Guaranteed Solution and its Finding in the Integer Programming Problems

Knyaz Sh. Mamecdov
Doctor of Mathematical Sciences, Professor
Institute of Control Systems of the Azerbaijan National Academy of Sciences
AZ1141, str. B.Vaqabzade 9
Baku, Azerbaijan

Nazim N. Mamedov
Graduate Student
Institute Control Systems of the Azerbaijan National Academy of Sciences
AZ1141, str. B.Vaqabzade 9
Baku, Azerbaijan

Abstract

In the work the definitions of the guaranteed solution and guaranteed suboptimal solution are given for the multiconstraint integer programming problem. A method is developed for finding these solitions. This method is based on the sequental changing of the right hand side of the system of constraints by the dichotomia ous principle.

Keywords: Integer knarck problem, guaranteed solution, suboptimal solution, guaranteed suboptimal solution, dichotomous, calculation experiments

Introduction

Mathematical models of many design and marketing problems related to decision making in economics and technics are usualy described by the different classes of discrete optimization problems.

Since these problems belong to the NP-integer class, i.e. to the class of "hard problems", finding the solutions of such large dimensional problems meets some serious difficulties. That is why various high speed methods have been developed to find suboptimal (approximate) solutions of such problems [1-10, etc].

In the work integer programming problem is considered as a class of discrete optimization problems. It is supposed that solution or any suboptimal solution of this problem is found by some known method [2, 3, 5, 7, 8, 10, etc]. Then the maximal value of the perfomance index is known. Usually this value is not satisfactory for the customer in the solution of the practical design problems. Other words the customer wishes to have more that this maximal value (that descibes the benefit). In this case it is natural to change the right hand side of the constraint conditions (the recourses) i.e. to increase them. Thus we come to the following problem:

To give minimal increase to the recourses which guarantees that the perfomance index will not be less than given value.

In the work the mathematical model of such problem is constructed and its solution algorithm is developed.

1. **Problem formulation.** Consider the following problem:

$$\sum_{j=1}^{n} c_j x_j \to \max \qquad (1)$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i , \qquad (i = \overline{1, m}), \qquad (2)$$

$$0 \le x_j \le d_j \ ,(j = \overline{1,n}) \text{ are integers.} \quad (3)$$

Here

$a_{ij} \ge 0,\ c_j > 0,\ b_i > 0,\ d_j > 0\ \ (i = \overline{1,m},\ j = \overline{1,n})$

are given.

In the work a method is developed for finding the quaranted solution of the problem (1)-(3) (definition of the quaranted solution and quaranted suboptimal solution is given below). The matter of this method is as follows.

Suppose that optimal solution $X^* = (x_1^*, x_2^*, \dots, x_n^*)$ of the problem (1)-(3) is found by any known method. Then value of the function (1) will be

$$f^* = \sum_{j=1}^{n} c_j x_j^*.$$

Assume that we want to increase the number f^* by $\Delta = \left[f^* \cdot \frac{p}{100}\right]$. Here p is ncrease percent of the quantity f^*. It is clear that in this case we have to change, indeed to increase the numbers $b_i\ (i = \overline{1,m})$ in the system (2). Thus we arrive to the problem: To find a minimal numbers $\delta_i\ (i = \overline{1,m})$ such that in the problem obtained by replacing the right hand side of the system (2) by $b_i + \delta_i\ (i = \overline{1,m})$ the maximal value of the function (1) be no less than $f^* + \Delta$.

Here is the mathematical formulation of this problem

$$\delta_i \to \min \ ,(i = \overline{1,m}) \quad (4)$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i + \delta_i \ ,(i = \overline{1,m}), \quad (5)$$

$$\sum_{j=1}^{n} c_j x_j \ge f^* + \Delta, \quad (6)$$

$$0 \le x_j \le d_j \ ,\ (j = \overline{1,n}) \quad (7)$$

are integers.

