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A new approach to stability analysis of neural networks with time-varying delay via novel Lyapunov–Krasovskii functional*

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In this paper, new delay-dependent stability criteria for asymptotic stability of neural networks with time-varying delays are derived. The stability conditions are represented in terms of linear matrix inequalities (LMIs) by constructing new Lyapunov–Krasovskii functional. The proposed functional has an augmented quadratic form with states as well as the nonlinear function to consider the sector and the slope constraints. The less conservativeness of the proposed stability criteria can be guaranteed by using convex properties of the nonlinear function which satisfies the sector and slope bound. Numerical examples are presented to show the effectiveness of the proposed method.

Keywords: neural networks, Lyapunov–Krasovskii functional, sector bound, time-delay

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1. Introduction

Neural networks have been investigated by many researchers because of their various applications such as pattern recognition, signal processing, content-addressable memory, and optimization.[1−3] On the other hand, time-delays are frequently encountered in the implementation of neural networks as well as in many fields of science and engineering, including physics, large-scale systems, complex networks, population dynamics, biology, economy, etc.[4] Therefore, the stability analysis of neural networks with time-delay has been extensively studied and lots of papers have derived various kinds of criteria for the stability problem.[5−15] In general, one is more concerned with the derivation of delay-dependent stability conditions because delay-dependent stability conditions are often less conservative than delay-independent ones when delays are small. In the field of the delay-dependent stability criteria, an important index for checking the conservatism of stability criteria is to increase the feasible region of stability criteria or to acqurie the maximum allowable bounds of time delays for guaranteeing the stability of the networks.

Recently, in order to reduce the conservatism of stability criteria, new Lyapunov–Krasovskii functional with the technique of free-weighting matrices and the discretization method was proposed.[12,14,15,16,17] In the existing results, the activation function of neural networks is assumed to be nondecreasing, bounded and globally Lipschitz. Since the globally Lipschitz condition can be expressed as the bounded sector condition, their methods focused on the delay-dependent stability criteria for the systems with only the sector bounded condition.[5−15]

One of important kinds of the nonlinear systems is the Lur'e system whose nonlinear element satisfies certain sector constraints. Since the absolute stability as a notion of global asymptotic stability was introduced, stability analysis for Lur'e systems has been extensively studied.[18] The stability criteria for the Lur'e systems with slope-restricted nonlinearity are also presented in Refs. [19]–[21].

In this paper, new delay-dependent stability conditions for neural networks with time-varying delays are investigated by considering the sector and slope bound constraints. Less conservative LMI stability

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conditions are obtained by using the new Lyapunov–Krasovskii functional subject to the equality constraints. The proposed functional is based on an augmented vector which consists of state vector and nonlinear functions of the systems. The nonlinear functions of neural networks are expressed as convex combinations of sector and slope bounds, so that the equality constraint is derived by using convex properties of the nonlinear function. The equality constraint is included to the LMI condition by using Finsler's lemma.[22] The LMI criterion obtained can be easily solved by effective convex optimization algorithms such as interior point algorithm.[23] Finally, numerical examples show that the proposed method is less conservative than those of the existing results.

In the sequel, the following notations will be used. $\boldsymbol{R}^n$ is the n-dimensional Euclidean space. $\boldsymbol{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. $*$ denotes the symmetric part. $X > 0$ ($X \geq 0$) means that $\boldsymbol{X}$ is a real symmetric positive definitive matrix (positive semi-definite). $\boldsymbol{I}$ denotes the identity matrix with an appropriate dimension. diag$\{\ldots\}$ denotes the block diagonal matrix. $\boldsymbol{A}^{\perp}$ is the orthogonal complement matrix of $\boldsymbol{A}$.

