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## ANALYTICAL SOLUTION OF A STOCHASTIC EUTROPHICATION MODEL

**Rao Qun** | Department of Hydrology and Water Resources  
**Rui Xiaofang** | Hohai University  
Nanjing, China

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*Based on the Vollenweider model, a stochastic eutrophication model was built. The changing process of total phosphorus concentration was regarded as a random process in the model. If the random properties of initial conditions are obtained, the analytical solution as well as the numerical solution can be found. Therefore the model provides the random properties of total phosphorus concentration, such as the mean and variance. The model is calibrated using the results of a monitoring program in Geheyan reservoir.*

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## INTRODUCTION

Eutrophication is a comprehensive environmental problem existing in every region in the world. In the last years, as engineers and scientists increase their understanding of eutrophication mechanisms and the technology of computers is improved, more and more eutrophication studies are focused on the mathematic modeling of eutrophication. Many deterministic eutrophication models are applied to predict and assess water quality. However, because there are many uncertainties related to water quality parameters, such as total phosphorus (TP) concentration input and sunshine, it is unreasonable to simulate the process of eutrophication with deterministic differential equations. In order to find more rational predictive results, stochastic differential equations are utilized to describe the variability of river water quality. The approach in this paper offers a relatively efficient and manageable methodology for developing stochastic versions of the Vollenweider model. By moving to a stochastic format, it may be possible to assess the eutrophication status now, and make future projections more reasonably.

## STOCHASTICEUTROPHICATIONMODEL

Vollenweider model is one of the most commonly applied deterministic nutrient response models, which can be denoted as:

$$\frac{dp}{dt} = \frac{Q_i p_i}{v} - \sigma p - \frac{Q_o p}{v} \quad (1)$$

where  $p$  = total phosphorus concentration in lake,  $v$  = the water volume in lake,  $\sigma$  = net sedimentation coefficient of TP,  $Q_i$  = inflow rate,  $Q_o$  = outflow rate,  $p_i$  = TP concentration of inflow. The paper considers the eutrophication procedure as a Markov process and simulates it with a stochastic differential equation.

Based on the Vollenweider model, which is widely used, the stochastic differential equation of total phosphorus concentration is as follows:

$$\frac{dp(t)}{dt} = \frac{Q_i p_i}{V} - \sigma p(t) - \frac{Q_o p(t)}{V} + W(t) \quad (2)$$

where  $p(t)$  is the random process of TP concentration in a lake, and  $W(t)$  is a white noise process.

The meanings of other coefficients in Equation (2) have been explained in Equation (1). Equation (2) is a typical Ito differential equation, whose solution procedure is a Markov process. In general, it is difficult to obtain the analytical solution of a stochastic differential equation. But for this special equation, not only a numerical solution, but also an analytical one can be found.

## ANALYTICALMETHOD

If  $\alpha = \sigma + \frac{Q_o}{V}$ ,  $\beta = \frac{Q_i p_i}{V}$ , then, Equation (2) can be simplified as:

$$dp(t) = (-\alpha \cdot p(t) + \beta)dt + dB(t)$$

where  $B(t)$ ,  $t \geq 0$  is Brownian movement or Wiener procedure, which has the following characteristic:

$$E[B(t)] = 0, E[B(t) - B(s)] = 0, \quad t, s \in T \quad (3)$$

$$E[B(t) - B(s)]^2 = |t - s|, \quad t, s \in T \quad (4)$$

The formal derivative of  $B(t)$  also has the same characteristic of  $B(t)$ . It can be written as follows:

$$dB(t)/dt = W(t), \quad t \geq 0$$

If  $Y(t) = p(t) \cdot e^{\alpha t}$ , according to the Ito differential principle, the following equation can be derived:

$$dY(t) = \left[ \alpha \cdot e^{\alpha t} \cdot p(t) + e^{\alpha t} (-\alpha \cdot p(t) + \beta) + \frac{1}{2} \cdot 0 \right] dt + e^{\alpha t} \cdot dB(t)$$

$$dY(t) = \beta \cdot e^{\alpha t} dt + e^{\alpha t} \cdot dB(t) \quad (5)$$

The integral of Equation (5) is as follows:

$$Y(t) - Y(t_0) = \int_{t_0}^t \beta \cdot e^{\alpha t} dt + (I) \int_{t_0}^t e^{\alpha t} dB(t)$$

where the integral (I) is the Ito integral.