2. *Finding of the Guaranteed Solution*

Note thtat that (4)-(7) is a multicriterial discrete optimization problem of special type. This problem may be solved theoratically by the known methods [11-15, etc]. But it is impossible to find optimal solution (the best solution in Pareto set) of such problems in large dimensional case in real time, since it belogs to the NP-integer class.

To find the optimal solution for the problem (4)-(7) we transform it to more suitable form. For this purpose we add new integer variables $y_i \ge 0,\ (i = \overline{1,m})$ to each $i-th\ (i = \overline{1,m})$ inequality of the system (5) turning them to the equalities, and then find expressions for $\delta_i\ ,\ (i = \overline{1,m})$. Thus

$$\sum_{j=1}^{n} a_{ij} x_j + y_i = b_i + \delta_i \ ,(i = \overline{1,m}),$$

or

$$\delta_i = \sum_{j=1}^{n} a_{ij} x_j + y_i - b_i \ ,(i = \overline{1,m}).$$

Now we give the definitions below as in [16].

Definition 1. Each n dimensional vector $X = (x_1, x_2, \dots, x_n)$ satisfying the system (5)-(7) by fixed Δ and $\delta_i\ ,\ (i = \overline{1,m})$ is called to be an admissible solution for the system (4)-(7) .

Definition 2. The admissible solution $X^z = (x_1^z, x_2^z, \dots, x_n^z)$ for the problem (4)-(7) corresponding to the minimal integer value of $\delta_i\ (i = \overline{1,m})$ is called to be a guaranteed solution for the problem (1)-(3).

The following theorem is proved.

Theorem 1. For the quaranted solution $X^z = (x_1^z, x_2^z, \dots, x_n^z)$ of the problem (1)-(3) the system of inequalities (5) should be satisfied as equalities, i.e. $y_i = 0 \quad (i = \overline{1,m})$ for this solution.

Proof. If to put the expressions for the variables δ_i, $(i = \overline{1,m})$ into the criteria (4) we obtain the problem

$$\delta_i = \sum_{j=1}^{n} a_{ij}x_j + y_i - b_i \to \min, \qquad (i = \overline{1,m}) \tag{8}$$

$$\sum_{j=1}^{n} c_j x_j \geq f^* + \Delta . \tag{9}$$

Here

$$0 \leq x_j \leq d_j, \ (j = \overline{1,n}), \tag{10}$$

$$y_i \geq 0, \qquad (i = \overline{1,m}) \tag{11}$$

are integers.

Multiplying both sides of the relations (8) and (9) by -1 we obtain

$$-\delta_i = -\sum_{j=1}^{n} a_{ij}x_j - y_i + b_i \to \max ,$$

$$-\sum_{j=1}^{n} c_j x_j \leq -f^* - \Delta ,$$

$$0 \leq x_j \leq d_j , \qquad (j = \overline{1,n}) \ and \ y_i \geq 0 , \qquad (i = \overline{1,m})$$

are integers.

Now we accept the replacements $x_j = d_j - t_j$, $(j = \overline{1,n})$. Here $0 \leq t_j \leq d_j$, $(j = \overline{1,n})$ are integers. Then

$$-\delta_i = \sum_{j=1}^{n} (-a_{ij})(d_j - t_j) - y_i + b_i = \sum_{j=1}^{n} a_{ij}t_j - \sum_{j=1}^{n} a_{ij}d_j - y_i + b_i,$$

$$\sum_{j=1}^{n} (-c_j)(d_j - t_j) \leq -f^* - \Delta \Rightarrow \sum_{j=1}^{n} c_j t_j \leq \sum_{j=1}^{n} c_j d_j - f^* - \Delta .$$

Thus we obtain the following problem

$$\sum_{j=1}^{n} a_{ij}t_j - \sum_{j=1}^{n} a_{ij}d_j - y_i + b_i \to max, \ (i = \overline{1,m}) \tag{12}$$

$$\sum_{j=1}^{n} c_j t_j \leq \sum_{j=1}^{n} c_j d_j - f^* - \Delta . \tag{13}$$

Here

$$0 \leq t_j \leq d_j , \ (j = \overline{1,n}) \tag{14}$$

and

$$y_i \geq 0 , \qquad (i = \overline{1,m}) \tag{15}$$

are integer numbers.