2. Problem statement and preliminaries

Consider the following neural networks with time-varying delays:

$$\dot{\boldsymbol{y}}(t) = -A\boldsymbol{y}(t)+W_0 g(\boldsymbol{y}(t))+W_1 g(\boldsymbol{y}(t-h(t))+b, \quad (1)$$

where $\boldsymbol{y}(t) \in \boldsymbol{R}^n$ is the neuron state vector, $g(x(t)) = [g_1(x_1(t)),\ldots,g_n(x_n(t))]^T \in \boldsymbol{R}^n$ denotes the neuron activation function, $g(x(t-h(t))) = [g_1(x_1(t-h(t))),\ldots,g_n(x_n(t-h(t)))]^T \in \boldsymbol{R}^n$, $\boldsymbol{A} = diag\{a_i\} \in \boldsymbol{R}^{n\times n}$ is the positive diagonal matrix, $\boldsymbol{W}_0 \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{W}_1 \in \boldsymbol{R}^{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons, and $\boldsymbol{b} = [b_1, b_2, \ldots, b_n]^T \in \boldsymbol{R}^n$ is the constant input vector.

The delay, $h(t)$, is the time-varying continuous function that satisfies:

$$0 \leq h(t) \leq \bar{h}, \quad \dot{h}(t) \leq h_{\rm D}. \quad (2)$$

The activation functions, $g_i(x_i(t)), i = 1,\ldots,n$, are assumed to be sector bounded, that is,

$$0 \leq \frac{g_i(\sigma_i)}{\sigma_i} \leq l_i, \quad \sigma_i, \in \boldsymbol{R}, \quad i = 1,\ldots,n \quad (3)$$

where l_i is constant.

Also, the activation functions are assumed to satisfy

$$0 \leq \frac{{\rm d}g_i(\sigma_i)}{{\rm d}\sigma_i} \leq l'_i, \quad \sigma_i \in \boldsymbol{R}, \quad i = 1,\ldots,n. \quad (4)$$

For simplicity, by using the Brouwer's fixed-point theorem,[24] the equilibrium point $y^* = [y_1^*,\ldots,y_n^*]^{\rm T}$ is shifted to the origin by the $x(\cdot) = y(\cdot) - y^*$. Therefore, system (5) is transformed into the following form:

$$\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 f(x(t-h(t)), \quad (5)$$

where $x(t)$ is the state vector and $f_i(x_i(t)) = g_i(x_i(t)+y_i^*) - g_i(y_i^*)$ with $f_i(0) = 0, i = 1,\ldots,n$.
The following conditions are derived from Eqs. (3) and (4):

$$0 \leq \frac{f_i(\sigma_i)}{\sigma_i} \leq l_i, \quad (6)$$

$$0 \leq \frac{df_i(\sigma_i)}{d\sigma_i} \leq l'_i, \quad \forall\, \sigma_i \neq 0, \quad i = 1,\cdots,n. \quad (7)$$

In order to use the convexity of the nonlinear function, we define

$$\Delta(x(t)) \stackrel{\triangle}{=} {\rm diag}\{\Delta_1(x_i(t)),\ldots,\Delta_n(x_i(t))\}, \quad (8)$$

$$\bar{\Delta}(x(t)) \stackrel{\triangle}{=} {\rm diag}\{\Delta'_1(x_i(t)),\ldots,\Delta'_n(x_i(t))\}, \quad (9)$$

where

$$\Delta_i(x(t)) \stackrel{\triangle}{=} \frac{f_i(x_i(t))}{x_i(t)}, \quad \Delta'_i(x(t)) \stackrel{\triangle}{=} \frac{{\rm d}f_i(x_i(t))}{{\rm d}x_i(t)}. \quad (10)$$

By the definition of $f(\cdot)$, the nonlinear function $f(\cdot)$ is represented as

$$f(x(t)) = \Delta(x(t))x(t), \quad \dot{f}(x(t)) = \bar{\Delta}(x(t))\dot{x}(t) \quad (11)$$

and the parameters $\Delta(\cdot)$ belongs to the following sets, respectively:

$$\Delta := \{\Delta(x(t)) \in \boldsymbol{R}^{m\times m} | \Delta(x(t)) \in Co\{0, L\}\}, \quad (12)$$

where Co denotes the convex hull and

$$L \stackrel{\triangle}{=} {\rm diag}\{l_1,\ldots,l_n\}, \quad \bar{L} \stackrel{\triangle}{=} {\rm diag}\{l'_1,\ldots,l'_n\}. \quad (13)$$

