Journal of Algebra 455 (2016) 137–163

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Periodic free resolutions from twisted matrix factorizations

Thomas Cassidy [a], Andrew Conner [b], Ellen Kirkman [c,*,1], W. Frank Moore [c]

[a] Department of Mathematics, Bucknell University, Lewisburg, PA 17837, United States
[b] Department of Mathematics, St. Mary's College of California, Moraga, CA 94575, United States
[c] Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, United States

A R T I C L E   I N F O

Article history:
Received 31 May 2015
Available online 1 March 2016
Communicated by Luchezar L. Avramov

MSC:
primary 16E05, 16E35, 16E65

Keywords:
Matrix factorization
Zhang twist
Singularity category
Minimal free resolution
Maximal Cohen–Macaulay

A B S T R A C T

The notion of a matrix factorization was introduced by Eisenbud in the commutative case in his study of bounded (periodic) free resolutions over complete intersections. Since then, matrix factorizations have appeared in a number of applications. In this work, we extend the notion of (homogeneous) matrix factorizations to regular normal elements of connected graded algebras over a field.
Next, we relate the category of twisted matrix factorizations to an element over a ring and certain Zhang twists. We also show that in the setting of a quotient of a ring of finite global dimension by a normal regular element, every sufficiently high syzygy module is the cokernel of some twisted matrix factorization. Furthermore, we show that in the noetherian AS-regular setting, there is an equivalence of categories between the homotopy category of twisted matrix factorizations and the singularity category of the hypersurface, following work of Orlov.

* Corresponding author.
E-mail address: kirkman@wfu.edu (E. Kirkman).

http://dx.doi.org/10.1016/j.jalgebra.2016.01.037

0. Introduction

The notion of a matrix factorization was introduced by Eisenbud [5] in his study of bounded (periodic) free resolutions over commutative complete intersections. Since then, matrix factorizations have appeared in a number of applications, including string theory, singularity categories, representation theory of Cohen–Macaulay modules, among others. The goal of the present article is to extend the notion of matrix factorization to the broader context of noncommutative algebras.

One motivation of such a study comes from attempting to understand the notion of a complete intersection in a noncommutative context. A perspective that one may take is that a graded noncommutative complete intersection is a quotient of a free algebra whose Hilbert series is 'as small as possible'. This is the perspective of Etingof–Ginzburg [6] which builds off of work of Anick [1]. Here, they show that Anick's original definition is connected to several other conditions, some of which go back to results of Golod–Shafarevich [8,9]. One drawback of this approach however, is that Artin–Schelter regular (hereafter abbreviated AS-regular) algebras of global dimension greater than 2 are not noncommutative complete intersections using this definition.

Another perspective is to ask that the Ext algebra of a graded algebra A behaves in a manner similar to that of a commutative graded (or local) complete intersection. This means that one seeks conditions that force $\mathrm{Ext}_A^*(k,k)$ be a noetherian k-algebra, or have finite Gelfand–Kirillov (GK) dimension. This is the perspective of the work of Kirkman–Kuzmanovich–Zhang [14]; they prove that if A is the quotient of an AS-regular algebra by a regular sequence of normalizing elements, then $\mathrm{Ext}_A^*(k,k)$ has finite GK dimension. Our main goal is to better understand the minimal free resolutions of modules over quotients of AS-regular by a *single* regular normal element.

To state our results below, we assume that A is a connected, $\mathbb{N}$-graded, locally finite dimensional algebra over a field k. We also fix a homogeneous normal regular element $f \in A_+ = \bigoplus_{n>0} A_n$ and set $B = A/(f)$. The regularity and normality of f provide us with a graded automorphism σ of A which we incorporate into the definition of matrix factorization. The use of σ to modify ring actions is the reason for our "twisted" terminology; see Definition 2.2.

Our first main result shows that just as in the commutative case, (reduced) twisted matrix factorizations give rise to (minimal) resolutions.

Theorem A. *(See Propositions 2.4, 2.9, 2.12.) A twisted left matrix factorization (φ, τ) of f gives rise to a complex $\mathbf{\Omega}(\varphi, \tau)$ of free left B-modules which is a graded free resolution of* $\operatorname{coker}\varphi$ *as a left B-module. If the twisted matrix factorization (φ, τ) is reduced (see Definition 2.7), then the graded free resolution is minimal. If the order of σ is finite, then the resolution is periodic of period at most twice the order of σ.*

[1] Ellen Kirkman was partially supported by the Simons Foundation Grant #208314.

As in the commutative case, one can consider the category of all twisted matrix factorizations of f over a ring A, which we call $TMF_A(f)$.

There is another context where twisting via an automorphism arises in the study of graded algebras: the Zhang twist [19]. The following theorem relates the category of twisted matrix factorizations of f over A to those over the Zhang twist of A with respect to a compatible twisting system ζ (which we denote A^ζ).

Theorem B. *(See Theorem 3.6.) Let $\zeta = \{\sigma^n \mid n \in \mathbb{Z}\}$ be the twisting system associated with the normalizing automorphism σ. Then the categories $TMF_A(f)$ and $TMF_{A^\zeta}(f)$ are equivalent.*

This result is somewhat surprising. If f is central in the Zhang twist A^ζ, then the complexes associated to matrix factorizations in $TMF_{A^\zeta}(f)$ will be periodic of period at most two, while those coming from matrix factorizations in $TMF_A(f)$ could have a longer period, depending on the order of σ. It should be noted that f is not necessarily central in the Zhang twist. This peculiarity is illustrated in Example 6.2.

A major result in [5, Theorem 5.2] that drives many of the applications of matrix factorizations is that, under appropriate hypotheses, every minimal graded free resolution is eventually given by a reduced matrix factorization. We extend this result in the following:

Theorem C. *(See Theorem 4.2.) Let A be a connected graded algebra of global dimension $n < \infty$, $f \in A_+$ be a homogeneous normal regular element and let $B = A/(f)$. Then for every finitely generated graded left B-module M, the $(n+1)^{st}$ left syzygy of M is the cokernel of some reduced twisted left matrix factorization of f over A.*

There is a suitable notion of homotopy in the category $TMF_A(f)$, and we denote the associated homotopy category $hTMF_A(f)$. Following Orlov's lead [17], we provide a triangulated structure on $hTMF_A(f)$ and prove the following Theorem.

Theorem D. *(See Theorem 5.8.) Let A be a left noetherian Artin-Schelter regular algebra, and let $f \in A_+$ be a homogeneous normal regular element. Then the homotopy category of twisted matrix factorizations of f over A is equivalent to the bounded singularity category of B.*

It should be noted that since the minimal resolution that comes from a twisted matrix factorization need not be periodic, some minor adjustments to Orlov's original argument must be made.

The paper is organized as follows: Section 1 covers preliminaries, as well as sets up notation regarding various twists that will be used for the remainder of the paper. Section 2 covers the definition of matrix factorization, as well as the proof of Theorem A. Section 3 contains the background regarding the Zhang functor, as well as the proof of Theorem B.

Section 4 includes the precise statement and proof of Theorem C. Section 5 contains the categorical considerations for Theorem D, and Section 6 contains some examples.

1. Preliminaries

The main results in this paper concern graded modules over graded rings, hence we work exclusively in that context. Let A be an $\mathbb{N}$-graded algebra over a field k. We assume A is locally finite-dimensional: $\dim_k A_i < \infty$ for all $i \in \mathbb{N}$, and connected: $A_0 = k$. Throughout, we work in the category A-GrMod of graded left A-modules with degree zero morphisms, though our definitions and results have obvious analogs for graded right modules. Throughout, 'graded k-algebra' means 'connected graded locally finite algebra over a field k'. It is well known that in this category, finitely generated projective A-modules are free. Rather than use the language of projective modules, we prefer to state our results using free modules.

Let σ be a degree zero graded algebra automorphism of A. For $M \in A$-GrMod, we write ${}^\sigma M$ for the graded left A-module with ${}^\sigma M = M$ as graded abelian groups and left A-action $a \cdot m = \sigma(a)m$. If $\varphi : M \to N$ is a degree zero homomorphism of graded left A-modules, ${}^\sigma\varphi = \varphi : {}^\sigma M \to {}^\sigma N$ is also a graded module homomorphism. It is straightforward to check that the functor ${}^\sigma(-)$ is an autoequivalence of A-GrMod, and that M is free if and only if ${}^\sigma M$ is free.

For any $n \in \mathbb{Z}$ and $M \in A$-GrMod, we write $M(n)$ for the shifted module whose degree i component is $M(n)_i = M_{i+n}$. The degree shift functor $M \mapsto M(n)$ is also easily seen to be an autoequivalence of A-GrMod which commutes with ${}^\sigma(-)$; that is, ${}^\sigma(M(n)) = ({}^\sigma M)(n)$.

Let $f \in A_d$ be a normal, regular homogeneous element of degree d, and let $\sigma : A \to A$ be the graded automorphism of A determined by the equation $af = f\sigma(a)$ for each $a \in A$. We call σ the *normalizing automorphism* of f and say f *is normalized by* σ. Note that f is normalized by σ if and only if left multiplication by f is a graded left module homomorphism $\lambda_f^M : {}^\sigma M(-d) \to M$ for all M. Moreover, $\lambda_f^N \circ {}^\sigma\varphi(-d) = \varphi \circ \lambda_f^M$ for any graded homomorphism $\varphi : M \to N$.

In this paper we are especially interested in periodic resolutions. We say a complex of degree zero morphisms $\mathbf{P} : \cdots \to P_2 \to P_1 \to P_0$ of graded left A-modules is *periodic of period* p if p is the smallest positive integer such that there exists an integer n and a morphism of complexes $t : \mathbf{P}(n) \to \mathbf{P}$ of (homological) degree $-p$ where $t : P_{i+p}(n) \to P_i$ is an isomorphism for all $i \geq 0$. Note the shift (n) is applied to the internal grading of each free module in the complex. If such an integer p exists, we say $\mathbf{P}$ is *periodic*. If there exists an integer $m \geq 0$ such that the truncated complex $\cdots \to P_{m+2} \to P_{m+1} \to P_m$ is periodic, we say $\mathbf{P}$ is *periodic after* m *steps*.

Free modules over noncommutative rings need not have a well-defined notion of rank. Even among those that do, not all satisfy the *rank conditions* (a) $f : A^n \to A^m$ an epimorphism $\Rightarrow n \geq m$ and (b) $f : A^n \to A^m$ a monomorphism $\Rightarrow n \leq m$ (though (b) implies (a), see [15, Proposition 1.22]). However, since we assume A is locally finite dimensional and morphisms preserve degree, the graded version of (b) clearly holds for graded free A-modules. Thus rank is well-defined for graded free A-modules. We adopt the usual convention that the zero module is free on the empty set.

We record a few straightforward facts about periodic complexes needed later.