In this problem the variables $y_i \quad (i = \overline{1,m})$ are integers beloging to the interval $0 \leq y_i \leq b_i \quad (i = \overline{1,m})$ and do not participate in the constraints (13). Since (12)-(15) is a maximization problem and the quatities $y_i \quad (i = \overline{1,m})$ are involved to the criteria (12) with the sign minus, it is clear that $y_i = 0 \quad (i = \overline{1,m})$ in the optimal solution. The theorem is proved.

To reduce the problem (12)-(14) to more simple form let us accept the denotations

$$A_i := b_i - \sum_{j=1}^{n} a_{ij} d_j, \quad (i = \overline{1,m}), \quad S := \sum_{j=1}^{n} c_j d_j - f^* - \Delta.$$

It is clear that A_i $(i = \overline{1,m})$ and S are constant numbers. Then the problem (12)-(15) turns to

$$\sum_{j=1}^{n} a_{ij} t_j + A_i \to max, \quad (i = \overline{1,m}) \tag{16}$$

$$\sum_{j=1}^{n} c_j t_j \le S, \tag{17}$$

$$0 \le t_j \le d_j, \ (j = \overline{1,n}) \text{ are integers.} \tag{18}$$

Note that in the problem (16)-(18) $a_{ij} \ge 0$, $c_j > 0$, $A_i \le 0$ $(i = \overline{1,m},\ j = \overline{1,n})$ and $S > 0$ are integer numbers. Obtained problem (16)-(18) is a multicriterial boolen programming problem with a single constraint. Solving the problem (16)-(18) by any known method [11] one can find its optimal solution $T^* = (t_1^*, t_2^*, \dots, t_n^*)$. Then making the replacement $x_j = d_j - t_j$, $(j = \overline{1,n})$ we can find the guaranteed solution $X^z = (x_1^z, x_2^z, \dots, x_n^z)$ of the problem (1)-(3).

3. *Finding of the Guaranteed Suboptimal Solution*

Note that the problem (16)-(18) is a multicriterial boolen programming problem and finding its optimal solution may take non real machine time. That is why here we give a definition of the guaraneed suboptimal solution and develop a method for its finding.

Definition. The admissible solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ that gives minimum to δ_i $(i = \overline{1,m})$ in the problem (4)-(7) by fixed $\Delta > 0$ is called to be a guaranteed suboptimal solution for the problem (1)-(3).

Note that this definition is transformation of the definition of the guaranteed suboptimal solution for the boolen programming problem given in [16].

Here is the description of the proposed method for the construction of the guaranteed suboptimal solution for the problem (1)-(3).

Suppose that some suboptimal solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ of the problem (1)-(3) and corresponding value of the function (1)

$$f^0 = \sum_{j=1}^{n} c_j x_j^s$$

is found by any known method [9-10]. We want to find new admissible solution $X = (x_1, x_2, \dots, x_n)$ that guarantees increasing the corresponding value of the function (1) at least by integer $\Delta^\delta > 0$. In particular may be taken $\Delta^\delta = \left[f^0 \frac{p}{100}\right]$. Here $p > 0$ is a given number and denotes the increasing percent of the number f^0. As a result we get the following problem corresponding to (4)-(7)

$$\delta_i \to \min, \ (i = \overline{1,m}) \tag{19}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i + \delta_i, \ (i = \overline{1,m}), \tag{20}$$

$$\sum_{j=1}^{n} c_j x_j \ge f^0 + \Delta^\delta, \tag{21}$$

where

$$0 \le x_j \le d_j, \ (j = \overline{1,n}) \tag{22}$$

are integer numbers.