The following lemmas will be utilized for deriving an LMI condition of the stability of system (5).
Lemma 1[4] For any constant matrix $\boldsymbol{W} \in \boldsymbol{R}^{n\times n} > 0$, a scalar $\tau > 0$ and a vector function $e(t) : \boldsymbol{R} \to \boldsymbol{R}^n$ such that the following integration is well defined, then

$$-\tau\int_{-\tau}^{0}\dot{e}^{\mathrm{T}}(t+\xi)W\dot{e}(t+\xi)\mathrm{d}\xi \leq [\,e(t)^{\mathrm{T}} \quad e(t-\tau)^{\mathrm{T}}\,]\begin{bmatrix}-W & W\\ W & -W\end{bmatrix}\begin{bmatrix}e(t)\\ e(t-\tau)\end{bmatrix}. \tag{14}$$

The following Finsler Lemma will be utilized to convert an inequality subject to an equality constraint into an inequality.

Lemma 2[22] Let $x \in \boldsymbol{R}^n$, $\Theta = \Theta^{\mathrm{T}} \in \boldsymbol{R}^{n\times n}$, and $\Gamma \in \boldsymbol{R}^{m\times n}$. The following statements are equivalent:

(i) $\xi^{\mathrm{T}}\Theta\xi < 0$ s.t. $\Gamma\xi = 0$, and $\forall xi \neq 0$,

(ii) ${\Gamma^{\perp}}^{\mathrm{T}}\Theta\Gamma^{\perp} < 0$, where $\Gamma^{\perp}$ is a matrix beloning to a null space of Γ.

3. Main results

In this section, we derive an LMI condition for systems (5) by using augmented variables and some useful lemmas.

For simplicity, define the augmented vectors

$$\begin{aligned}
x_a(t) &= [\,x^{\mathrm{T}}(t) \quad f^{\mathrm{T}}(x(t))\,]^{\mathrm{T}},\\
\zeta(t) &= \Big[x^{\mathrm{T}}(t)\;\; f^{\mathrm{T}}(x(t))\;\; x^{\mathrm{T}}(t-h(t))\;\; f^{\mathrm{T}}(x(t-h(t)))\;\; x^{\mathrm{T}}(t-\bar{h}/2)\;\; f^{\mathrm{T}}(x(t-\bar{h}/2))\\
&\qquad x^{\mathrm{T}}(t-\bar{h})\;\; f^{\mathrm{T}}(x(t-\bar{h}))\;\; \dot{x}^{\mathrm{T}}(t)\;\; \dot{f}^{\mathrm{T}}(x(t))\Big]^{\mathrm{T}},
\end{aligned} \tag{15}$$