$$p(t) \cdot e^{\alpha t} = p(t_0) \cdot e^{\alpha t_0} - \frac{\beta}{\alpha} e^{\alpha t_0} + \frac{\beta}{\alpha} e^{\alpha t} + \int_{t_0}^t e^{\alpha t} dB(t) \quad (6)$$

$$E[p(t)] = E[p(t_0)] \cdot e^{-\alpha(t-t_0)} + \frac{\beta}{\alpha} - \frac{\beta}{\alpha} e^{-\alpha(t-t_0)} + E\left[ \int_{t_0}^t e^{-\alpha(t-\tau)} dB(\tau) \right] \quad (7)$$

According to the definition of the Ito integral, the following formula can be obtained.

$$E[p(t)] = e^{-\alpha(t-t_0)} \cdot E[p(t_0)] + \frac{\beta}{\alpha} e^{-\alpha t_0} + E\left[ \lim_{\Delta n \rightarrow 0} \sum_{k=0}^{n-1} e^{-\alpha(t-t_k)} \cdot (B(t_{k+1}) - B(t_k)) \right]$$

(Note: the meaning of *l.i.m* is different from the ordinal limit symbol, which is used to represent limit in the mean).

$$E[p(t)] = e^{-\alpha(t-t_0)} \cdot E[p(t_0)] + \frac{\beta}{\alpha} e^{-\alpha t_0} + \lim_{\Delta n \rightarrow 0} E\left[ \sum_{k=0}^{n-1} e^{-\alpha(t-t_k)} \cdot (B(t_{k+1}) - B(t_k)) \right]$$

$$E[p(t)] = e^{-\alpha(t-t_0)} \cdot E[p(t_0)] + \frac{\beta}{\alpha} e^{-\alpha t_0} + \lim_{\Delta n \rightarrow 0} \sum_{k=0}^{n-1} e^{-\alpha(t-t_k)} E[B(t_{k+1}) - B(t_k)]$$

Under the assumption of Equations (3) and (4), it can be determined that:

$$\lim_{\Delta n \rightarrow 0} \sum_{k=0}^{n-1} e^{-\alpha(t-t_k)} \cdot E[B(t_{k+1}) - B(t_k)] = 0$$

$$E[p(t)] = e^{-\alpha(t-t_0)} \cdot E[p(t_0)] + \frac{\beta}{\alpha} - \frac{\beta}{\alpha} e^{-\alpha(t-t_0)} \tag{8}$$

If  $p(t)=p$ ,  $p(t_0)=p_0$ , then  $D(p) = E[p - E(p)]^2$ .

According to Equations (5) and (7):

$$D(p) = E\left[ p_0 \cdot e^{-\alpha(t-t_0)} + \int_{t_0}^t e^{-\alpha(t-\tau)} dB(\tau) - E(p_0) \cdot e^{-\alpha(t-t_0)} \right]^2$$

$$D(p) = e^{-2\alpha(t-t_0)} \cdot D(p_0) + E\left[ \int_{t_0}^t e^{-\alpha(t-\tau)} dB(\tau) \right]^2$$

$$D(p) = e^{-2\alpha(t-t_0)} \cdot D(p_0) + e^{-2\alpha t} \cdot E\left[ \int_{t_0}^t e^{\alpha\tau} dB(\tau) \right]^2$$

