphi_{zt_{s-l}}(y)\leq\|h\|_{s,q_r}\varepsilon^{s-l}.
\end{aligned}$$

□

Finally, (2.8) follows easily from (4.1). Clearly if $T\in\mathrm{Diff}^\infty(X,X)$ we have a complete series of adapted spaces.

Appendix A. **Proof of Sub-Lemma 3.2.** Let us choose a periodic point $x_*\in$ Fix (T^n) and limit our considerations to the associated inverse branch, that, by a slight abuse of notation, I will designate simply by Φ_n^{-1}. Again we will use the map F_z introduced in (3.4). Clearly $D_z\Phi_n^{-1}=(\mathbf{Id}-D_{x_z}T^n)^{-1}=(\mathbf{Id}-D_{x_z}F_z)^{-1}$, where $\Phi_n(x)=z$.

To study the regularities properties at a given point z_0, small enough, it is convenient to perform an affine change of coordinates $\Lambda(z_0)$ such that $\Lambda(z_0)(x_{z_0})=0$, $|(D_x\Lambda)|_{\mathcal{C}^0}+|(D_x\Lambda)^{-1}|_{\mathcal{C}^0}\leq C$, and[15]

$$D_{\Lambda(z_0)(x_z)}\widetilde{F}_{z,z_0}=\begin{pmatrix}A(z,z_0) & B(z,z_0)\\ C(z,z_0) & D(z,z_0)\end{pmatrix} \tag{A.1}$$

[15]Here, and in the following, given $I\subset\mathbb{R}^q$ and a function f from I to some Banach algebra $\mathbb{B}$, by $|\cdot|_{\mathcal{C}^p}$ we mean the norm $\sup_{z\in I;\,|\alpha|\leq p} 2^{p-|\alpha|}\|\partial^\alpha f(z)\|_{\mathbb{B}}$ so that $\mathcal{C}^p(I,\mathbb{B})$ is itself a Banach algebra.

where $\widetilde{F}_{z,z_0} := \Lambda(z_0)\circ F_z\circ\Lambda(z_0)^{-1}$; $\|A(z_0,z_0)\|, \|D(z_0,z_0)^{-1}\| \le \lambda^{-n}$ and $B(z_0,z_0) = C(z_0,z_0) = 0$. In other words, in the new coordinates, $\{(\xi,0)\}$ corresponds to the stable manifold at z_0 and $\{(0,\eta)\}$ to the unstable one. In such coordinates,[16]

$$D_z\Phi_n^{-1}\big|_{z=z_0} = \begin{pmatrix}(\mathbf{Id}-A)^{-1} & 0\\ 0 & -(\mathbf{Id}-D^{-1})^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{Id} & 0\\ 0 & D^{-1}\end{pmatrix}. \tag{A.2}$$

Given the simpler structure of $\widetilde{F}_{z_0,z_0}$ it would be much easier to study its regularity rather than the one of F_{z_0}. Yet, the two are equivalent only if the change of coordinates $\Lambda(z_0)$ is uniformly $\mathcal{C}^r$. To prove the latter is our first task.

We start by computing the derivatives of $x_z^i := T^i x_z$ with respect to z:

$$\frac{\partial x_z^i}{\partial z} = D_{x_z}T^i(\mathbf{Id} - D_{x_z}T^n)^{-1}.$$

It is convenient to use in the tangent space at x_z^i the coordinates pushed forward from the tangent space of x_z. In such coordinates holds

$$D_{x_z^{i-1}}T =: \begin{pmatrix}A_i(z,z_0) & B_i(z,z_0)\\ C_i(z,z_0) & D_i(z,z_0)\end{pmatrix} \tag{A.3}$$

where $\|A_i\|, \|D_i^{-1}\| \le \lambda^{-1}$ and $B_i(z_0,z_0) = C_i(z_0,z_0) = 0$. Hence,

$$\frac{\partial x_z^i}{\partial z}\Big|_{z=z_0} = \begin{pmatrix}\prod_{j=1}^i A_i & 0\\ 0 & \prod_{j=i+1}^n D_i^{-1}\end{pmatrix}\begin{pmatrix}(\mathbf{Id}-A)^{-1} & 0\\ 0 & -(\mathbf{Id}-D^{-1})^{-1}\end{pmatrix}, \tag{A.4}$$

which readily implies $|\partial_z x_z^i|_{\mathcal{C}^0} \le \lambda^{-i}C$, for the stable coordinates, and $|\partial_z x_z^i|_{\mathcal{C}^0} \le \lambda^{i-n}C$ for the unstable ones.