and the matrices

$$\begin{aligned}
\bar{\boldsymbol{A}} &= \begin{bmatrix}0&0&0&0&0&0&0&0&I&0\\0&0&0&0&0&0&0&0&0&I\end{bmatrix}, \quad \mathcal{I}_1 = \begin{bmatrix}I&0&0&0&0&0&0&0&0&0\\0&I&0&0&0&0&0&0&0&0\end{bmatrix},\\
\mathcal{I}_2 &= \begin{bmatrix}0&0&I&0&0&0&0&0&0&0\\0&0&0&I&0&0&0&0&0&0\end{bmatrix}, \quad \mathcal{I}_3 = \begin{bmatrix}I&0&0&0&0&0&0&0&0&0\\0&I&0&0&0&0&0&0&0&0\\0&0&0&0&I&0&0&0&0&0\\0&0&0&0&0&I&0&0&0&0\end{bmatrix},\\
\mathcal{I}_4 &= \begin{bmatrix}0&0&0&0&I&0&0&0&0&0\\0&0&0&0&0&I&0&0&0&0\\0&0&0&0&0&0&I&0&0&0\\0&0&0&0&0&0&0&I&0&0\end{bmatrix}, \quad \mathcal{I}_5 = [\,0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;I\,],\\
\mathcal{I}_6 &= [\,0\;\;I\;\;0\;\;-I\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\,], \quad \mathcal{I}_7 = [\,0\;\;0\;\;0\;\;I\;\;0\;\;-I\;\;0\;\;0\;\;0\;\;0\,],\\
\mathcal{I}_8 &= [\,0\;\;0\;\;0\;\;0\;\;0\;\;I\;\;0\;\;-I\;\;0\;\;0\,], \quad \mathcal{I}_9 = [\,0\;\;I\;\;0\;\;0\;\;0\;\;-I\;\;0\;\;0\;\;0\;\;0\,],\\
\mathcal{I}_{10} &= [\,0\;\;0\;\;0\;\;-I\;\;0\;\;I\;\;0\;\;0\;\;0\;\;0\,], \quad \mathcal{I}_{11} = [\,0\;\;0\;\;0\;\;I\;\;0\;\;0\;\;0\;\;-I\;\;0\;\;0\,],\\
\boldsymbol{\Gamma} &= \begin{bmatrix}-A & W_0 & 0 & W_1 & 0&0&0&0 & -I & 0\\ \Delta & -I & 0 & 0 & 0&0&0&0&0&0\\ -\bar{\Delta}A & \bar{\Delta}W_0 & 0 & \bar{\Delta}W_1 & 0&0&0&0&0&-I\end{bmatrix}.
\end{aligned} \tag{16}$$

Theorem 1 System (5) is asymptotically stable, if there exist positive symmetric matrices $\boldsymbol{P}^{2n\times 2n}$, $\boldsymbol{Q}_1^{4n\times 4n}$, $\boldsymbol{Q}_2^{4n\times 4n}$, $\boldsymbol{R}_1^{n\times n}$ and $\boldsymbol{R}_2^{n\times n}$ such that the following LMIs hold:

$$N_{\Gamma}^{\mathrm{T}}(\Sigma_1+\Sigma_2+\Sigma_3)N_{\Gamma} < 0, \quad N_{\Gamma}^{\mathrm{T}}(\Sigma_1+\Sigma_2+\Sigma_4)N_{\Gamma} < 0, \tag{17}$$

where N_{Γ} is orthogonal complement of Γ and

$$\begin{aligned}
\Sigma_1 &= \mathcal{I}_1 P\bar{A} + \bar{A}^{\mathrm{T}}P\mathcal{I}_1^{\mathrm{T}}\\
\Sigma_2 &= \mathcal{I}_1^{\mathrm{T}}\boldsymbol{Q}_1\mathcal{I}_1 - (1-h_{\mathrm{D}})\mathcal{I}_2^{\mathrm{T}}\boldsymbol{Q}_1\mathcal{I}_2 + \mathcal{I}_3^{\mathrm{T}}\boldsymbol{Q}_2\mathcal{I}_3 - \mathcal{I}_4^{\mathrm{T}}\boldsymbol{Q}_2\mathcal{I}_4\\
\Sigma_3 &= (\bar{h}/2)^2\mathcal{I}_5^{\mathrm{T}}(R_1+R_2)\mathcal{I}_5 - \mathcal{I}_6^{\mathrm{T}}R_1\mathcal{I}_6 - \mathcal{I}_7^{\mathrm{T}}R_1\mathcal{I}_7 - \mathcal{I}_8^{\mathrm{T}}R_2\mathcal{I}_8\\
\Sigma_4 &= (\bar{h}/2)^2\mathcal{I}_5^{\mathrm{T}}(R_1+R_2)\mathcal{I}_5 - \mathcal{I}_9^{\mathrm{T}}R_1\mathcal{I}_9 - \mathcal{I}_{10}^{\mathrm{T}}R_2\mathcal{I}_{10} - \mathcal{I}_{11}^{\mathrm{T}}R_2\mathcal{I}_{11}.
\end{aligned} \tag{18}$$