If  $B(t_i) = B_i, \Delta B_i = B(t_{i+1}) - B(t_i), \Delta t_i = t_{i+1} - t_i$ ,

$$\text{Then } \left[ \int_{t_0}^t e^{\alpha\tau} dB(\tau) \right]^2 = \lim_{\Delta n \rightarrow 0} \sum_{i=0}^{n-1} e^{\alpha\tau} \cdot (B_{i+1} - B_i) \cdot \lim_{\Delta n \rightarrow 0} \sum_{i=0}^{n-1} e^{\alpha\tau} \cdot (B_{i+1} - B_i)$$

$$= \lim_{\Delta n \rightarrow 0} \left[ \sum_{i=0}^{n-1} e^{\alpha\tau} \cdot (B_{i+1} - B_i) \right]^2$$

$$= \lim_{\Delta n \rightarrow 0} \left[ \sum_{i=0}^{n-1} e^{\alpha\tau_i} \cdot \Delta B_i^2 + \sum_{i \neq j} e^{\alpha\tau_i} \cdot e^{\alpha\tau_j} \cdot \Delta B_i \cdot \Delta B_j \right]$$

$$E\left[ \int_{t_0}^t e^{\alpha\tau} dB(\tau) \right]^2 = \lim_{\Delta n \rightarrow 0} E\left( \sum_{i=0}^{n-1} e^{2\alpha\tau_i} \cdot \Delta B_i^2 \right) + \lim_{\Delta n \rightarrow 0} E\left( \sum_{i \neq j} e^{\alpha\tau_i} \cdot e^{\alpha\tau_j} \cdot \Delta B_i \cdot \Delta B_j \right)$$

$$E\left( e^{\alpha\tau_i} \cdot e^{\alpha\tau_j} \cdot \Delta B_i \cdot \Delta B_j \right) = e^{\alpha\tau_i} \cdot e^{\alpha\tau_j} \cdot E(\Delta B_i) \cdot E(\Delta B_j) = 0$$

$$E\left[ \int_{t_0}^t e^{\alpha\tau} dB(\tau) \right]^2 = \lim_{\Delta n \rightarrow 0} E\left( \sum_{i=0}^{n-1} e^{2\alpha\tau_i} \cdot \Delta B_i^2 \right)$$

so:  $E\left[ \int_{t_0}^t e^{\alpha\tau} dB(\tau) \right]^2 = \lim_{\Delta n \rightarrow 0} \sum_{i=0}^{n-1} e^{2\alpha\tau_i} \cdot \Delta t_i = \int_{t_0}^t e^{2\alpha\tau} d\tau$

$$D(p) = e^{-2\alpha(t-t_0)} \cdot D(p_0) - \frac{e^{-2\alpha(t-t_0)} - 1}{2\alpha} \tag{9}$$

### NUMERICAL METHOD

First, the model is derived in a discrete form as the following:

$$p(t + \Delta t) = p(t) + (-\alpha p(t) + \beta)\Delta t + w(t)\Delta t$$

$$E[p(t)] = E(p_t), E[p(t + \Delta t)] = E(p_{t+\Delta t})$$

$$E(p_{t+\Delta t}) = E[p(t) + (-\alpha p(t) + \beta)\Delta t + w(t)\Delta t]$$

$$E(p_{t+\Delta t}) = E(p_t)(1 - \alpha\Delta t) + \beta\Delta t + E[w(t)]\Delta t \tag{10}$$