Lemma 1.1. *Let* $\mathbf{P}$ *be a complex of graded free left* A*-modules.*

(1) *If* $\mathbf{P}$ *is periodic and there exists an integer* $N > 0$ *such that* $\operatorname{rank} P_j = \operatorname{rank} P_N$ *for all* $j \geq N$*, then* $\operatorname{rank} P_j = \operatorname{rank} P_0$ *for all* $j \geq 0$.
(2) *Let* $\widetilde{\mathbf{P}}$ *be a complex of graded left* A*-modules which is periodic of period* p*. If there exist an integer* n *and an isomorphism of complexes* $t : \widetilde{\mathbf{P}}(n) \to \mathbf{P}$*, then* $\mathbf{P}$ *is also periodic of period* p*.*

We say a resolution $(\mathbf{P}_\bullet, d_\bullet)$ is minimal if $\operatorname{im} d_i \subset A_+ P_{i-1}$ for all i, where $A_+ = \bigoplus_{n>0} A_n$. Recall that every bounded below, graded module over a connected, $\mathbb{N}$-graded, locally finite-dimensional k-algebra has a minimal graded free resolution. This resolution is unique up to non-unique isomorphism of complexes (see, for example, [18, Proposition 1.4.2]).

Lemma 1.2. *Let* σ *be a degree zero graded automorphism of* A*. Let* $\mathbf{P}$ *be a minimal graded free resolution of a bounded below, graded left* A*-module* M*. If* ${}^{\sigma}M \cong M$ *as graded modules, then the complexes* ${}^{\sigma}\mathbf{P}$ *and* $\mathbf{P}$ *are isomorphic.*

Proof. First note that ${}^{\sigma}\mathbf{P}$ is a minimal graded free resolution of ${}^{\sigma}M$. Since ${}^{\sigma}M \cong M$, the comparison theorem shows that ${}^{\sigma}\mathbf{P}$ and $\mathbf{P}$ are isomorphic. □

2. Twisted matrix factorizations

The key to our study is the notion of a twisted matrix factorization. As will be evident, the notion can in fact be defined over any ring containing a normal, regular element. Indeed some results, such as Proposition 2.4, are readily seen to hold in this generality by forgetting the grading.

Throughout this section, we let A be a connected, $\mathbb{N}$-graded, locally finite-dimensional algebra over a field k. Let $f \in A_d$ be a normal, regular homogeneous element of degree d and let σ be its degree zero normalizing automorphism. Let B be the quotient algebra $A/(f)$. We will use $\overline{(\ )}$ to denote quotients modulo (f), and will denote by $\overline{\sigma}$ the degree zero automorphism of B induced by σ.

Definition 2.1. Let A, f, d and σ be as above. We define the functor ${}^{\mathrm{tw}}M$ to be the composite $M \mapsto {}^{\sigma}M(-d)$. We will use ${}^{\mathrm{tw}^{-1}}M$ to denote its inverse $M \mapsto {}^{\sigma^{-1}}M(d)$.

Before going further, we address a potential source of confusion. Let M be a left B-module (and hence a left A-module). We will write ${}^{\mathrm{tw}}M$ to mean ${}_A({}^{\sigma}M(-d))$ and

$\overline{^{\rm tw}}M$ to mean ${}_B({}^\sigma M(-d))$. The reader should note that $\overline{^{\rm tw}}M \cong B \otimes_A ({}^{\rm tw}M)$ and likewise ${}_A(\overline{^{\rm tw}}M) \cong {}^{\rm tw}M$. We denote repeated application of ${}^{\rm tw}(-)$ by ${}^{\rm tw^2}(-)$, ${}^{\rm tw^3}(-)$, etc. and likewise for $\overline{^{\rm tw}}(-)$.

We are now in position to define the main object of study in this paper.

Definition 2.2. A *twisted left matrix factorization* of f over A is an ordered pair of maps of finitely generated graded free left A-modules $(\varphi : F \to G, \tau : {}^{\rm tw}G \to F)$ such that $\varphi\tau = \lambda_f^G$ and $\tau\,{}^{\rm tw}\varphi = \lambda_f^F$, where ${}^{\rm tw}\varphi$ is the induced map ${}^{\rm tw}\varphi : {}^{\rm tw}F \to {}^{\rm tw}G$. Note that ${}^{\rm tw}G$ is free whenever G is.

Remark 2.3. The homomorphisms ${}^{\rm tw}\varphi : {}^{\rm tw}F \to {}^{\rm tw}G$ and $\varphi : F \to G$ are identical on the underlying abelian groups. If we fix bases for F and G, and keep the same bases for ${}^{\rm tw}F$ and ${}^{\rm tw}G$, the matrices of φ and ${}^{\rm tw}\varphi$ with respect to these bases are different. The matrix of ${}^{\rm tw}\varphi$ is obtained by applying σ^{-1} to each entry of the matrix of φ.

Our definition is a generalization of the familiar notion from commutative algebra that incorporates the automorphism σ. A more general version of our definition in the context of an abelian category is given in Ballard et al. [3, Definition 2.3].

Note that if any of (φ, τ), $({}^{\rm tw}\varphi, {}^{\rm tw}\tau)$ or $(\tau, {}^{\rm tw}\varphi)$ is a twisted matrix factorization, then the other two are as well. It is easy to see that if (φ, τ) is a twisted matrix factorization, then both φ and τ are injective since f is regular, and hence rank $F =$ rank G.

We call the twisted factorization (φ, τ) where $\varphi = \tau : 0 \to 0$ the *irrelevant factorization*. We call a twisted factorization (φ, τ) *trivial* if $\varphi = \lambda_f^A$ or $\tau = \lambda_f^A$.

Paralleling the commutative case, twisted matrix factorizations provide a general construction of resolutions.

Proposition 2.4. *Let $(\varphi : F \to G, \tau : {}^{\rm tw}G \to F)$ be a twisted left matrix factorization of a regular element $f \in A$ with normalizing automorphism σ. Then the complex*

$$\mathbf{\Omega}(\varphi,\tau) : \cdots \to \overline{^{\rm tw}}^2\overline{G} \xrightarrow{\overline{^{\rm tw}}\overline{\tau}} \overline{^{\rm tw}}\overline{F} \xrightarrow{\overline{^{\rm tw}}\overline{\varphi}} \overline{^{\rm tw}}\overline{G} \xrightarrow{\overline{\tau}} \overline{F} \xrightarrow{\overline{\varphi}} \overline{G}$$

is a resolution of $M =$ coker φ by free left B-modules.

The proof of Proposition 2.4 is a straightforward generalization of the commutative case.

Proof. Since $\varphi\tau = \lambda_f^G$, we see that $f(\text{coker } \varphi) = 0$ so coker $\overline{\varphi} =$ coker φ.

We prove exactness at $\overline{^{\rm tw}}^i\overline{F}$, exactness at $\overline{^{\rm tw}}^i\overline{G}$ being analogous. Let K be a graded free A-module and $\kappa : K \to {}^{\rm tw^i}F$ an A-module map such that $\overline{\kappa}$ is a B-module surjection onto $\ker(\overline{^{\rm tw}}^i\overline{\varphi})$. Then $\operatorname{im}({}^{\rm tw^i}\varphi\kappa) \subseteq f({}^{\rm tw^i}G)$ and we can define an A-module map $h : K \to {}^{\rm tw}({}^{\rm tw^i}G) = {}^{\rm tw^{i+1}}G$ by $h(x) = g$ where $g \in {}^{\rm tw^i}G$ satisfies ${}^{\rm tw^i}\varphi \circ \kappa(x) = fg$. For any $a \in A$ we have ${}^{\rm tw^i}\varphi \circ \kappa(ax) = afg = f\sigma(a)g$ so h is A-linear. Since f is regular, $\overline{h}$ does not depend on the choice of g. We have

$$\begin{aligned}\lambda_f^{{}^{\mathrm{tw}^i}F} \circ \kappa &= {}^{\mathrm{tw}^{i-1}}\tau \circ {}^{\mathrm{tw}^i}\varphi \circ \kappa \\ &= {}^{\mathrm{tw}^{i-1}}\tau \circ \lambda_f^{{}^{\mathrm{tw}^i}G} \circ h \\ &= \lambda_f^{{}^{\mathrm{tw}^i}F} \circ {}^{\mathrm{tw}^i}\tau \circ h.\end{aligned}$$

Again, since f is regular, $\overline{\kappa} = {}^{\overline{\mathrm{tw}}^i}\overline{\tau} \circ \overline{h}$, hence $\ker {}^{\overline{\mathrm{tw}}^i}\overline{\varphi} = \mathrm{im}\,\overline{\kappa} \subseteq \mathrm{im}\,{}^{\overline{\mathrm{tw}}^i}\overline{\tau}$. □

In light of Proposition 2.4, we make the following natural definition.

Definition 2.5. A morphism $(\varphi, \tau) \to (\varphi', \tau')$ of twisted left matrix factorizations of f over A is a pair $\Psi = (\Psi_G, \Psi_F)$ of degree zero module homomorphisms $\Psi_G : G \to G'$ and $\Psi_F : F \to F'$ such that the following diagram commutes.

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ {\scriptstyle \Psi_F}\downarrow & & \downarrow{\scriptstyle \Psi_G} \\ F' & \xrightarrow[\varphi']{} & G' \end{array}$$

A morphism Ψ is an *isomorphism* if Ψ_G and Ψ_F are isomorphisms.

The regularity of f guarantees that $(\Psi_G, \Psi_F) : (\varphi, \tau) \to (\varphi', \tau')$ is a morphism if and only if

$$(\Psi_F, {}^{\mathrm{tw}}\Psi_G) : (\tau, {}^{\mathrm{tw}}\varphi) \to (\tau', {}^{\mathrm{tw}}\varphi')$$

is. It is clear that (Ψ_G, Ψ_F) is a morphism if and only if

$$({}^{\mathrm{tw}}\Psi_G, {}^{\mathrm{tw}}\Psi_F) : ({}^{\mathrm{tw}}\varphi, {}^{\mathrm{tw}}\tau) \to ({}^{\mathrm{tw}}\varphi', {}^{\mathrm{tw}}\tau')$$

is. We leave the straightforward proof of the next Proposition to the reader.

Proposition 2.6. *A morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ of twisted matrix factorizations of f over A induces a chain morphism of complexes $\mathbf{\Omega}(\Psi) : \mathbf{\Omega}(\varphi, \tau) \to \mathbf{\Omega}(\varphi', \tau')$. Twisted matrix factorizations (φ, τ) and (φ', τ') are isomorphic if and only if the complexes $\mathbf{\Omega}(\varphi, \tau)$ and $\mathbf{\Omega}(\varphi', \tau')$ are chain isomorphic.*

Definition 2.7. We define the *direct sum* of twisted matrix factorizations (φ, τ) and (φ', τ') to be $(\varphi \oplus \varphi', \tau \oplus \tau')$. We call (φ, τ) *reduced* if it is not isomorphic to a factorization having a trivial direct summand.

Next we show that a normal, regular homogeneous element gives rise to twisted matrix factorizations in many cases of interest. However, see Example 6.1.

Construction 2.8. Let M be a finitely generated graded left B-module with $\mathrm{pd}_A M = 1$. Let $0 \to F \xrightarrow{\varphi} G \to 0$ be a minimal graded resolution of M by graded free left A-modules. We have the commutative diagram

$$\begin{array}{ccc} {}^{\mathrm{tw}}F & \xrightarrow{{}^{\mathrm{tw}}\varphi} & {}^{\mathrm{tw}}G \\ \lambda_f^F \downarrow & & \downarrow \lambda_f^G \\ F & \xrightarrow[\varphi]{} & G. \end{array}$$

Since M is a B-module, $f_A M = 0$, hence im $\lambda_f^G \subseteq$ im φ. Thus by graded projectivity there exists a lift $\tau : {}^{\mathrm{tw}}G \to F$ such that $\lambda_f^G = \varphi\tau$:

$$\begin{array}{ccc} {}^{\mathrm{tw}}F & \xrightarrow{{}^{\mathrm{tw}}\varphi} & {}^{\mathrm{tw}}G \\ \lambda_f^F \downarrow & \overset{\tau}{\swarrow} & \downarrow \lambda_f^G \\ F & \xrightarrow[\varphi]{} & G. \end{array}$$

Note that since f is regular, λ_f^G is injective, so τ is injective. Next observe that $\varphi\tau{}^{\mathrm{tw}}\varphi - \varphi\lambda_f^F = 0$ and since φ is injective, $\tau{}^{\mathrm{tw}}\varphi = \lambda_f^F$.