Proof Take a candidate of the Lyapunov–Krasovskii functional as follows:

$$V = V_1 + V_2 + V_3, \tag{19}$$

where

$$V_1 = x_{\rm a}(t)^{\rm T} P x_{\rm a}(t), \tag{20}$$

$$V_2 = \int_{t-h(t)}^{t} x_{\rm a}^{t}(s) Q_1 x_{\rm a}(s){\rm d}s + \int_{t-\bar{h}/2}^{t} \begin{bmatrix} x_{\rm a}(s) \\ x_{\rm a}(s-\bar{h}/2) \end{bmatrix}^{\rm T} Q_2 \begin{bmatrix} x_{\rm a}(s) \\ x_{\rm a}(s-\bar{h}/2) \end{bmatrix} {\rm d}s, \tag{21}$$

$$V_3 = (\bar{h}/2)\int_{t-\bar{h}/2}^{t}\int_{s}^{t} \dot{f}^{\rm T}(x(u)) R_1 \dot{f}(x(u)){\rm d}u{\rm d}s + (\bar{h}/2)\int_{t-\bar{h}}^{t-\bar{h}/2}\int_{s}^{t} \dot{f}^{\rm T}(x(u)) R_2 \dot{f}(x(u)){\rm d}u{\rm d}s, \tag{22}$$

then the time derivative of V_1 will be obtained by using the definition in Eqs. (15) and (16) to be

$$\dot{V}_1 = \dot{\boldsymbol{x}}_{\rm a}(t)^{\rm T} \boldsymbol{P} \boldsymbol{x}_{\rm a}(t) + \boldsymbol{x}_{\rm a}(t)^{\rm T} \boldsymbol{P} \dot{\boldsymbol{x}}_{\rm a}(t) = \boldsymbol{\zeta}^{\rm T}(t) \varSigma_1 \boldsymbol{\zeta}(t). \tag{23}$$

Also, by using the augmented vector $\boldsymbol{\zeta}(t)$ and augmented matrices in Eq. (16), the time derivative of V_2 is derived to be:

$$\begin{aligned}\dot{V}_2 &= \boldsymbol{x}_{\rm a}^{\rm T}(t) Q_1 \boldsymbol{x}_{\rm a}(t) - (1-\dot{h}(t))\boldsymbol{x}_{\rm a}(t-h(t))^{\rm T} Q_1 \boldsymbol{x}_{\rm a}(t-h(t)) \\ &\quad + \begin{bmatrix} \boldsymbol{x}_{\rm a}(t) \\ \boldsymbol{x}_{\rm a}(t-\bar{h}/2) \end{bmatrix}^{\rm T} Q_2 \begin{bmatrix} \boldsymbol{x}_{\rm a}(t) \\ \boldsymbol{x}_{\rm a}(t-\bar{h}/2) \end{bmatrix} - \begin{bmatrix} \boldsymbol{x}_{\rm a}(t-\bar{h}/2) \\ \boldsymbol{x}_{\rm a}(t-\bar{h}) \end{bmatrix}^{\rm T} Q_2 \begin{bmatrix} \boldsymbol{x}_{\rm a}(t-\bar{h}/2) \\ \boldsymbol{x}_{\rm a}(t-\bar{h}) \end{bmatrix} \\ &\le \boldsymbol{x}_{\rm a}^{\rm T}(t) Q_1 \boldsymbol{x}_{\rm a}(t) - (1-h_{\rm D})\boldsymbol{x}_{\rm a}(t-h(t))^{\rm T} Q_1 \boldsymbol{x}_{\rm a}(t-h(t)) + \zeta^{\rm T}(t)\mathcal{I}_3^{\rm T} Q_2 \mathcal{I}_3 \zeta(t) - \zeta^{\rm T}(t)\mathcal{I}_4^{\rm T} Q_2 \mathcal{I}_4 \zeta(t) \\ &= \zeta^{\rm T}(t) \varSigma_2 \zeta(t). \end{aligned} \tag{24}$$