To find the small and non negative integer values of δ_i , $(i = \overline{1,m})$ we first write their admissible maximal values corresponding to the solution $X^s = (d_1, d_2, \dots, d_n)$

$$\delta_i^0 = \sum_{j=1}^{n} a_{ij} d_j - \mathrm{b}_i, \quad (i = \overline{1,m}).$$

We use the dichotomous principle to minimize the parameters δ_i $(i = \overline{1,m})$. To do this we fix the given numbers b_i $(i = \overline{1,m})$ because in the dichotomous process these numbers are changed. We denote $\hat{\mathrm{b}}_i := \mathrm{b}_i, \mathrm{b}_i' := \mathrm{b}_i,$

$$\mathrm{b}_i := \sum_{j=1}^{n} a_{ij} d_j , \qquad (i = \overline{1,m})$$

and $\delta_i^1 := \left[\frac{\delta_i^0}{2}\right]$ $(i = \overline{1,m})$. Here $[\, z \,]$ denotes integral part of the number z.

If $\delta_i^1 = 0$ $(i = \overline{1,m})$ then the minimization process of the numbers δ_i , $(i = \overline{1,m})$ is finished and $\min_i \delta_i^0, (i = \overline{1,m})$.

Otherwise taking $\mathrm{b}_i := \hat{\mathrm{b}}_i + \delta_i^1, (i = \overline{1,m})$ we find the next solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ of the problem (1)-(3) and corresponding value of the function (1)

$$f^s = \sum_{j=1}^{n} c_j x_j^s \ .$$

If $f^s \geq f^0 + \Delta^\delta$, then we take $\delta_i^0 := \delta_i^1$, $\delta_i^1 := \left[\frac{\delta_i^0}{2}\right]$, $\mathrm{b}_i := \hat{\mathrm{b}}_i + \delta_i^1$ $(i = \overline{1,m})$, otherwise if $f^0 + \Delta^\delta$, then we take $\delta_i^0 := \delta_i^1$, $\delta_i^1 := \left[\frac{\delta_i^0}{2}\right]$, $\hat{\mathrm{b}}_i := \mathrm{b}_i$, $\mathrm{b}_i := \hat{\mathrm{b}}_i + \delta_i^1$ $(i = \overline{1,m})$. Then we find the next suboptimal solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ of the problem (1)-(3) and corresponding value of the function (1). We continue this process for all i $(i = \overline{1,m})$ untill we have $\delta_i^1 = 0$. Thus the quantities δ_i, $(i = \overline{1,m})$ satisfying the conditions of the problem (19)-(22) will be $\delta_i = \mathrm{b}_i - \mathrm{b}_i'$.

The last solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ will be guaranteed suboptimal solution of the problem (1)-(3).

Note that the the process of minimization of the quantities δ_i $(i = \overline{1,m})$ was necessary for construction of the guaranteed suboptimal solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ of the problem (1)-(3). To find the increasing (decreasing) norm of the given recourses b_i, $(i = \overline{1,m})$ we put the constructed suboptimal solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ into the system (20) and find

$$\delta_i^s = \sum_{j=1}^{n} a_{ij} x_j^s - \mathrm{b}_i' \ ; \quad (i = \overline{1,m}) \ . \qquad (23)$$

which indeed is a final changing norm of the right hand side $b_i, (i = \overline{1,m})$.

Other words the new values of the given quantities b_i , $(i = \overline{1,m})$ by guaranteed suboptimal solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ are $\mathrm{b}_i^s = \mathrm{b}_i' + \delta_i^s$, $(i = \overline{1,m})$. As one can see from the formula (23) the quantities δ_i^s $(i = \overline{1,m})$ may take negative values also. It shows that if $\delta_i^s < 0$ for some i then the recourse b_i corresponding to the system (2) must decrease by δ_i^s, otherwise must increase δ_i^s units. This situation is often met in the calculation experiments.