Applying ${}^{\mathrm{tw}}(-)$ to this diagram, and applying $B \otimes_A -$ yields the maps that appear in the complex $\mathbf{\Omega}(\varphi, \tau)$ of Proposition 2.4:

$$\cdots \to B \otimes_A {}^{\mathrm{tw}}F \xrightarrow{1\otimes{}^{\mathrm{tw}}\varphi} B \otimes_A {}^{\mathrm{tw}}G \xrightarrow{1\otimes\tau} B \otimes_A F \xrightarrow{1\otimes\varphi} B \otimes_A G \to 0.$$

Proposition 2.9. *Using the notation of Construction 2.8, the complex $\mathbf{\Omega}(\varphi, \tau)$ is exact. It is a minimal graded free resolution of ${}_BM$ if and only if M has no B-free direct summand.*

Proof. Exactness follows from Proposition 2.4. Since $F \xrightarrow{\varphi} G$ is a minimal resolution of ${}_AM$, the complex ${}^{\mathrm{tw}^i}F \xrightarrow{{}^{\mathrm{tw}^i}\varphi} {}^{\mathrm{tw}^i}G$ is a minimal resolution of ${}_A{}^{\mathrm{tw}^i}M$ for all $i \geq 0$. Thus we have $\mathrm{im}({}^{\mathrm{tw}^i}\varphi) \subseteq A_+({}^{\mathrm{tw}^i}G)$ for all $i \geq 0$. It follows that $\mathrm{im}(1 \otimes {}^{\mathrm{tw}^i}\varphi) \subseteq B_+(B \otimes_A {}^{\mathrm{tw}^i}G)$ for all $i \geq 0$. Thus it suffices to consider the maps $1 \otimes {}^{\mathrm{tw}^i}\tau$.

Now, ${}_BM$ has a B-free direct summand if and only if the twisted module ${}^{\overline{\mathrm{tw}}^i}M$ does. Since $\mathbf{\Omega}(\varphi, \tau)$ is exact, for each $i \geq 0$ we have

$$\mathrm{im}(1 \otimes {}^{\mathrm{tw}^i}\tau) \cong \mathrm{coker}(1 \otimes {}^{\mathrm{tw}^i}\varphi) \cong B \otimes_A {}^{\mathrm{tw}^i}M \cong {}^{\overline{\mathrm{tw}}^i}M.$$

Since $\mathrm{im}(1 \otimes {}^{\mathrm{tw}^i}\tau)$ is contained in the radical $B_+{}^{\overline{\mathrm{tw}}^{i-1}}\overline{F}$ if and only if no basis for the free B-module ${}^{\overline{\mathrm{tw}}^i}\overline{F}$ intersects $\mathrm{im}(1 \otimes {}^{\mathrm{tw}^i}\tau)$, the result follows. $\square$

Corollary 2.10. *Under the hypotheses and notation of Construction 2.8, the complex* $\overline{\mathrm{tw}}^n\boldsymbol{\Omega}(\varphi,\tau)$ *is a graded free resolution of* $\overline{\mathrm{tw}}^n M$ *for any integer* n.

We can also express minimality of the resolution $\boldsymbol{\Omega}(\varphi,\tau)$ in terms of the twisted matrix factoriaztion.

Lemma 2.11. *Let* A *be a graded* k*-algebra and* $f \in A_+$ *a homogeneous normal regular element. A twisted matrix factorization* (φ,τ) *of* f *over* A *is reduced if and only if* $\boldsymbol{\Omega}(\varphi,\tau)$ *is a minimal graded free resolution.*

Proof. The complex $\boldsymbol{\Omega}(\varphi,\tau)$ is minimal if and only if $\operatorname{coker}(1\otimes {}^{\mathrm{tw}^i}\varphi)$ and $\operatorname{coker}(1\otimes {}^{\mathrm{tw}^i}\tau)$ have no B-free direct summands for all $i \geq 0$. Since the functor ${}^{\mathrm{tw}}(-)$ preserves direct sums and free modules, the complex $\boldsymbol{\Omega}(\varphi,\tau)$ is minimal if and only if $\operatorname{coker}(1\otimes\varphi) = \operatorname{coker}\varphi$ and $\operatorname{coker}(1\otimes\tau) = \operatorname{coker}\tau$ have no B-free direct summands. The latter holds if and only if (φ,τ) is not isomorphic to a twisted factorization (φ',τ') where φ' or τ' has λ_f^A as a summand. □

Next we turn to periodicity. Clearly the complex $\boldsymbol{\Omega}(\varphi,\tau)$ is periodic of period at most $2n$ if σ has finite order n. In practice, the period is often less than $2n$ (see Section 6 for some examples). Even when $|\sigma| = \infty$, the complex may be periodic, as we show in the next proposition.

Proposition 2.12. *The complex* $\boldsymbol{\Omega}(\varphi,\tau)$ *is periodic of period at most* $2n$ *if and only if* ${}^{\bar\sigma^n}M \cong M$ *as* B*-modules for some* $n > 0$. *In particular, if* ${}^{\bar\sigma}M \cong M$, *then* $\boldsymbol{\Omega}(\varphi,\tau)$ *has period at most 2.*

Proof. Suppose there exists an integer $n > 0$ such that ${}^{\bar\sigma^n}M \cong M$. By Lemma 1.2 and Corollary 2.10, $\boldsymbol{\Omega}(\varphi,\tau)$ is periodic of period at most $2n$.

Conversely, suppose there exists an integer $p > 0$, an integer N, and a degree $-p$ morphism of complexes $\Phi : \boldsymbol{\Omega}(\varphi,\tau)(N) \to \boldsymbol{\Omega}(\varphi,\tau)$ such that $\Phi_{i+p} : \boldsymbol{\Omega}_{i+p}(N) \to \boldsymbol{\Omega}_i$ is an isomorphism for all $i \geq 0$. Since Φ^2 also has this property, we may assume $p = 2n$ is even.

By construction, minimal generators of $\boldsymbol{\Omega}_{i+p}$ can be taken to be minimal generators of $\boldsymbol{\Omega}_i$ with degrees shifted up by nd. It follows that $N = nd$. Thus the following diagram commutes:

$$\begin{array}{ccc} B\otimes_A {}^{\sigma^n}F & \xrightarrow{1\otimes{}^{\sigma^n}\varphi} & B\otimes_A {}^{\sigma^n}G \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ B\otimes_A F & \xrightarrow{1\otimes\varphi} & B\otimes_A G. \end{array}$$

Therefore

$${}^{\bar\sigma^n}M \cong \operatorname{coker}(1\otimes{}^{\sigma^n}\varphi) \cong \operatorname{coker}(1\otimes\varphi) \cong M. \quad \square$$

3. Equivalent categories of twisted matrix factorizations

We denote by $TMF_A(f)$ the category whose objects are all twisted left matrix factorizations of f over A and whose morphisms are defined as in Definition 2.5. As previously mentioned, $TMF_A(f)$ has a zero object and all finite direct sums. Since morphisms are pairs of module maps, monomorphisms and epimorphisms are determined componentwise. Thus we have the following obvious fact.

Proposition 3.1. *$TMF_A(f)$ is an abelian category.*

Proposition 2.6 shows that forming the resolution $\mathbf{\Omega}(\varphi, \tau)$ defines a functor from the abelian category of twisted matrix factorizations of f over A to the abelian category of complexes of finitely generated graded free B-modules.

In [19, Theorem 1.2], Zhang completely characterized pairs of graded k-algebras whose categories of graded modules are equivalent. With that characterization in mind, we consider the question of when categories of twisted matrix factorizations are equivalent.

The following easy fact is useful later.

Proposition 3.2.

(1) *For any scalar $\nu \in k^\times$, the categories $TMF_A(f)$ and $TMF_A(\nu f)$ are equivalent.*
(2) *Let $\phi : A \to A$ be a graded automorphism of A. Then $TMF_A(f) \approx TMF_A(\phi(f))$.*

Proof. For (1), the functors $(\varphi, \tau) \mapsto (\varphi, \nu\tau)$ and $(\varphi, \tau) \mapsto (\varphi, \nu^{-1}\tau)$ are easily seen to be inverse equivalences. For (2), first observe that $\phi\sigma\phi^{-1}$ is the normalizing automorphism for $\phi(f)$. Applying the functor ${}^{\phi^{-1}}(-)$ to any twisted matrix factorization of f over A produces the desired equivalence. □

We briefly recall the basic definitions underlying Zhang's graded Morita equivalence and encourage the interested reader to see [19] for more details.

A (left) *twisting system* for A is a set $\zeta = \{\zeta_n \mid n \in \mathbb{Z}\}$ of graded k-linear automorphisms of A such that $\zeta_n(\zeta_m(x)y) = \zeta_{m+n}(x)\zeta_n(y)$ for all $n, m, \ell \in \mathbb{Z}$ and $x \in A_\ell$, $y \in A_m$. For example, if ϕ is a graded k-linear automorphism of A, then setting $\zeta_n = \phi^n$ for all $n \in \mathbb{Z}$ gives a twisting system.

Given a twisting system ζ, the *Zhang twist* of A is the graded k-algebra A^ζ where ${}^\zeta A = A$ as graded k-vector spaces and for all $x \in A_\ell$ and $y \in A_m$, multiplication in ${}^\zeta A$ is given by $x * y = \zeta_m(x)y$. Likewise, if M is a graded left A-module, the twisted left ${}^\zeta A$-module ${}^\zeta M$ has the same underlying graded vector space as M, and for $m \in M_n$ and $z \in A_\ell$, $z * m = \zeta_n(z)m$. Finally, we note that if $\varphi : M \to N$ is a degree zero homomorphism of graded left A-modules, $\varphi : {}^\zeta M \to {}^\zeta N$ is also a degree zero homomorphism of graded left ${}^\zeta A$-modules.

Remark 3.3. Aside from the use of the letter ζ, the notation for the twisted module is identical to that used for the functor ${}^\sigma(-)$ above. However, the notions are not the same. One important difference is that for an integer n, the free left A-modules ${}^{\sigma}(A(n))$ and $({}^{\sigma}A)(n)$ are identical, whereas the free left ${}^{\zeta}A$-modules ${}^{\zeta}(A(n))$ and $({}^{\zeta}A)(n)$ – which have the same underlying graded vector space – are generally not identical, but are isomorphic via the map ζ_{-n}. In light of this subtlety, the following simple lemma is not entirely trivial.