Because  $D(p_{t+\Delta t}) = E[(p(t + \Delta t) - E(p_{t+\Delta t}))^2]$

$$= E[(p(t) + (-\alpha p(t) + \beta)\Delta t - E[p(t) + (-\alpha p(t) + \beta)\Delta t])^2]$$

$$= E[(p(t)(1 - \alpha \cdot \Delta t) - E(p_t)(1 - \alpha \cdot \Delta t) + (w(t) - E(w(t)))\Delta t]^2$$

$$= (1 - \alpha \cdot \Delta t)^2 E[p(t) - E(p_t)]^2 + E^2[w(t) - E(w(t))]\Delta t^2$$

$$+ 2(1 - \alpha \cdot \Delta t)E[(p(t) - E(p_t))(w(t) - E(w(t)))\Delta t]$$

then  $D(p_{t+\Delta t}) = (1 - \alpha \cdot \Delta t)^2 D(p(t)) + D(w(t))\Delta t^2 + 2\Delta t(1 - \alpha \cdot \Delta t)D_{p,w}$

If it is assumed that  $w(t)$  is independent of  $p(t)$ , then the above equation can be summarized as:

$$D(p_{t+\Delta t}) = (1 - \alpha \cdot \Delta t)^2 D(p(t)) + D(w(t))\Delta t^2 \tag{11}$$

If the mean and the variance of the initial TP concentration are determined, Equations (10) and (11) can be solved.

### APPLICATION OF THE MODEL

The Geheyan reservoir, a typical channel reservoir, is located in northwest Hubei province. Total phosphorus data collected on the reservoir was used to calibrate the stochastic model. The data consisted of nine TP observations over a period of 16 days. The model was applied to the Geheyan reservoir using a daily time step over the period of the study.

The initial TP concentration is a discrete stochastic variable. The initial mean can be determined by the average of the nine individual observations collected in the reservoir.

$$E(p_0) = \sum_{i=1}^n \frac{p_i}{n} \tag{12}$$

and the initial variance of is determined as

$$D(p_0) = \frac{\sum_{i=1}^n (p_i - E(p_0))^2}{n} \tag{13}$$

where  $p_i$  is the  $i$ th observed TP concentration on any sampling day,  $n$  is the number of observations on any sampling day. The calibration simulation was initialized with a mean TP of 0.037mg/l and a variance of 0.0014(mg/l)<sup>2</sup>.

### RESULTS

Table 1 shows that there is a little difference between the values computed by two methods and the observed values.

Table 1. The Computed Results

Time (d)	Observed value (mg/l)	Analytical method		Numeric value method	
		Mean (mg/l)	Variance (mg/l) <sup>2</sup>	Mean (mg/l)	Variance (mg/l) <sup>2</sup>
1	0.047	0.042	0.0013	0.041	0.0023
2	0.046	0.039	0.0015	0.037	0.0012
3	0.061	0.051	0.0024	0.057	0.0014
4	0.032	0.037	0.0026	0.040	0.0016
5	0.063	0.049	0.0024	0.047	0.0043
6	0.042	0.048	0.0011	0.052	0.0021
7	0.035	0.037	0.0019	0.039	0.0030
8	0.032	0.046	0.0022	0.032	0.0012
9	0.085	0.035	0.0089	0.031	0.0091
10	0.048	0.041	0.0011	0.048	0.0011
11	0.029	0.035	0.0026	0.035	0.0046
12	0.071	0.066	0.0009	0.061	0.0023
13	0.073	0.061	0.0024	0.069	0.0021
14	0.069	0.058	0.0015	0.057	0.0025
15	0.051	0.056	0.0008	0.063	0.0014
16	0.037	0.041	0.0019	0.040	0.0009

### CONCLUSION

Using the theory of stochastic differential equations to study the uncertain change of water quality is one of the important area of stochastic eutrophication modeling. However, as it is very difficult to solve this kind of model, it is often solved using a numerical method. This paper presents the stochastic differential equation model of the Vollenweider model for eutrophication, which can be solved with an analytical method, as well as a numerical method. It provides a direct, simple, efficient computing tool for analyzing the uncertainties associated with eutrophication.

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#### ADDRESS FOR CORRESPONDENCE

Rao Qun  
 Department of Hydrology and Water Resources  
 Hohai University  
 Nanjing 210098  
 China

Email: raoqun@263.net

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