The construction of the guaranteed suboptimal solution by proposed here method allows one to compare given two general recourses

$$\sum_{i=1}^{n} \mathrm{b}_i' \ and \ \sum_{i=1}^{n} \mathrm{b}_i^s.$$

Suppose that

$$S = \sum_{i=1}^{n} (b_i^s - b_i') .$$

If $S > 0$ then the initial recourses should be increased, otherwise should be decreased. The experiments show that in such cases the general recourses decrease.

Note. It is claer that the proposed here method for finding of the guaranteed suboptimal solution is a numerical method to solving of the multicriterial integer programming problem.

5. Results of the Numerical Experimets

We made some numerical experiments on the problems with different dimensions to demonstrate the quality of the proposed method for finding of the guaranteed suboptimal solution. The problem data are taken as the random numbers satisfying the following conditions

$$0 < c_j \leq 999, \ 0 \leq a_{ij} \leq 99, \quad b_i = \left[0.4 \sum_{j=1}^{n} a_{ij} d_j \right], \quad 1 \leq d_j \leq 9, \quad (i = \overline{1,m}, \quad j = \overline{1,n}).$$

The results of the experiments are given in the table below. The suboptimal solution of the consideren problem is found by the method given in [10].

In the table we acceopt the denotations:

1. m - number of constraints
2. n - number of variables
3. f^s – value of the function (1) corresponding to the suboptimal solution of (1)-(3)
4. $p(\%)$ - increasing percent of the value f^s
5. Δ^s - increasing of the value f^s
6. $f^s(\delta)$ – value of the function (1) by the guaranteed suboptimal solution
7. $\tilde{S}$ - sum of the right hand side of the system (2) considering minimal values of the increments δ_i, $(i = \overline{1,m})$ i.e.
$$\tilde{S} = \sum_{i=1}^{m} (b_i + \delta_i^{min})$$
8. S - sum of the right hand sides of the system (2) in the initial problem (1)-(3) i.e.
$$S = \sum_{i=1}^{m} b_i$$
9. S^* - sum of the right hand sides of the system (2) by the guaranteed suboptimal solution $X^s = (x_1^s, x_2^s, \dots, x_n^s)$ i.e.
$$S^* = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j^s$$
10. $\underline{S}$ - sum of the right hand sides of (2) by initial suboptimal solution $X^0 = (x_1^0, x_2^0, \dots, x_n^0)$ i.e.
$$\underline{S} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j^0$$
11. $\delta_{min} = \min_i \delta_i^{min}$, $\delta_{max} = \max_i \delta_i^{min}$, $\delta_{ort} = \frac{1}{m} \sum_{i=1}^{m} \delta_i^{min}$
12. $\Delta(f^s)$ - increment of the additional benefit corresponding to the added average unit expense i.e.

$$\Delta(f^s) = \frac{f^s(\delta) - f^s}{\delta_{ort}}$$

Table 1. Results of the Minimization Process of δ_i , $(i = \overline{1,m})$

m x n	10 x 100				10 x 500			
f^s	7938				51120			
p(%)	1	2	3	5	1	2	3	5
$\Delta^s = \left[f^s \cdot \frac{p}{100}\right]$	79	158	238	396	511	1022	1533	2556
$f^s(\delta) - f^s$	145	204	288	411	577	1104	1546	2862
$\tilde{S} - S$	195	484	652	977	1287	2582	3660	6903
$S^* - \underline{S}$	221	325	610	963	1392	2583	3808	6877
δ_{min}	-50	-68	-40	-29	94	182	349	578
δ_{max}	81	144	177	216	223	317	416	790
δ_{ort}	22	32	61	96	139	258	380	687
$\Delta(f^s)$	6.59	6.37	4.72	4.28	4.15	4.28	4.07	4.17