Lemma 3.4. *Let A be a graded k-algebra and let $f \in A_d$ be a normal regular homogeneous element with normalizing automorphism σ. Let $\zeta = \{\zeta_n \mid n \in \mathbb{Z}\}$ be a left twisting system.*

(1) *If $\zeta_n(f) = c^n f$ for some $c \in k^{\times}$ and for all $n \in \mathbb{Z}$, then f is normal and regular in ${}^{\zeta}A$ with normalizing automorphism $\hat{\sigma}(a) = c^{-\deg a}\sigma\zeta_d(a)$.*
(2) *If ζ further satisfies $\zeta_n\sigma\zeta_d = \sigma\zeta_{n+d}$ for all $n \in \mathbb{Z}$ we have*

$$ {}^{\zeta}({}^{\mathrm{tw}}A) \cong {}^{\hat{\mathrm{tw}}}({}^{\zeta}A) := {}^{\hat{\sigma}}({}^{\zeta}A)(-d) $$

as free left ${}^{\zeta}A$-modules.

If the twisting system ζ is "algebraic," meaning $\zeta_n\zeta_m = \zeta_{n+m}$ for all $n, m \in \mathbb{Z}$, the additional hypothesis of (2) becomes $\sigma\zeta_n = \zeta_n\sigma$ for all $n \in \mathbb{Z}$. In the common case where $\zeta_n = \phi^n$ for a k-linear automorphism $\phi : A \to A$, one needs only that $\sigma\phi = \phi\sigma$.

Proof. Let $a \in A_n$ be an arbitrary homogeneous element. To prove (1), we have

$$ a * f = \zeta_d(a)f = f\sigma\left(\zeta_d(a)\right) = \zeta_n^{-1}(f) * \sigma\zeta_d(a) = c^{-n}f * \sigma\zeta_d(a) = f * \hat{\sigma}(a) $$

Thus f is normal in ${}^{\zeta}A$. The equation also shows the regularity of f in ${}^{\zeta}A$ follows from the regularity of f in A, so $\hat{\sigma}$ is the normalizing automorphism.

For (2), first observe that $a \mapsto c^{\deg a}a$ defines a graded algebra automorphism λ_c of ${}^{\zeta}A$. For any graded left ${}^{\zeta}A$-module M, $M \cong {}^{\lambda_c}M$ via the map $m \mapsto c^{\deg m}m$ which we also denote λ_c.

Now, ${}^{\zeta}({}^{\mathrm{tw}}A)$ and ${}^{\hat{\mathrm{tw}}}({}^{\zeta}A)$ have the same underlying graded vector space as A. We compute the left ${}^{\zeta}A$ action on both modules. With a as above and $b \in A_m$, ${}^{\zeta}A$ acts on ${}^{\hat{\mathrm{tw}}}({}^{\zeta}A)$ by

$$ a \bullet b = \hat{\sigma}(a) * b = \zeta_m(\hat{\sigma}(a))b = \zeta_m c^{-\deg a}\sigma\zeta_d(a)b = c^{-\deg a}\zeta_m\sigma\zeta_d(a)b $$

and on ${}^{\zeta}({}^{\mathrm{tw}}A)$ by

$$ a \bullet b = \zeta_{m+d}(a) \cdot b = \sigma\zeta_{m+d}(a)b $$

since $b \in A_m = A^{\sigma}(-d)_{m+d}$. Thus ${}^{\lambda_c}({}^{\hat{\mathrm{tw}}}({}^{\zeta}A)) = {}^{\zeta}({}^{\mathrm{tw}}A)$ and the result follows. $\square$

For completeness, we mention the left module version of Zhang's theorem on graded Morita equivalence.

Theorem 3.5. *(See [19, Theorem 1.2].) Let k be a field and let A and A' be graded k-algebras with $A_1 \neq 0$. Then $A \cong {}^{\zeta}A'$ for some twisting system ζ if and only if the categories A-GrMod and A'-GrMod are equivalent.*

The equivalence is given by $M \mapsto {}^{\zeta}M$ for any graded A'-module M and is the identity on morphisms. We have the following.

Theorem 3.6. *Let A be a graded k-algebra, and let $f \in A_d$ a normal regular homogeneous element of degree d with normalizing automorphism σ. Let $\zeta = \{\zeta_n \mid n \in \mathbb{Z}\}$ be a twisting system such that for all $n \in \mathbb{Z}$, $\zeta_n \sigma \zeta_d = \sigma \zeta_{n+d}$ and $\zeta_n(f) = c^n f$ for some $c \in k^\times$. Then the categories $TMF_A(f)$ and $TMF_{{}^{\zeta}A}(f)$ are equivalent.*

Proof. By Proposition 3.2, it suffices to prove that $TMF_A(f)$ is equivalent to $TMF_{{}^{\zeta}A}(c^d f)$.

Let $(\varphi : F \to G, \tau : {}^{\mathrm{tw}}G \to F)$ be a twisted left matrix factorization of f over A. Let $\lambda_c : {}^{\widehat{\mathrm{tw}}}({}^{\zeta}G) \to {}^{\lambda_c}({}^{\widehat{\mathrm{tw}}}({}^{\zeta}G))$ be the graded isomorphism $m \mapsto c^{\deg m} m$ as in the proof of Lemma 3.4. By Lemma 3.4(2) and the note preceding Remark 3.3,

$$(\varphi : {}^{\zeta}F \to {}^{\zeta}G, \tau\lambda_c : {}^{\widehat{\mathrm{tw}}}({}^{\zeta}G) \to {}^{\zeta}F)$$

is a twisted matrix factorization of $c^d f$ over ${}^{\zeta}A$. The functoriality of Zhang's category equivalence ${}^{\zeta}(-)$ ensures any morphism $(\alpha, \beta) : (\varphi, \tau) \to (\varphi', \tau')$ of twisted factorizations over A remains a morphism over ${}^{\zeta}A$. This defines a functor $TMF_A(f) \to TMF_{{}^{\zeta}A}(f)$.

The inverse equivalence is given by applying the inverse twisting system $\zeta^{-1} = \{\zeta_n^{-1} \mid n \in \mathbb{Z}\}$ to a twisted matrix factorization over ${}^{\zeta}A$ and replacing λ_c by $\lambda_{c^{-1}}$ in the above construction. $\square$

Corollary 3.7. *The equivalence given in the preceding theorem completes a commutative diagram of functors*

$$\begin{array}{ccc} TMF_A(f) & \longrightarrow & TMF_{{}^{\zeta}A}(f) \\ \Big\downarrow{\scriptstyle \mathrm{coker}} & & \Big\downarrow{\scriptstyle \mathrm{coker}} \\ A\text{-}GrMod & \xrightarrow{\ Z\ } & {}^{\zeta}A\text{-}GrMod \end{array}$$

where Z denotes Zhang's equivalence of categories, and coker *sends the twisted matrix factorization (φ, τ) to* $\operatorname{coker}\varphi$.

We do not know an example of a twisting system ζ where f remains normal and regular in ${}^{\zeta}A$ but $TMF_A(f)$ and $TMF_{{}^{\zeta}A}(f)$ are inequivalent.

In some cases, a normal, regular element can become central in an appropriate Zhang twist. By Proposition 2.12, twisted matrix factorizations of a central element produce resolutions with period at most 2. Example 6.2 below illustrates the following important subtlety.

Proposition 3.8. *The period of a periodic minimal free resolution need not be invariant under a Zhang twist.*

4. Noncommutative hypersurfaces

One way to easily see the bijection between periodic minimal free resolutions and reduced matrix factorizations over commutative rings is via the Auslander–Buchsbaum formula. While a version of the Auslander–Buchsbaum formula has been developed for left noetherian AS-regular algebras by Jørgensen [11, Theorem 3.2], there is a more elementary argument that does not require this hypothesis that is suitable for our needs.

Proposition 4.1. *Let f be a normal regular element in a ring A, and let $B = A/fA$. If*

$$0 \to M_j \to P_{j-1} \to \cdots \to P_0 \to M_0 \to 0$$

is an exact sequence of left B-modules with $1 \leq j \leq \operatorname{pd}_B(M_0)$ and P_i projective for $i = 0, \dots, j-1$, then

$$\operatorname{pd}_A(M_j) = \max\{1, \operatorname{pd}_A(M_0) - j\}.$$

Proof. Let M be a nonzero B-module. Then M is annihilated by f, so M is not a submodule of a free A-module. Therefore $\operatorname{pd}_A M \geq 1$. If M is projective as a B-module, it is a direct summand of a direct sum of copies of B, thus $\operatorname{pd}_A M \leq \operatorname{pd}_A B = 1$, hence $\operatorname{pd}_A M = 1$.

The claim is clear when $\operatorname{pd}_A M_0$ is infinite, so we may assume that $\operatorname{pd}_A M_0 = p < \infty$. By splicing together the short exact sequences linking M_j and M_0, it suffices to prove the claim when $j = 1$.

If $p = 1$, let $g : P_0 \to M_0$ be the given map, and let $h : Q \to P_0$ be a surjection with Q a projective A-module. Let $K = \ker g$ and $L = \ker gh$. One then has a commutative diagram of left A-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & Q & \xrightarrow{h} & P_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g'} & & \| & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L & \longrightarrow & Q & \xrightarrow{gh} & M & \longrightarrow & 0. \end{array}$$

The snake lemma implies that $\ker g' = 0$ and $\operatorname{coker} g' \cong \ker g \cong M_1$, so there is an exact sequence of left A-modules

$$0 \to K \to L \to M_1 \to 0.$$

Since $\operatorname{pd}_A(P_0) = 1 = \operatorname{pd}_A(M_0)$, both K and L are projective left A-modules. This implies that $\operatorname{pd}_A M_1 \leq 1$ and hence $\operatorname{pd}_A M_1 = 1$.

When $p \geq 2$ and N is an A-module, the connecting map $\mathrm{Ext}_A^n(M_1, N) \to \mathrm{Ext}_A^{n+1}(M_0, N)$ defined by the exact sequence $0 \to M_1 \to P_0 \to M_0 \to 0$ is an isomorphism for $n = p$ and an epimorphism for $n = p - 1$. Therefore $\mathrm{pd}_A M_1 = p - 1$. □

Theorem 4.2. *Let A be a graded algebra of global dimension $n < \infty$, $f \in A_+$ a homogeneous normal regular element and let σ be its normalizing automorphism. Set $B = A/(f)$. If*

$$\mathbf{Q} : \cdots \to Q_2 \to Q_1 \to Q_0$$

is a minimal graded free left B-module resolution of a finitely generated graded left B-module M, then

(1) *The truncated complex $\cdots \to Q_{n+2} \to Q_{n+1}$ is chain isomorphic to $\mathbf{\Omega}(\varphi, \tau)$ for some reduced twisted left matrix factorization (φ, τ).*

Assuming further that $|\sigma| < \infty$, one has that

(2) *$\mathbf{Q}$ becomes periodic of period at most $2|\sigma|$ after $n + 1$ steps.*
(3) *$\mathbf{Q}$ is periodic (of period at most $2|\sigma|$) if and only if $\mathrm{pd}_A(M) = 1$ and M has no graded free B-module summand.*
(4) *Every periodic minimal graded free left module resolution over B has the form $\mathbf{\Omega}(\varphi, \tau)$ for some reduced twisted left matrix factorization (φ, τ) of f over A.*

Proof. Let $\mathbf{P} \to M$ be a minimal free resolution of M over A, and let $\Omega_i(M) := \mathrm{im}(P_i \to P_{i-1})$ denote the i-th syzygy of M. By Proposition 4.1, we have that $\mathrm{pd}_A(\Omega_i(M)) = 1$ for some $0 \leq i \leq d$. If $\Omega_i(M) = \Omega_i'(M) \oplus F$ where F is a graded free B-module and $\Omega_i'(M)$ has no free summand, then $\mathrm{pd}_A(\Omega_i'(M)) = 1$. By Construction 2.8 and Proposition 2.9 there exists a twisted left matrix factorization (φ, τ) such that $\mathbf{\Omega}(\varphi, \tau)$ is a periodic minimal graded free resolution of $\Omega_i'(M)$. If $F[i]$ denotes the free module F viewed as a complex concentrated in homological degree i, it follows that

$$\widetilde{\mathbf{Q}} : \mathbf{\Omega}(\varphi, \tau) \oplus F[i] \to Q_{i-1} \to \cdots \to Q_0$$

is a minimal graded free resolution of M (or, if $i = 0$, $\mathbf{\Omega}(\varphi, \tau) \oplus F$ is a resolution). By uniqueness of minimal resolutions, $\widetilde{\mathbf{Q}} \cong \mathbf{Q}$. Truncating each complex at homological degree $i + 1$ and recalling that if (φ, τ) is a twisted matrix factorization, so are $({}^{\mathrm{tw}}\varphi, {}^{\mathrm{tw}}\tau)$ and $(\tau, {}^{\mathrm{tw}}\varphi)$, we have established (1).