Next, the time-varying delay $h(t)$ can be considered in two intervals, i.e. $[0, \bar{h}/2)$ and $[\bar{h}/2, \bar{h}]$, thus the time derivative of V_3 is obtained separately in the two intervals as indicated below.

(i) **Case 1** $0 \le h(t) \le \bar{h}/2$

For this case, we have

$$\begin{aligned}\dot{V}_3 &\le \dot{f}^{\rm T}(x(t))(\bar{h}/2)^2(R_1+R_2)\dot{f}(x(t)) - \bar{h}/2\int_{t-h(t)}^{t} \dot{f}^{\rm T}(x(s)) R_1 \dot{f}(x(s)){\rm d}s \\ &\quad - \bar{h}/2\int_{t-\bar{h}/2}^{t-h(t)} \dot{f}^{\rm T}(x(s)) R_1 \dot{f}(x(s)){\rm d}s - \bar{h}/2\int_{t-\bar{h}}^{t-\bar{h}/2} \dot{f}^{\rm T}(x(s)) R_2 \dot{f}(x(s)){\rm d}s. \end{aligned} \tag{25}$$

By Lemma 1, the upper bound of the integral term can be estimated to be

$$\begin{aligned}\dot{V}_3 &\le \dot{f}^{\rm T}(x(t))(\bar{h}/2)^2(R_1+R_2)\dot{f}(x(t)) \\ &\quad + \begin{bmatrix} f(x(t)) \\ f(x(t-h(t))) \end{bmatrix}^{\rm T} \begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1 \end{bmatrix} \begin{bmatrix} f(x(t)) \\ f(x(t-h(t))) \end{bmatrix} \\ &\quad + \begin{bmatrix} f(x(t-h(t))) \\ f(x(t-\bar{h}/2)) \end{bmatrix}^{\rm T} \begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1 \end{bmatrix} \begin{bmatrix} f(x(t-h(t))) \\ f(x(t-\bar{h}/2)) \end{bmatrix} + \begin{bmatrix} f(x(t-\bar{h}/2)) \\ f(x(t-\bar{h})) \end{bmatrix}^{\rm T} \begin{bmatrix} -R_2 & R_2 \\ R_2 & -R_2 \end{bmatrix} \begin{bmatrix} f(x(t-\bar{h}/2)) \\ f(x(t-\bar{h})) \end{bmatrix} \\ &= \zeta^{\rm T}(t) \varSigma_3 \zeta(t). \end{aligned} \tag{26}$$