Table 2. Results of the Minimization Process of δ_i , $(i = \overline{1,m})$

m x n	10 x 1000				10 x 2000			
f^s	18471				38589			
p(%)	1	2	3	5	1	2	3	5
$\Delta^s = \left[f^s \cdot \frac{p}{100}\right]$	184	369	554	923	386	771	1157	1929
$f^s(\delta) - f^s$	257	409	606	979	390	833	1441	2261
$\tilde{S} - S$	251	352	689	1031	335	676	1357	2033
$S^* - \underline{S}$	252	433	790	962	326	714	1550	2266
δ_{min}	-7	10	48	29	-42	18	108	195
δ_{max}	67	102	144	152	75	111	219	300
δ_{ort}	25	43	79	96	32	71	155	226
$\Delta(f^s)$	10.28	9.51	7.67	10.19	12.18	11.73	9.3	10.01

Table 3. Results of the Minimization Process of δ_i , $(i = \overline{1,m})$

m x n	20 x 100				20 x 500			
f^s	7690				50230			
p(%)	1	2	3	5	1	2	3	5
$\Delta^s = \left[f^s \cdot \frac{p}{100}\right]$	77	153	230	384	502	1004	1506	2511
$f^s(\delta) - f^s$	147	159	268	513	724	1010	1624	3003
$\tilde{S} - S$	477	654	975	1970	3455	4313	6917	13844
$S^* - \underline{S}$	508	1237	1124	2274	2652	4422	6971	13429
δ_{min}	-63	-49	-60	8	-41	104	187	426
δ_{max}	113	171	143	221	227	331	456	792
δ_{ort}	25	61	56	113	132	221	348	671
$\Delta(f^s)$	5.88	2.61	4.79	4.54	5.48	4.57	4.67	4.48

Table 4. Results of the Minimization Process of δ_i , $(i = \overline{1,m})$

m x n	20 x 1000				20 x 2000			
f^s	15515				30408			
p(%)	1	2	3	5	1	2	3	5
$\Delta^s = \left[f^s \cdot \frac{p}{100}\right]$	155	310	465	775	304	608	912	1520
$f^s(\delta) - f^s$	163	368	491	837	352	750	1051	1553
$\tilde{S} - S$	241	679	1015	1702	681	1373	2054	2798
$S^* - \underline{S}$	175	694	838	1706	588	1278	1930	3394
δ_{min}	-84	-53	-48	-33	-133	-46	-30	102
δ_{max}	104	105	128	188	179	198	199	308
δ_{ort}	8	34	41	85	29	63	96	169
$\Delta(f^s)$	20.38	10.82	11.98	9.85	12.14	11.91	10.95	9.19

Table 5. Results of the Minimization Process of δ_i , $(i = \overline{1,m})$

m x n	50 x 100				50 x 500			
f^s	9002				48352			
p(%)	1	2	3	5	1	2	3	5
$\Delta^s = \left[f^s \cdot \frac{p}{100}\right]$	90	180	270	450	483	967	1450	2417
$f^s(\delta) - f^s$	129	208	352	523	784	1053	1590	3210
$\tilde{S} - S$	835	2097	3405	4240	8599	12890	17221	34463
$S^* - \underline{S}$	172	1961	2408	4522	7708	10134	17659	37110
δ_{min}	-72	1	-77	-4	32	57	201	521
δ_{max}	69	98	139	230	337	341	569	1041
δ_{ort}	3	39	48	90	154	202	353	742
$\Delta(f^s)$	43	5.33	7.33	5.81	5.09	5.21	4.51	4.33

Table 6. Results of the Minimization Process of δ_i , $(i = \overline{1,m})$

m x n	50 x 1000				50 x 2000			
f^s	12796				26897			
p(%)	1	2	3	5	1	2	3	5
$\Delta^s = \left[f^s \cdot \frac{p}{100}\right]$	127	255	383	639	269	537	806	1344
$f^s(\delta) - f^s$	149	309	436	769	412	626	963	1457
$\tilde{S} - S$	2531	3414	3815	5109	1697	3517	5114	7258
$S^* - \underline{S}$	208	650	1698	3653	2456	3389	5789	8611
δ_{min}	131	-156	-149	-89	-44	-113	-9	54
δ_{max}	131	144	154	220	169	211	309	323
δ_{ort}	4	13	33	73	49	67	115	172
$\Delta(f^s)$	37.25	23.77	13.21	10.53	8.40	9.34	8.37	8.47