If $|\sigma| < \infty$, the resolution $\widetilde{\mathbf{Q}}$ is periodic of period at most $2|\sigma|$ after $i + 2$ steps and rank $\widetilde{Q}_j =$ rank $\widetilde{Q}_{i+2}$ for all $j \geq i+2$. This proves (2). Setting $i = 0$ and $\Omega_i'(M) = \Omega_i(M)$, we also obtain the “if” direction of (3).

Now suppose that $\mathbf{Q}$ is periodic of period p. If $\Omega_i(M)$ has a free summand, rank $Q_{p+i+1} =$ rank $Q_{i+1} >$ rank Q_{i+2}. But this is impossible, since $\mathbf{Q}$ and $\widetilde{\mathbf{Q}}$ are isomorphic minimal free resolutions. Thus $\Omega_i(M)$ has no free direct summand and

$\widetilde{\mathbf{Q}} : \mathbf{\Omega}(\varphi, \tau) \to Q_i \to \cdots \to Q_0$ is a minimal free resolution of M. By Lemma 1.1, rank $\widetilde{Q}_j =$ rank $\widetilde{Q}_0$ for all $j \geq 0$, so M has no free direct summand.

By graded periodicity, $M = \operatorname{coker}(Q_1 \to Q_0)$ is isomorphic to $\operatorname{coker}(1 \otimes {}^{\sigma^m}\varphi)$ or $\operatorname{coker}(1 \otimes {}^{\sigma^m}\tau)$ for some m. Since both maps lift to injective maps of free A-modules, $\operatorname{pd}_A(M) = 1$. This completes the proof of (3). Since $({}^{\mathrm{tw}^m}\varphi, {}^{\mathrm{tw}^m}\tau)$ and $({}^{\mathrm{tw}^m}\tau, {}^{\mathrm{tw}^{m+1}}\varphi)$ are also twisted matrix factorizations, (4) follows as well. □

Taking A to be the polynomial ring $k[x_1, \ldots, x_n]$, we recover a graded version of [5, Theorem 6.1] as a special case of Theorem 4.2. We remark that the analogous theorem in [5] relies on the existence of regular sequences of length depth (A), whereas our proof necessarily avoids this assumption.

As a first corollary, we have the following useful fact.

Corollary 4.3. *Let A, f, and B as in Theorem 4.2. Assume $|\sigma| < \infty$. If $(\mathbf{Q}, \partial)$ is a minimal graded free left B-module resolution of a finitely generated module, then im ∂_k has no free summands for $k \geq d + 1$.*

We also see that resolutions of the trivial module ${}_Bk$ have a very rigid structure. Note we do not need to assume $|\sigma| < \infty$.

Corollary 4.4. *Let A, f, and B as in Theorem 4.2. There exists a minimal graded free resolution of the trivial B-module ${}_Bk$ which becomes periodic of period at most 2 after $d + 1$ steps.*

Proof. Let $\mathbf{Q}$ be a minimal graded free resolution of ${}_Bk$. By Theorem 4.2, there exists a twisted left matrix factorization (φ, τ) of f such that $\mathbf{\Omega}(\varphi, \tau)$ is a minimal graded free resolution of the $(d + 1)$-st syzygy $\Omega_{d+1}({}_Bk)$.

Let $\bar{\sigma} : B \to B$ be the automorphism induced by σ. Clearly, ${}^{\bar{\sigma}}({}_Bk) \cong {}_Bk$, so there exists a chain isomorphism $\Phi : {}^{\bar{\sigma}}\mathbf{Q} \to \mathbf{Q}$ by Lemma 1.2. By the 5-Lemma, Φ_{d+1} restricts to a graded B-module isomorphism

$$ {}^{\bar{\sigma}}(\Omega_{d+1}({}_Bk)) \cong \Omega_{d+1}({}_Bk). $$

The result follows from Proposition 2.12. □

Recall that a graded free resolution $(\mathbf{P}_\bullet, d_\bullet)$ is called *linear* if P_i is generated in degree i for all $i \geq 0$.

Corollary 4.5. *Let A, f, and B be as in Theorem 4.2. Additionally assume f is quadratic. Then the truncation $\mathbf{Q}_{\geq d+1}(d + 1)$ is a linear free resolution of ker ∂_d.*

5. Homotopy category of twisted matrix factorizations

In [4, Theorem 4.4.1], Buchweitz established an equivalence between the stable category of maximal Cohen–Macaulay modules over a noetherian ring B with finite left and right injective dimensions and a quotient of the bounded derived category of modules over B now called the singularity category. He noted the equivalence also holds in the graded case. In [17, Theorem 3.10], Orlov proved that if B is a graded factor algebra of a finitely generated, connected, $\mathbb{N}$-graded noetherian k-algebra A of finite global dimension by a central regular element W, then the singularity category of graded B-modules is equivalent to a category Orlov called "the category of graded D-branes of type B for the pair (B, W)." In this section we extend Orlov's result to factors of left noetherian AS-regular algebras by regular, normal elements. Much of Orlov's work goes through with the obvious necessary changes. The key difference is that $|\sigma|$ need not be finite in our case, so we cannot appeal to periodicity of a resolution.

Before stating our main result, we recall some definitions.

Definition 5.1. Let A be a graded k-algebra. Then A is *Artin-Schelter regular* of dimension n if

(1) $\text{gl.dim}(A) = n < \infty$
(2) $\text{GKdim}(A) = n$
(3) $\text{Ext}^i_A(k, A) = \delta_{i,n} k$

We frequently abbreviate this condition as AS-regular. We also note that the results below in which A is an AS-regular algebra do not require the assumption that the Gelfand-Kirillov dimension is finite.

In this section, we focus on the case that A is a left noetherian AS-regular algebra of dimension n. We continue to let $f \in A_+$ be a normal, regular homogeneous element with normalizing automorphism σ, and let $B = A/(f)$. We will also continue to consider left modules over B.

We begin by recalling the definition of maximal Cohen–Macaulay modules.

Definition 5.2. A finitely generated graded module M over a graded left noetherian k-algebra B of finite left and right injective dimension is called *maximal Cohen–Macaulay* if and only if $\text{Ext}^i_B(M, B) = 0$ for $i \neq 0$.

Lemma 5.3. *Let A be a left noetherian, AS-regular algebra. Let $f \in A_+$ be a homogeneous normal regular element and let $B = A/(f)$. Then for any finitely generated graded left B-module M, $\text{pd}_A(M) = 1$ if and only if $\text{Ext}^i_B(M, B) = 0$ for all $i \neq 0$.*

Proof. We have $\text{pd}_A(M) = 1$ if and only if $\text{Ext}^i_A(M, A) = 0$ for all $i > 1$. One direction of this is clear. The other is Jørgensen's Ext-vanishing theorem [12, Theorem 2.3]. Let

$d = \deg f$. Since $0 \to A(-d) \xrightarrow{f} A \to B \to 0$ is a minimal graded free resolution of ${}_AB$, we see that $\mathrm{Ext}_A(B, A)$ is concentrated in homological degree 1 and $\mathrm{Ext}^1_A(B, A) \cong B(d)$ as graded left B-modules. Then the change of rings spectral sequence

$$\mathrm{Ext}^p_B(M, \mathrm{Ext}^q_A(B, A)) \Rightarrow \mathrm{Ext}^{p+q}_A(M, A)$$

shows $\mathrm{Ext}^i_B(M, B) = 0$ for $i \neq 0$ if and only if $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 1$. □

There is a natural functor of abelian categories $\mathcal{C} : TMF_A(f) \to B$-GrMod given on objects by $(\varphi, \tau) \mapsto \mathrm{coker}\ \varphi$. (Recall from the proof of Proposition 2.4 that coker φ is a B-module.) A morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ induces a well-defined map $\psi : \mathrm{coker}\ \varphi \to \mathrm{coker}\ \varphi'$ by $\pi'\Psi_G\pi^{-1}$ where $\pi : G \to \mathrm{coker}\ \varphi$ and $\pi' : G' \to \mathrm{coker}\ \varphi'$ are the canonical projections, and π^{-1} is any section of π.

The functor $\mathcal{C}$ is not essentially surjective[2]; objects in the image of $\mathcal{C}$ are finitely generated B-modules M such that $\mathrm{pd}_A(M) = 1$. By Lemma 5.3, if A is left noetherian and AS-regular, the image of $\mathcal{C}$ consists of maximal Cohen–Macaulay modules.

We denote the full subcategory of maximal Cohen–Macaulay modules in B-GrMod by $MCM(B)$. Following [4], we define the category of *stable* maximal Cohen–Macaulay modules, which we denote $\underline{MCM}(B)$, to have the same objects as $MCM(B)$, but for $M, N \in \underline{MCM}(B)$,

$$\mathrm{Hom}_{\underline{MCM}(B)}(M, N) = \mathrm{Hom}_B(M, N)/R$$

where R is the subspace of morphisms which factor through a graded free B-module.

As in [4] and [17], let $D^b(B)$ be the bounded derived category of finitely generated graded left B-modules. A complex in $D^b(B)$ is called *perfect* if it is isomorphic in $D^b(B)$ to a complex of finitely generated graded free modules. Perfect complexes form a full, triangulated subcategory $D^b_{perf}(B)$ of $D^b(B)$. The *singularity category of* B is defined to be the quotient category $D^b_{sg}(B) = D^b(B)/D^b_{perf}(B)$.

As noted in [4, p. 16], the composition $MCM(B) \to D^b(B) \to D^b_{sg}(B)$, where the first functor takes a module to its trivial complex, factors uniquely through the quotient $MCM(B) \to \underline{MCM}(B)$, yielding a functor $\mathcal{G} : \underline{MCM}(B) \to D^b_{sg}(B)$. Buchweitz showed that $\mathcal{G}$ is an exact equivalence. The equivalence $\mathcal{G}$ induces a triangulated structure on $\underline{MCM}(B)$. (It is possible to describe the triangulated structure independently, see [4], but as we will not need it, we omit any details.)