(ii) **Case 2** $\bar{h}/2 \le h(t) \le \bar{h}$

$$\begin{aligned}\dot{V}_3 &\le \dot{f}^{\rm T}(x(t))(\bar{h}/2)^2(R_1+R_2)\dot{f}(x(t)) - \bar{h}/2\int_{t-\bar{h}/2}^{t} \dot{f}^{\rm T}(x(s)) R_1 \dot{f}(x(s)){\rm d}s \\ &\quad - \bar{h}/2\int_{t-h(t)}^{t-\bar{h}/2} \dot{f}^{\rm T}(x(s)) R_2 \dot{f}(x(s)){\rm d}s - \bar{h}/2\int_{t-\bar{h}}^{t-h(t)} \dot{f}^{\rm T}(x(s)) R_2 \dot{f}(x(s)){\rm d}s \\ &\le \dot{f}^{\rm T}(x(t))(\bar{h}/2)^2(R_1+R_2)\dot{f}(x(t)) \\ &\quad + \begin{bmatrix} f(x(t)) \\ f(x(t-\bar{h}/2)) \end{bmatrix}^{\rm T} \begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1 \end{bmatrix} \begin{bmatrix} f(x(t)) \\ f(x(t-\bar{h}/2)) \end{bmatrix} \\ &\quad + \begin{bmatrix} f(x(t-\bar{h}/2)) \\ f(x(t-h(t))) \end{bmatrix}^{\rm T} \begin{bmatrix} -R_2 & R_2 \\ R_2 & -R_2 \end{bmatrix} \begin{bmatrix} f(x(t-\bar{h}/2)) \\ f(x(t-h(t))) \end{bmatrix} \\ &\quad + \begin{bmatrix} f(x(t-h(t))) \\ f(x(t-\bar{h})) \end{bmatrix}^{\rm T} \begin{bmatrix} -R_2 & R_2 \\ R_2 & -R_2 \end{bmatrix} \begin{bmatrix} f(x(t-h(t))) \\ f(x(t-\bar{h})) \end{bmatrix} \\ &= \zeta^{\rm T}(t) \varSigma_4 \zeta(t). \end{aligned} \tag{27}$$

Now, using the convex representation, we can establish the equality constraints as follows:

$$\begin{aligned} f(x(t)) &= \varDelta(x(t))x(t), \\ \dot{f}(x(t)) &= -\bar{\varDelta}Ax(t) + \bar{\varDelta}W_0 f(x(t)) \\ &\quad + \bar{\varDelta}W_1 f(x(t-h(t))). \end{aligned} \tag{28}$$

Then, there exist parameter α_k and $\bar{\alpha}_k$, where $\alpha_k > 0$, $\bar{\alpha}_k > 0$ for $k = 1, 2$, $\sum_{k=1}^{2} \alpha_k = 1$, and $\sum_{k=1}^{2} \bar{\alpha}_k = 1$ such that $\varDelta$ and $\bar{\varDelta}$ can be expressed as a convex combination of the vertex values as follows:

$$\varDelta = \sum_{k=1}^{2} \alpha_k \varDelta^k, \quad \text{and} \quad \bar{\varDelta} = \sum_{k=1}^{2} \bar{\alpha}_k \bar{\varDelta}^k, \tag{29}$$

where $\varDelta^1 = 0$, $\varDelta_2 = L$, $\bar{\varDelta}^1 = 0$, $\bar{\varDelta}^2 = \bar{L}$.
Using Eq. (28) and the augmented vector $\zeta(t)$, we obtain the equality constraint, $\varGamma\zeta(t) = 0$.

Hence, the derivative of the Lyapunov function (19) is

$$\begin{aligned} &\zeta(t)^{\mathrm{T}}(\varSigma_1 + \varSigma_2 + \varSigma_3)\zeta(t) < 0, \\ &\zeta(t)^{\mathrm{T}}(\varSigma_1 + \varSigma_2 + \varSigma_4)\zeta(t) < 0, \end{aligned} \tag{30}$$

such that

$$\varGamma\zeta(t) = 0. \tag{31}$$

By applying the Finsler lemma to inequality (30) and Eq. (31), we obtain the inequality (17). This completes our proof. ■

By the convexities of the $\varDelta(\cdot)$ and $\bar{\varDelta}(\cdot)$, Theorem 1 can be described as follows.

Corollary 1 The system (5) is asymptotically stable, if there exist positive symmetric matrices $\boldsymbol{P}^{2n\times 2n}$, $\boldsymbol{Q}_1^{4n\times 4n}$, $\boldsymbol{Q}_2^{4n\times 4n}$, $\boldsymbol{R}_1^{n\times n}$ and $\boldsymbol{R}_2^{n\times n}$ such that the following LMIs hold:

$$\begin{aligned} &N_{\varGamma}^{\mathrm{T}}(i,j)(\varSigma_1 + \varSigma_2 + \varSigma_3)N_{\varGamma}(i,j) < 0, \\ &N_{\varGamma}^{\mathrm{T}}(i,j)(\varSigma_1 + \varSigma_2 + \varSigma_4)N_{\varGamma}(i,j) < 0, \end{aligned} \tag{32}$$