5. Conclusion

As one can see from the table 1. the increase of the maximal value of the function (1) (maximal benefit) at least few times by the initial suboptimal solution is provided by relatively less increase of the right hand side of the systenm (2) (expenses). The avarage additional increment of the function corresponding to the minimal value of δ_i $(i = \overline{1,m})$ is minimum 2 and maximum 43 times. This fact is important in the solution of the practical problems. It is shown that raising $(1\% - 5\%)$ of the the increment percent of the found initial maximal value of the function leads to decreasing of the increment of the function. It shows that in practical problems it is suitable to increase the maximal benefit only few percents.

References

Sergienko I.V. Mathematical models and solution methods for the discrete optimization problems, Kiev, Naukova Dumka, 1988, 472p. (in Russian).

Sergienko I.V. Shilo V.P. Problems of discrete optimization: problems, solution methods, investigations, Kiev, Naukova Dumka, 2003, 264p. (in Russian).

Kovolyov M.M. Disgrete optimization (integer programming), Moscow, URSA, 2003 , 191p. (in Russian).

Sigal I.Kh., Ivanova A.P. Introduction to the practical discrete optimization, Moscow, FIZMATLIT, 2007. (in Russian).

Kellerer H., Pferschy U., Pisinger D. Knapsack problems. Berlin-Heidelberg: Springer-Verlag, 2004, 546p.

Babaev Dj, Mamedov K.Sh, Mehdiyev M.G. Methods to construction of the suboptimal solutions to the multidimensional knapsack problem, J. Cal. Math. and Math, Ph., 1978 , N.6, pp.1443-1453. (in Russian).

Sergienko I.V. Lebedeva T.T., Roshin V.A. Approximal methods to solution of the discrete optimization problems, Kiev, Naukovo Dumka, 1980, 276 p. (in Russian).

Martello S., Toth P. Knapsack problems. Algorithm and Computers Implementations, J.Willey $ Sons: New York Chichester, 1990, 296p.

Mamedov K.Sh., Musayeva T.M. Construction of the approximate solution for the multidimensional knapsack problem and finding the up estimation of the optimum, J. Aut. Cal. Tech., 2004, N.5, pp.72-82 (in Russian).

Mamedov K.Sh., Huseynov S.Y. Methods to construction of the suboptimal solutions for the integer programming problem and their sequential improvement, J. Aut. Cal. Tech., 2007, N.6, pp.20-31 (in Russian).

Shtoyer R. Multicriterial optimization. Theory and applied calculations, Moscow, Radio and Communications, 1992, 504 p. (in Russian).

Yemelichev V.A., Klichko V.G. On a stability of the pareto optimum of the boolen programming vector problem, J. Discrete mathematics, 1999, V.11, N.4, pp.27-32.

Yemelichev V.A., Yanushkevich O.A. Efficiency condition of the solutuion of the problem of discrete vector optimization, Inf. Bul. Of the Assos. Of Math. Prog., 2001, N 9, V.1, pp.53-57.

Semyonova G.V., Kolechkina L.N., Nagornaya A.N. An approach to the solution of the discrete vector optimization on the combinatorial set of permutations, Cybernetics ans system analysis, 2008, N.3, pp.158–172. (in Russian).

Vladimir Emelichev, Dmitry Podkopaev. Quantitative stability analysis for vector problems of 0-1 Programming, Discrete Optimization, vol.7, 2010, p.48-63.

Mamedov K.Sh., Mamedov N.N. Definition of the guaranteed solution and its finding in the boolen programming problem, Reports of ANAS, 2012, N.6, pp.19-26.