Thus it is natural to consider a "stable" version of the category $TMF_A(f)$. To motivate the definition, we remark that the category $TMF_A(f)$ is equivalent to the category of doubly-infinite sequences of graded free A-module homomorphisms of the form

$$\cdots \to {}^{\mathrm{tw}}F \xrightarrow{{}^{\mathrm{tw}}\varphi} {}^{\mathrm{tw}}G \xrightarrow{\tau} F \xrightarrow{\varphi} G \xrightarrow{{}^{\mathrm{tw}^{-1}}\tau} {}^{\mathrm{tw}^{-1}}F \to \cdots$$

[2] A functor $F : C \to D$ is *essentially surjective* if every object in D is isomorphic to $F(c)$ for some object c in C.

whose compositions are multiplication by f, and whose morphisms are maps of sequences

$$\Psi = (\dots, {}^{\mathrm{tw}}\Psi_F, {}^{\mathrm{tw}}\Psi_G, \Psi_F, \Psi_G, {}^{\mathrm{tw}^{-1}}\Psi_F, \dots)$$

satisfying the necessary commutation relations. We adopt the structure of the homotopy category of such sequences of graded free A-modules.

Definition 5.4. A morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ is *null homotopic* if there exists a pair (s, t) of degree zero module homomorphisms $s : G \to F'$ and $t : F \to {}^{\mathrm{tw}}G'$ such that ${}^{\mathrm{tw}}\Psi_G = {}^{\mathrm{tw}}\varphi' {}^{\mathrm{tw}}s + t\tau$ and $\Psi_F = \tau' t + s\varphi$.

We denote by $hTMF_A(f)$ the quotient (homotopy) category of $TMF_A(f)$ with the same objects, and whose morphisms are equivalence classes of morphisms in $TMF_A(f)$ modulo null homotopic morphisms. Observe that taking $s : A \to A$ to be the identity map and $t : A \to {}^{\mathrm{tw}}A$ to be zero shows the identity map $(\mathrm{id}_A, \lambda_f^A) \to (\mathrm{id}_A, \lambda_f^A)$ is null homotopic. Thus $(\mathrm{id}_A, \lambda_f^A) \cong 0$ in $hTMF_A(f)$.[3]

A similar calculation shows $(\lambda_f^A, \mathrm{id}_{{}^{\mathrm{tw}}A}) \cong 0$ in this category. More generally we have the following.

Lemma 5.5. *If $(\varphi, \tau) \in TMF_A(f)$ such that* $\operatorname{coker} \varphi$ *is a graded free B-module, then $(\varphi, \tau) \cong 0$ in $hTMF_A(f)$.*

Proof. The lemma is trivial if $\varphi = 0$, so suppose $\varphi \neq 0$ and $M = \operatorname{coker} \varphi = \operatorname{coker} (B \otimes_A F \xrightarrow{1\otimes\varphi} B \otimes_A G)$ is a graded free left B-module. Let $\psi : M \to B \otimes_A G$ be a graded splitting of the canonical projection π, viewed as a map of graded left A-modules. Since $B \otimes_A G$ is isomorphic (as a left A-module) to the cokernel of $\lambda_f^G : {}^{\mathrm{tw}}G \to G$, lifting ψ gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Psi_F} & & \downarrow{\scriptstyle \Psi_G} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & {}^{\mathrm{tw}}G & \xrightarrow{\lambda_f^G} & G & \longrightarrow & B \otimes_A G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & F & \xrightarrow{\varphi} & G & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Since $\pi\psi = \mathrm{id}_M$, there exists $s : G \to F$ such that $\mathrm{id}_F - \tau\Psi_F = s\varphi$ and $\mathrm{id}_G - \Psi_G = \varphi s$ by the comparison theorem. (The only A-module homomorphism $M \to G$ is the zero map.) If we now set $t = \Psi_F$, the morphism $(\Psi_G, \tau\Psi_F)$ of twisted matrix factorizations

[3] Recall that objects (φ, τ), (φ', τ') are isomorphic in $hTMF_A(f)$ if and only if there exist maps Φ and Ψ between them such that $\mathrm{id}_{(\varphi,\tau)} - \Phi\Psi$ and $\mathrm{id}_{(\varphi',\tau')} - \Psi\Phi$ are null homotopic.

is chain homotopic to the identity map on (φ, τ) via the pair (s, t). Indeed, it is clear that $\mathrm{id}_F = \tau t + s\varphi$. To see that $\mathrm{id}^{\mathrm{tw}}G = {}^{\mathrm{tw}}\varphi\, {}^{\mathrm{tw}}s + t\tau$, it suffices to show ${}^{\mathrm{tw}}\Psi_G = t\tau$. This follows from the equalities

$$0 = {}^{\mathrm{tw}}\Psi_G\, {}^{\mathrm{tw}}\varphi - \lambda_f^{{}^{\mathrm{tw}}G}\, {}^{\mathrm{tw}}\Psi_F = {}^{\mathrm{tw}}\Psi_G\, {}^{\mathrm{tw}}\varphi - \Psi_F \lambda_f^F = {}^{\mathrm{tw}}\Psi_G\, {}^{\mathrm{tw}}\varphi - \Psi_F \tau\, {}^{\mathrm{tw}}\varphi$$

and the injectivity of φ. □

Remark 5.6. A fact that is implicit in the previous proof is that $TMF_A(f) \to MCM(B)$, and hence the composite $\underline{\mathcal{C}} : TMF_A(f) \to \underline{MCM}(B)$, is a full functor.

Our next objective is to establish the following fact.

Proposition 5.7. *The category $hTMF_A(f)$ is a triangulated category.*

We begin with a few definitions. The translation functor on $hTMF_A(f)$ is given by $(\varphi, \tau)[1] = (-{}^{\mathrm{tw}^{-1}}\tau, -\varphi)$ on objects and by $\Psi[1] = ({}^{\mathrm{tw}^{-1}}\Psi_F, \Psi_G)$ on morphisms. For any morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ the *mapping cone* of Ψ is the pair

$$C(\Psi) = (\gamma : F' \oplus G \to G' \oplus {}^{\mathrm{tw}^{-1}}F, \quad \delta : {}^{\mathrm{tw}}G' \oplus F \to F' \oplus G)$$

where

$$\gamma = \begin{pmatrix} \varphi' & 0 \\ \Psi_G & -{}^{\mathrm{tw}^{-1}}\tau \end{pmatrix} \qquad \text{and} \qquad \delta = \begin{pmatrix} \tau' & 0 \\ \Psi_F & -\varphi \end{pmatrix}.$$

By the above matrix notation, we mean that the maps γ and δ are given as follows on ordered pairs:

$$\gamma(x', y) = \left(\varphi'(x') + \Psi_G(y), -{}^{\mathrm{tw}^{-1}}\tau(y)\right) \qquad \delta(y', x) = (\tau'(y') + \Psi_F(x), -\varphi(x))\,.$$

It is straightforward to check that this pair is a twisted matrix factorization and there exist canonical inclusion and projection morphisms $i : (\varphi', \tau') \to C(\Psi)$ and $p : C(\Psi) \to (\varphi, \tau)[1]$. Moreover, given a commutative square of twisted factorizations

$$\begin{array}{ccc} (\varphi, \tau) & \xrightarrow{\Psi} & (\varphi', \tau') \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi'} \\ (\gamma, \delta) & \xrightarrow[\Phi]{} & (\gamma', \delta') \end{array}$$

an easy diagram chase shows $(\Pi'_G \oplus {}^{\mathrm{tw}^{-1}}\Pi_F, \Pi'_F \oplus \Pi_G)$ defines a morphism $C(\Psi) \to C(\Phi)$. Note that the complex $\mathbf{\Omega}(C(\Psi))$ is the mapping cone of the induced morphism of complexes $\mathbf{\Omega}(\Psi) : \mathbf{\Omega}(\varphi, \tau) \to \mathbf{\Omega}(\varphi', \tau')$.

We define a *standard triangle* to be any sequence of maps in $hTMF_A(f)$

$$(\varphi,\tau) \xrightarrow{\Psi} (\varphi',\tau') \xrightarrow{i} C(\Psi) \xrightarrow{p} (\varphi,\tau)[1].$$

We define a *distinguished triangle* to be any triangle

$$(\varphi,\tau) \xrightarrow{\Psi} (\varphi',\tau') \xrightarrow{\Psi'} (\varphi'',\tau'') \xrightarrow{\Psi''} (\varphi,\tau)[1]$$

isomorphic to a standard triangle. For any twisted factorization (φ,τ), the triangle

$$(\varphi,\tau) \xrightarrow{\mathrm{id}} (\varphi,\tau) \to 0 \to (\varphi,\tau)[1]$$

is distinguished. To see this, consider the diagram

$$\begin{array}{ccccccc} (\varphi,\tau) & \xrightarrow{\mathrm{id}} & (\varphi,\tau) & \longrightarrow & 0 & \longrightarrow & (\varphi,\tau)[1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ (\varphi,\tau) & \xrightarrow{\mathrm{id}} & (\varphi,\tau) & \xrightarrow{i} & C(\mathrm{id}) & \xrightarrow{p} & (\varphi,\tau)[1] \end{array}$$

and note that $\mathrm{id}_{C(\mathrm{id})}$ is null homotopic via the pair

$$s : G \oplus {}^{\mathrm{tw}^{-1}}F \to F \oplus G \qquad t : F \oplus G \to {}^{\mathrm{tw}}G \oplus F$$

both given by $(x, y) \mapsto (0, x)$. Precomposing this homotopy with the canonical inclusion $(\varphi,\tau) \to C(\mathrm{id})$ shows that i is null homotopic. Thus the diagram commutes in $hTMF_A(f)$, and is hence an isomorphism of triangles in $hTMF_A(f)$.

To show $hTMF_A(f)$ is triangulated, it remains to show distinguished triangles are closed under rotations and that the octahedral axiom holds. The argument very closely follows the proof of [7, Theorem IV.1.9]. We discuss only rotations of distinguished triangles in detail, leaving the translation of the remainder of the proof from [7] to the interested reader.

To verify the class of distinguished triangles is closed under rotations, it suffices to consider standard triangles.