An hyperplane E in the stable direction is uniquely determined by a linear operator $U : \mathbb{R}^{d_s} \to \mathbb{R}^{d_u}$: $E = \{(\xi, U\xi) \;:\; \xi \in \mathbb{R}^{d_s}\}$. A simple computation shows that, by defining

$$H(z,z_0,U) := (C(z,z_0) + D(z,z_0)U)(A(z,z_0) + B(z,z_0)U)^{-1},$$

the stable hyperplane for $\widetilde{F}_z$ at the point $\Lambda(z_0)(x_z)$ is determined by the fixed point of $H(z_0,z,U(z)) = U(z)$.[17] Applying the implicit function theorem, since by construction $U(z_0) = 0$, yields

$$\partial_z U\big|_{z=z_0} = -(\mathbf{Id} - G)^{-1}(D^{-1}\partial_z C\big|_{z=z_0}), \tag{A.5}$$

where $G : GL(d_s,d_u) \to GL(d_s,d_u)$ is defined by $G(V) := D^{-1}VA$. In addition,

$$\partial_{z_i} C\big|_{z=z_0} = \sum_{p=1}^{d}\sum_{k=1}^{n}\left[\prod_{j>k} D_j\right] c_{2,p}(x_{z_0}^{k-1})\left[\prod_{j<k} A_j\right]\frac{\partial (x_z^{k-1})_p}{\partial z_i}\Big|_{z=z_0} \tag{A.6}$$

where

$$\partial_{x_p} D_x T = \begin{pmatrix}a_{2,p}(x) & b_{2,p}(x)\\ c_{2,p}(x) & d_{2,p}(x).\end{pmatrix}$$

This, by equations (A.1) and (A.3) implies $|U|_{\mathcal{C}^1} \le C$. Since the exact same argument can be carried out for the unstable space, remembering (A.4) we have $|\Lambda|_{\mathcal{C}^1} + |D_{x_{z_0}}\Lambda|_{\mathcal{C}^1} + |D_{x_{z_0}}\Lambda^{-1}|_{\mathcal{C}^1} \le C$. We can thus conclude the argument by induction: let $l \le r$ and suppose that $|x_z^i|_{\mathcal{C}^l} \le C$ and $|D_x\Lambda|_{\mathcal{C}^l} + |D_x\Lambda^{-1}|_{\mathcal{C}^l} \le C$. Then by (A.3) follows $A_i(z,z), B_i(z,z), C_i(z,z), D_i(z,z)$, seen as functions of z have $\mathcal{C}^l$

[16]Here and in the following I suppress the dependence on z, z_0 when non confusion arises.

[17]In fact, it is known that $U(z)$ is $\mathcal{C}^{r-1}$, e.g. see [22, Propositions 1, 2], yet here we need explicit estimates. This forces us to redo the argument.

norms equibounded. In turn, by (A.4), it follows $|x_z^i|_{\mathcal{C}^{l+1}} \leq C$. Equations, (A.5) and (A.6), imply then $|D_x\Lambda|_{\mathcal{C}^{l+1}} + |D_x\Lambda^{-1}|_{\mathcal{C}^{l+1}} \leq C$, provided $l+1 \leq r$. This proves that the change of coordinates $\Lambda(z_0)$ is uniformly $\mathcal{C}^r$.

It is now easy to verify that $\tilde{F}_z$ is $\mathcal{C}^r$ and, remembering (A.2), the first inequality of the sub-lemma can be readily proven. To conclude note that

$$\det(D\Phi_n^{-1}) = -\det(\mathbf{Id} - A)^{-1}\det(\mathbf{Id} - D^{-1})^{-1}\det(D^{-1}).$$

Next, since given any smooth function $\Delta(z)$ with values in the invertible matrices,

$$\partial_{z_i}\det\Delta(z) = \lim_{h\to 0}\frac{\det(\mathbf{Id} + [\Delta(z+he_i) - \Delta(z)]\Delta(z)^{-1}) - 1}{h}\det(\Delta(z))$$
$$= \lim_{h\to 0}\frac{e^{\operatorname{Tr}\ln(\mathbf{Id}+[\partial_{z_i}\Delta(z)]\Delta(z)^{-1}h)} - 1}{h}\det(\Delta(z)) = \operatorname{Tr}([\partial_{z_i}\Delta(z)]\Delta(z)^{-1})\cdot\det(\Delta(z)),$$

also in view of (A.4), holds true

$$\|\det(D\Phi_n^{-1})\|_{\mathcal{C}^s} \leq C|\det(D^{-1})|.$$

Finally, since $|\ln\det D|_{\mathcal{C}^1} \leq C$, it follows $\|\det(D^{-1})\|_{\mathcal{C}^0} \leq |\det(D^{-1}(x_*))|$ and the lemma.

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E-mail address: `liverani@mat.uniroma2.it`