where $N_{\varGamma}(i,j)$ is the orthogonal complement of $\varGamma(i,j)$ and

$$\varGamma(i,j) = \varGamma = \begin{bmatrix} -A & W_0 & 0 & W_1 & 0 & 0 & 0 & 0 & -I & 0 \\ \varDelta^i & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\bar{\varDelta}^j A & \bar{\varDelta}^j W_0 & 0 & \bar{\varDelta}^j W_1 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}, \quad i,\ j = 1,\ 2. \tag{33}$$

Remark 1 In order to reduce the conservatism in searching for the maximum allowable delay which guarantees that the delayed neural networks are asymptotically stable, Kwon[14] used a method with fractional delay interval and different free-weighting matrices in the upper bounds of integral terms. In the present paper, we choose a different Lyapunov–Krasovskii's functional with equality constraints instead of any free-weighting matrices in obtaining upper bounds. Through two numerical examples, we show that the proposed method provides the improved results compared with the recent ones in Refs. [11]–[14].

4. Numerical examples

Example 1 In order to show the effectiveness of the proposed method, consider the neural networks studied in Ref. [14], decribed with

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, \\ \boldsymbol{W}_0 &= \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}, \\ \boldsymbol{W}_1 &= \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}, \\ \boldsymbol{L} &= \mathrm{diag}\{0.1137,\ 0.1279,\ 0.7994,\ 0.2368\}. \end{aligned} \tag{34}$$

The slope bound is given by $\bar{L} = L$. The maximum allowable time-delay bound $\bar{h}$ is found by solving the LMIs in Corollary 1 for different values of $h_{\rm D}$.

Table 1 shows that the proposed method is less conservative than the existing ones in Refs. [11]–[14]. For instance, when $h_{\rm D} = 0.1$, the maximum delay bound[14] was 3.7525, while the proposed method is 3.9594. Also, when $h_{\rm D} = 0.9$, $\bar{h}$ was 2.2760,[14] while our result is 2.5447.

Table 1. Comparison among the values of maximum time delay bound $\bar{h}$, obtained at different values of $h_{\rm D}$ in Example 1.

$h_{\rm D}$	0.1	0.5	0.9
Li *et al.*[11]	3.2819	2.2261	1.6035
Chen *et al.*[12]	3.3428	2.5421	2.0867
Kwon *et al.*[14]	3.7525	2.7353	2.2760
Theorem 1	3.9594	2.9576	2.5447

Example 2 Consider neural networks (5) with the following parameters:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \boldsymbol{W}_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$
$$\boldsymbol{W}_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{L} = \mathrm{diag}\{0.4, 0.8\}. \tag{35}$$

By applying Corollary 1 to the above system when the values of $h_{\rm D}$ are 0.8 and 0.9, separately, the delay bounds are shown in Table 2 when the slope bound is given by $\bar{L} = 0.99L$. Table 2 gives the comparison results on the maximum delay bound allowed via the methods in recent studies. One can see that our result is less conservative than existing results.

Table 2. Comparison among the values of maximum time delay bound $\bar{h}$, obtained at different values of $h_{\rm D}$ in Example 2.

$h_{\rm D}$	0.8	0.9
Chen *et al.*[12]	2.3534	1.6050
Kwon *et al.*[14]	2.8854	1.9631
Theorem 1	2.9101	2.0430

5. Conclusion

In this paper, the stability problem of time-delayed neural networks with nonlinearities is considered. Based on the Finsler's lemma and Lyapunov stability theory, a new delay-dependent stability criterion is derived in terms of LMIs by using the augmented Lyapunov–Krasovskii functional which consists of states and nonlinear functions. The new criterion is less conservative than the existing ones. The effectiveness of the proposed method is illustrated by numerical examples.

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