Let

$$(\varphi,\tau) \xrightarrow{\Psi} (\varphi',\tau') \xrightarrow{i} C(\Psi) \xrightarrow{p} (\varphi,\tau)[1]$$

be a standard triangle. To see the rotated triangle

$$(\varphi',\tau') \xrightarrow{i} C(\Psi) \xrightarrow{p} (\varphi,\tau)[1] \xrightarrow{-\Psi[1]} (\varphi',\tau')[1]$$

is distinguished, first observe that $C(i)$ is given by the pair

$$(F' \oplus G) \oplus G' \xrightarrow{\begin{pmatrix} \varphi' & 0 & 0 \\ \Psi_G & -{}^{\mathrm{tw}^{-1}}\tau & 0 \\ \mathrm{id} & 0 & -{}^{\mathrm{tw}^{-1}}\tau' \end{pmatrix}} \left(G' \oplus {}^{\mathrm{tw}^{-1}}F\right) \oplus {}^{\mathrm{tw}^{-1}}F'$$

$$\left({}^{\mathrm{tw}}G' \oplus F\right) \oplus F' \xrightarrow{\begin{pmatrix} \tau' & 0 & 0 \\ \Psi_F & -\varphi & 0 \\ \mathrm{id} & 0 & -\varphi' \end{pmatrix}} (F' \oplus G) \oplus G'.$$

Let $\Theta : (\varphi, \tau)[1] \to C(i)$ be the morphism defined by the pair

$$\Theta_{\mathrm{tw}^{-1}F} : {}^{\mathrm{tw}^{-1}}F \xrightarrow{(0\ \mathrm{id}\ -{}^{\mathrm{tw}^{-1}}\Psi_F)} G' \oplus {}^{\mathrm{tw}^{-1}}F \oplus {}^{\mathrm{tw}^{-1}}F',$$

$$\Theta_G : G \xrightarrow{(0\ \mathrm{id}\ -\Psi_G)} F' \oplus G \oplus G'.$$

This gives a diagram

$$\begin{array}{ccccccc} (\varphi', \tau') & \xrightarrow{i} & C(\Psi) & \xrightarrow{p} & (\varphi, \tau)[1] & \xrightarrow{-\Psi[1]} & (\varphi', \tau')[1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\varphi', \tau') & \xrightarrow{i} & C(\Psi) & \xrightarrow{j} & C(i) & \xrightarrow{q} & (\varphi', \tau')[1] \end{array}$$

where j and q are the canonical morphisms for the mapping cone $C(i)$. The first and last squares are easily seen to commute.

The middle square, however, commutes only up to homotopy. The morphism $j - \Theta p$ is seen to be null homotopic via the pair of maps

$$s : G' \oplus {}^{\mathrm{tw}^{-1}}F \to F' \oplus G \oplus G' \qquad t : F' \oplus G \to {}^{\mathrm{tw}}G' \oplus F \oplus F'$$

both given by $(x, y) \mapsto (0, 0, x)$. To see that Θ is an isomorphism in $hTMF_A(f)$, let $\pi : C(i) \to (\varphi, \tau)[1]$ be the canonical projection. Then $\pi\Theta$ is the identity on $(\varphi, \tau)[1]$ and $\mathrm{id}_{C(i)} - \Theta\pi$ is seen to be null homotopic by precomposing the pair (s, t) above with the projection $C(i) \to C(\Psi)$. This shows the class of distinguished triangles is closed under rotations.

Theorem 5.8. *Let A be a left noetherian AS-regular algebra, $f \in A_+$ a homogeneous normal regular element, and $B = A/(f)$. Then the categories $hTMF_A(f)$, $\underline{MCM}(B)$, and $D^b_{sg}(B)$ are equivalent.*

Proof. Since $\mathcal{G}$ is known to be an exact equivalence, it suffices to show $hTMF_A(f) \approx \underline{MCM}(B)$.

The functor $\underline{\mathcal{C}} : TMF_A(f) \to \underline{MCM}(B)$ factors through the projection to $hTMF_A(f)$ to complete the commutative diagram of functors

$$\begin{array}{ccccc} TMF_A(f) & \xrightarrow{\mathcal{C}} & MCM(B) & \longrightarrow & D^b(B) \\ \downarrow & \searrow^{\underline{\mathcal{C}}} & \downarrow & & \downarrow \\ hTMF_A(f) & \xrightarrow{\mathcal{F}} & \underline{MCM}(B) & \xrightarrow{\mathcal{G}} & D^b_{sg}(B). \end{array}$$

To see this, it is enough to show that any null homotopic morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ induces the zero map in $\underline{MCM}(B)$. Specifically, we show the induced map ψ : coker $\varphi \to$ coker φ' factors through the graded free module $B \otimes_A G'$. The morphism Ψ factors as

$$(\varphi, \tau) \xrightarrow{\Phi} (\gamma, \delta) \xrightarrow{\Pi} (\varphi', \tau')$$

through the twisted "horseshoe" factorization (γ, δ) where

$$\gamma : {}^{\mathrm{tw}}G' \oplus F' \xrightarrow{\begin{pmatrix} -\tau' & 0 \\ \mathrm{id} & \varphi' \end{pmatrix}} F' \oplus G',$$
$$\delta : {}^{\mathrm{tw}}F' \oplus {}^{\mathrm{tw}}G' \xrightarrow{\begin{pmatrix} -{}^{\mathrm{tw}}\varphi' & 0 \\ \mathrm{id} & \tau' \end{pmatrix}} {}^{\mathrm{tw}}G' \oplus F',$$

$\Phi_G = (s, \Psi_G)$, $\Phi_F = (t, \Psi_F)$ and Π is the canonical projection onto the second factor.

We claim coker $\gamma = B \otimes_A G'$. For any $x \in F'$ and $y \in G'$, $(x, y) = (0, y - \varphi'(x))$ in coker γ. Thus there is a surjection $G' \twoheadrightarrow$ coker γ. The kernel of this surjection consists of $z \in G'$ such that $z = \varphi'\tau'(w) = fw$ for some $w \in {}^{\mathrm{tw}}G'$. So coker $\gamma = G'/fG' = B \otimes_A G'$.

Now the induced maps coker $\varphi \xrightarrow{\phi} B \otimes_A G' \xrightarrow{\pi}$ coker φ' show the map ψ induced by Ψ factors through a graded free module, hence is the zero map in $\underline{MCM}(B)$. Thus the functor $\mathcal{F}$ is well-defined.

The triangulated structure on $\underline{MCM}(B)$ is induced by $\mathcal{G}$, so to prove $\mathcal{F}$ is an exact functor, it suffices to check that $\mathcal{G}F$ is exact. By Proposition 2.9, $\boldsymbol{\Omega}((\varphi, \tau)[1])$ is exact. Thus

$$0 \to \text{coker } (-\varphi) \to B \otimes_A {}^{\mathrm{tw}^{-1}}F \to \text{coker } (-{}^{\mathrm{tw}^{-1}}\tau) \to 0$$

is a short exact sequence in B-GrMod, and hence

$$\text{coker } (-\varphi) \to B \otimes_A {}^{\mathrm{tw}^{-1}}F \to \text{coker } (-{}^{\mathrm{tw}^{-1}}\tau) \to (\text{coker } (-\varphi))[1]$$

is a distinguished triangle in $D^b_{sg}(B)$. Since $B \otimes_A {}^{\mathrm{tw}^{-1}}F$ is graded free, the first two morphisms are zero. Rotating the triangle yields

$$\operatorname{coker}(-{}^{\mathrm{tw}^{-1}}\tau) \cong (\operatorname{coker}\,(-\varphi))[1] \cong (\operatorname{coker}\,\varphi)[1] = (F(\varphi,\tau))[1]$$

in $D^b_{sg}(B)$. Thus we have a natural isomorphism $(\mathcal{F}(\varphi,\tau))[1] \cong \mathcal{F}((\varphi,\tau)[1])$. That $\mathcal{F}$ takes a standard triangle in $hTMF_A(f)$ to a distinguished triangle in $D^b_{sg}(B)$ follows from this natural isomorphism, the fact that for a morphism $\Psi : (\varphi,\tau) \to (\varphi',\tau')$, $\mathbf{\Omega}(C(\Psi))$ is the mapping cone of $\mathbf{\Omega}(\Psi) : \mathbf{\Omega}(\varphi,\tau) \to \mathbf{\Omega}(\varphi',\tau')$, and the usual property of mapping cones fitting into long exact sequences in homology.

By Construction 2.8, $\underline{\mathcal{C}}$ is surjective on objects of $\underline{MCM}(B)$, hence the same is true of $\mathcal{F}$. Since $\underline{\mathcal{C}}$ is full by Remark 5.6, $\mathcal{F}$ is as well.

To see that $\mathcal{F}$ is injective on objects, we show $\mathcal{GF}$ is. Suppose $\mathcal{GF}(\varphi,\tau) \cong 0$ in $D^b_{sg}(B)$. Then $M = \operatorname{coker}\,\varphi$ admits a finite length graded free B-module resolution, so $\operatorname{Ext}^i_B(M,N) = 0$ for all N and all $i \gg 0$. By Proposition 2.9, $\mathbf{\Omega}(\varphi,\tau)$ is a graded free B-module resolution. Thus for some n, $\operatorname{Ext}^i_B(\operatorname{coker}(1 \otimes {}^{\mathrm{tw}^n}\varphi), N) = 0$ for all N and all $i > 0$. That is, $\operatorname{coker}(1 \otimes {}^{\mathrm{tw}^n}\varphi)$ is graded free. As noted in the proof of Proposition 2.9, $\operatorname{coker}(1 \otimes {}^{\mathrm{tw}^n}\varphi) \cong {}^{\overline{\mathrm{tw}}^n}M$. Since M is free if and only if ${}^{\overline{\mathrm{tw}}}M$ is, M is graded free. By Lemma 5.5, $(\varphi,\tau) \cong 0$ in $hTMF_A(f)$.

That $\mathcal{F}$ is faithful now follows from the triangulated structure (see [17, Theorem 3.9]). □

Zhang proves in [19, Theorem 1.3] that, among other properties, being noetherian, AS regular or AS-Gorenstein is invariant under graded Morita equivalence. Thus, in the case where A is left noetherian and AS-regular, the equivalence theorems of Section 3 imply equivalences of the corresponding categories of singularities.

6. Examples

Example 6.1. Let V be a finite-dimensional vector space over a field k with skew-symmetric, nondegenerate form ω. Assume $\dim V = 2n \geq 4$ and let $\mathfrak{h}$ be the corresponding Heisenberg Lie algebra and $U(\mathfrak{h})$ its universal enveloping algebra. Then $U(\mathfrak{h})$ can be presented by generators $x_1,\dots,x_n,y_1,\dots,y_n$ subject to the relations

$$[x_i,x_j] = [y_i,y_j] = 0$$
$$[x_i,y_j] = 0 \text{ for } i \neq j$$
$$[x_1,y_1] = [x_2,y_2] = \cdots = [x_n,y_n].$$

Since $\mathfrak{h}$ is a Lie algebra of dimension $2n+1$, $U(\mathfrak{h})$ is Artin-Schelter regular of dimension $2n+1$. The element $f = [x_1,y_1]$ is central and regular, and $B = U(\mathfrak{h})/(f) \cong k[x_1,\dots,x_n,y_1,\dots,y_n]$ is a commutative polynomial ring. By Hilbert's Syzygy Theorem, every finitely generated left B-module has a finite minimal graded free resolution. Thus there exist no nontrivial reduced twisted left matrix factorizations of f.

Example 6.2. Let $A = k[x, y][w; \zeta]$ be a graded Ore extension of a commutative polynomial ring in two variables by a graded automorphism ζ, where $wg = \zeta(g)w$ for all $g \in k[x, y]$. Then w^2 is regular and its normalizing automorphism is $\sigma = \zeta^{-2}$. After choosing bases, we define homomorphisms $\varphi : F \to G$ and $\tau : {}^{\mathrm{tw}}G \to F$ of graded free left A-modules via right multiplication by the matrices

$$[\varphi] = \begin{pmatrix} w & -\zeta(x) \\ 0 & w \end{pmatrix} \qquad \text{and} \qquad [\tau] = \begin{pmatrix} w & \zeta^2(x) \\ 0 & w \end{pmatrix}.$$

Note $[{}^{\mathrm{tw}}\varphi] = \begin{pmatrix} w & -\zeta^3(x) \\ 0 & w \end{pmatrix}$. A straightforward verification shows that $\varphi\tau = \lambda_{w^2}$ and $\tau\,{}^{\mathrm{tw}}\varphi = \lambda_{w^2}$. (We remind the reader that since we work with left modules, the composition is computed by multiplying matrices in the opposite order.) Examining for periodicity, we see that the minimal resolution $\mathbf{\Omega}(\varphi, \tau)$ is periodic of period p if and only if $\zeta^p(x) = cx$ for some integer p and scalar c. This example suggests a useful method for constructing twisted factorizations with desired properties. For example, if $\zeta(x) = x+y$ and $\zeta(y) = qy$ where q is a primitive n-th root of unity, then the resolution is periodic of period n.

As another example, taking $\zeta(x) = (x + y)/2$ and $\zeta(y) = y/2$ we obtain a resolution which is not periodic. But it is interesting to note that since $\zeta^n(x) \to 0$ as $n \to \infty$, the limiting matrix $\begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}$ defines a minimal resolution which is periodic of period 1. In this sense, the resolution becomes periodic after infinitely many steps.

In any case, extend ζ to a graded automorphism of A by $\zeta(w) = w$. Let $Z = \{\zeta^n \mid n \in \mathbb{Z}\}$ be the associated twisting system. The twisted multiplication in ${}^{Z}A$ gives

$$g * w^2 = \zeta^2(g)w^2 = w^2 g = w^2 * g$$

for all $g \in A$ so w is central in ${}^{Z}A$. By Proposition 2.12, every twisted matrix factorization of w^2 over ${}^{Z}A$ gives rise to a minimal graded free resolution of period at most 2.

Example 6.3. Let $A = k\langle x, y, z\rangle/\langle r_1, r_2, r_3\rangle$ where

$$r_1 = yz + zy - x^2$$
$$r_2 = xz + zx - y^2$$
$$r_3 = xy + yx - z^2.$$

The algebra A is a nondegenerate 3-dimensional Sklyanin algebra. The element $g = 2(y^3+xyz-yxz-x^3)$ (the factor of 2 is only to clean up the twisted matrix factorization) is central and regular in A, so $\sigma = \mathrm{id}_A$. Let

$$\varphi = \begin{pmatrix} x & y & z & 0 \\ -yz-2x^2 & -yx & zx-xz & x \\ xy-2yx & xz & -x^2 & y \\ -y^2-zx & x^2 & -xy & z \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} -zy & -x & z & y \\ zx-xz & z & -y & x \\ xy & y & x & -z \\ 2xyz-4x^3 & -2x^2 & 2y^2 & 2(xy-yx) \end{pmatrix}$$

be matrices with entries in A. One can check (it is not trivial) that $\tau\varphi = \varphi\tau = gI_4$. This matrix factorization produces a minimal resolution of the second syzygy module in a minimal resolution of the trivial module ${}_Bk$. Indeed if we put

$$M_2 = \begin{pmatrix} -x & z & y \\ z & -y & x \\ y & x & -z \\ -2x^2 & 2y^2 & 2(xy-yx) \end{pmatrix} \qquad M_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

then

$$\cdots \xrightarrow{\overline{\varphi}} B(-5)^3 \oplus B(-6) \xrightarrow{\overline{\tau}} B(-3) \oplus B(-4)^3 \xrightarrow{\overline{\varphi}}$$
$$B(-2)^3 \oplus B(-3) \xrightarrow{M_2} B(-1)^3 \xrightarrow{M_1} B$$

is a minimal graded free left B-module resolution of ${}_Bk$.

Example 6.4. Let $A = k_q[x, y]$ be the skew polynomial ring where $yx = qxy$ for some fixed $q \in k^\times$. Let g be the graded automorphism of A given by $g(x) = \lambda x$ and $g(y) = \lambda^{-1}y$ where λ is a primitive n-th root of unity. Let $G = \langle g \rangle$, the cyclic group of order n, act on A with invariant subring A^G. Classically (when $q = 1$), this is an A_n Kleinian singularity. It is not hard to check that A^G is generated by $X := x^n$, $Y := xy$, and $Z := y^n$, and $A^G \cong C/(\omega)$, where

$$C = k\langle X, Y, Z\rangle / \langle YX - q^n XY, ZX - q^{n^2} XZ, ZY - q^n YZ\rangle$$

is a skew polynomial ring and $\omega := XZ - q^{-\binom{n}{2}} Y^n$ is a regular normal element of C [13, Case 2.2]. Let C be graded by setting $\deg X = \deg Z = n$, and $\deg Y = 2$. Note that C is noetherian and AS-regular of dimension 3 and one has relations

$$\omega X = q^{n^2} X\omega, \;\; \omega Y = Y\omega, \;\text{ and }\; \omega Z = q^{-n^2} Z\omega.$$

The sets $M_j = \{a \in A \mid g(a) = \lambda^j a\}$ for $0 \le j < n$ are graded left $R = A^G$ modules, generated by x^j and y^{n-j}. Note that $M_0 = R$, and henceforth assume $j \neq 0$.

As a module over C, a minimal resolution of M_j has the form

$$0 \longrightarrow C(-2n+j) \oplus C(-n-j) \xrightarrow{G_j} C(-j) \oplus C(-n+j) \xrightarrow{D} M_j \longrightarrow 0,$$

where the maps are given by right multiplication by[4]:

[4] We adopt the usual convention that $\binom{k}{l} = 0$ for $k < l$.

$$D := \begin{pmatrix} x^j \\ y^{n-j} \end{pmatrix} \text{ and } G_j := \begin{pmatrix} -q^{-\binom{n-j}{2}}Y^{n-j} & q^{(n-j)j}X \\ -Z & q^{nj-\binom{j}{2}}Y^j \end{pmatrix}.$$

Thus $\mathrm{pd}_C M_j = 1$, and hence M_j is a maximal Cohen–Macaulay R-module. It is worth noting that when $q = 1$, the M_j form a complete set of maximal Cohen–Macaulay R-modules [16, Example 5.25].

Next, observe that

$$G_{n-j}G_j = \begin{pmatrix} -q^{(n-j)j}\omega & 0 \\ 0 & -q^{(n-j)j+n^2}\omega \end{pmatrix} = G_jG_{n-j}.$$

This shows $0 \to M_{n-j}(-n) \xrightarrow{\overline{G}_j} R(-j) \oplus R(-n+j) \xrightarrow{D} M_j \to 0$, where $\overline{G}_j$ is the R-module map induced on coker G_{n-j} by G_j, is an exact sequence of R-modules. So a minimal graded R-module resolution of M_j is periodic of period at most 2 for every $0 < j < n$. (When $n = 2$, the resolution has period 1.) With a small adjustment, we obtain a complex arising from a twisted matrix factorization of ω. Let

$$\Delta := \begin{pmatrix} -1 & 0 \\ 0 & -q^{n^2} \end{pmatrix}, \quad N_j := G_j\Delta^{-1} = \begin{pmatrix} q^{-\binom{n-j}{2}}Y^{n-j} & -q^{(n-j)j-n^2}X \\ Z & -q^{nj-\binom{j}{2}-n^2}Y^j \end{pmatrix},$$

and $P_{n-j} := q^{-(n-j)j}G_{n-j}$. Then we have

$$P_{n-j} := \begin{pmatrix} -q^{-\binom{j}{2}-j(n-j)}Y^j & X \\ -q^{-(n-j)j}Z & q^{(n-j)^2-\binom{n-j}{2}}Y^{n-j} \end{pmatrix},$$

and

$${}^{\mathrm{tw}}N_j = \begin{pmatrix} q^{-\binom{n-j}{2}}Y^{n-j} & -q^{(n-j)j}X \\ q^{-n^2}Z & -q^{nj-\binom{j}{2}-n^2}Y^j \end{pmatrix}.$$

Finally we have $P_{n-j}N_j = \omega I = {}^{\mathrm{tw}}N_jP_{n-j}$ as desired. We note that $|\sigma| = |q|$, which can be an arbitrary positive integer or infinite.

Acknowledgments

Many computations were performed using the `NCAlgebra` package written by Conner and Moore for Macaulay2 [10]; it provides a Macaulay2 interface to the Bergman [2] system and was invaluable in performing many of the calculations. We also would like to thank the anonymous referee for comments regarding our exposition and for providing a simplified version of Proposition 4.1.

References

[1] David J. Anick, Noncommutative graded algebras and their Hilbert series, J. Algebra 78 (1) (1982) 120–140, MR 677714 (84g:16001).
[2] Jørgen Backelin, Svetlana Cojocaru, Victor Ufnarovski, Bergman, a system for computations in commutative and non-commutative algebra, Available at: http://servus.math.su.se/bergman/.
[3] Matthew Ballard, Dragas Deliu, David Favero, M. Umut Isik, Ludmil Katzarkov, Resolutions in factorization categories, arXiv:1212.3264v2.
[4] Ragnar-Olaf Buchweitz, Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings, Available at: https://tspace.library.utoronto.ca/handle/1807/16682.
[5] David Eisenbud, Homological algebra on a complete intersection, with an application to group representations, Trans. Amer. Math. Soc. 260 (1) (1980) 35–64, MR 570778 (82d:13013).
[6] Pavel Etingof, Victor Ginzburg, Noncommutative complete intersections and matrix integrals, in: Special Issue: In Honor of Robert D. MacPherson. Part 3, Pure Appl. Math. Q. 3 (1) (2007) 107–151, MR 2330156 (2008b:16044).
[7] Sergei I. Gelfand, Yuri I. Manin, Methods of Homological Algebra, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, MR 1950475 (2003m:18001).
[8] E.S. Golod, Homology of the Shafarevich complex, and noncommutative complete intersections, Fundam. Prikl. Mat. 5 (1) (1999) 85–95, MR 1800120 (2001k:16012).
[9] E.S. Golod, I.R. Šafarevič, On the class field tower, Izv. Ross. Akad. Nauk Ser. Mat. 28 (1964) 261–272, MR 0161852 (28 #5056).
[10] Daniel R. Grayson, Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at: http://www.math.uiuc.edu/Macaulay2/.
[11] Peter Jørgensen, Non-commutative graded homological identities, J. Lond. Math. Soc. (2) 57 (2) (1998) 336–350, MR 1644217 (99h:16010).
[12] Peter Jørgensen, Ext vanishing and infinite Auslander–Buchsbaum, Proc. Amer. Math. Soc. 133 (5) (2005) 1335–1341 (electronic), MR 2111939 (2005k:16018).
[13] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Invariant theory of finite group actions on down-up algebras, Transform. Groups 20 (1) (2015) 113–165, MR 3317798.
[14] E. Kirkman, J. Kuzmanovich, J.J. Zhang, Noncommutative complete intersections, J. Algebra 429 (2015) 253–286, MR 3320624.
[15] T.Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999, MR 1653294 (99i:16001).
[16] Graham J. Leuschke, Roger Wiegand, Cohen–Macaulay Representations, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012, MR 2919145.
[17] Dmitri Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, in: Algebra, Arithmetic, and Geometry: In Honor of Yu. I. Manin, vol. II, in: Progr. Math., vol. 270, Birkhäuser Boston Inc., Boston, MA, 2009, pp. 503–531, MR 2641200 (2011c:14050).
[18] Alexander Polishchuk, Leonid Positselski, Quadratic Algebras, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005, MR MR2177131.
[19] J.J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. Lond. Math. Soc. (3) 72 (2) (1996) 281–311, MR 1367080 (96k:16078).