Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 759–766

Quantum Groups and Cabibbo Mixing

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Treating the issue of hadron masses and mass relations by the use of quantum groups $U_q(su_n)$ taken as hadron flavor symmetries suggests, at least in the case of baryons, a direct connection of the deformation parameter q with the Cabibbo angle. We discuss possible manifestations of the Cabibbo mixing implied by such connection, including unusual ones.

1 Introduction

The standard model of fundamental particles and forces incorporates an important concept of quark mixing [1] usually described by the CKM matrix, which in the Wolfenstein's form looks as

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho-{\rm i}\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-{\rm i}\eta) & -A\lambda^2 & 1 \end{pmatrix}.$$

When keeping the first two quark families, only the parameter $\lambda \approx 0.22$ (the sine of Cabibbo angle $\theta_{\rm C}$) persists. Cabibbo mixing [1] is known to play basic role in describing weak decays of mesons and baryons by means of flavor changing (e.g., strangeness changing) quark currents. Our goal is to discuss some other, unusual implications of the Cabibbo mixing which are deduced on the base of adopting instead of the flavor groups $SU(n)$ their quantum counterparts $U_q(su(n))$. Note that the idea of using [2] in place of $SU(n)$ the quantum groups (or q-deformed algebras) $U_q(su_n)$ and their representations [3] in order to treat hadronic flavor symmetries is natural, easily justifiable and very fruitful, leading to many interesting implications [2, 4]. The most important one is the possibility to link the q-parameter with the angle $\theta_{\rm C}$, and this enables to find novel connections for the concept of Cabibbo mixing. Say, the earliest modified (improved) version of the Gell-Mann–Okubo (GMO) mass relation for the $SU(3)$ octet baryons $\frac{1}{2}^+$ derived in a particular version [5] of 'diquark-quark' model (the LTK model), involves certain parameters characterizing diquark and separate quark. The alternative improved version of GMO for octet baryons is the extremely precise mass sum rule (MSR) obtained with $U_q(su_n)$. It exploits fixed value of the q-parameter linked in a simple way to $\theta_{\rm C}$, and this fact allows to find direct connection, through the fixed q, of the parameters of LTK diquark-quark model to the angle $\theta_{\rm C}$. Besides this, we discuss the connections of the $\theta_{\rm C}$ with such concepts as anyonic statistics parameter from the anyonic picture, as intercepts of the two- and three-pion (-kaon) correlation functions drawn in recent experiments on relativistic heavy ion collisions.

2 Baryon mass sum rules from q-deformation

2.1 Mass relations for octet and decuplet baryons

Consider first $\frac{1}{2}^+$ baryons from $SU(3)$ octet. Using $U_q(su_n)$, $n = 2, 3, 4$, and $U_q(u_{4,1})$ (the latter playing the role of dynamical q-algebra), the q-deformed mass relation for octet baryons

$$[2]M_N + \frac{[2]M_\Xi}{[2]-1} = [3]M_\Lambda + \Big(\frac{[2]^2}{[2]-1} - [3]\Big)M_\Sigma + \frac{A_q}{B_q}\,(M_\Xi + [2]M_N - [2]M_\Sigma - M_\Lambda) \qquad (1)$$

was obtained [6]. The pair of polynomials A_q, B_q (of the q-deuce $[2]_q \equiv q + q^{-1}$), with nonoverlapping sets of zeros, results from calculation in a particular dynamical representation (irrep) of $U_q(u_{4,1})$; different irreps lead to differing pairs A_q, B_q in (1). Each A_q possesses the mandatory factor $([2]_q - 2)$, i.e. 'classical' zero $q = 1$, and nontrivial zeros. As particular cases, the q-analog (1) yields the familiar [7] Gell-Mann–Okubo (GMO) mass relation $M_N + M_\Xi = \frac{3}{2}M_\Lambda + \frac{1}{2}M_\Sigma$ in the 'classical' limit $q = 1$, and two new mass sum rules of improved accuracy [4, 8]

$$M_N + \frac{1+\sqrt{3}}{2}M_\Xi = \frac{2M_\Lambda}{\sqrt{3}} + \frac{9-\sqrt{3}}{6}M_\Sigma \qquad (0.22\ \%), \tag{2}$$

$$M_N + \frac{M_\Xi}{[2]_{q_7}-1} = \frac{M_\Lambda}{[2]_{q_7}-1} + M_\Sigma \qquad (0.07\ \%) \tag{3}$$

which correspond to A_q and $\tilde{A}_q$ with respective zeros $q_6 = e^{i\pi/6}$ and $q_7 = e^{i\pi/7}$. Among the admissible irreps of $U_q(u_{4,1})$ or its 'compact' counterpart $U_q(u_5)$ there exist infinite series of irreps producing infinite set of MSRs numbered by integer m $(6 \le m < \infty)$, each given by (1) with q_m put for q, where $q_m = e^{i\pi/m}$ guarantees vanishing of $\frac{A_q}{B_q}$. Each MSR in this set *agrees better with data* than the standard GMO one. Thus, a 'discrete choice' (instead of usual fitting) becomes possible; the q-polynomial A_q due to zero q_m serves as *defining* polynomial for the corresponding MSR. It is the value $q_7 = e^{i\pi/7}$ and the MSR (3) that provides the best choice.

For decuplet baryons $\frac{3}{2}^+$, 1st order $SU(3)$ breaking yields well-known [7] equal spacing rule (ESR) for isoplets from **10**-plet. However, $M_{\Sigma^*} - M_\Delta$, $M_{\Xi^*} - M_{\Sigma^*}$ and $M_\Omega - M_{\Xi^*}$ significantly deviate from ESR: 152.6 MeV $\leftrightarrow$ 148.8 MeV $\leftrightarrow$ 139.0 MeV. The other known relation [9, 5]

$$(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*})/2 = M_{\Xi^*} - M_{\Sigma^*} \tag{4}$$

accounts 1st, 2nd orders in $SU(3)$-breaking and holds only slightly better than the ESR.

On the contrary, the use of q-algebras $U_q(su_n)$ gives nice improvement. Evaluation of decuplet masses with particular irreps of the dynamical $U_q(u_{4,1})$ yields the q-deformed mass relation [10]

$$(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*})/(2\cos\theta_{\mathbf{10}}) = M_{\Xi^*} - M_{\Sigma^*}, \qquad q = \exp(i\theta_{\mathbf{10}}). \tag{5}$$

This mass relation is *universal* – it results from any admissible irrep (containing $U_q(su_3)$-decuplet embedded in *20*-plet of $U_q(su_4)$) of the dynamical $U_q(u_{4,1})$. With empirical masses, equation (5) holds remarkably for $\theta_{\mathbf{10}} \simeq \frac{\pi}{14}$. It is argued that $\theta_{\mathbf{10}} = \theta_C$, see [4] and also the next subsection.

2.2 Linking the *q*-parameter to the Cabibbo angle

We compare the generalization [11] $f_\pi^2 m_\pi^2 + 3f_\eta^2 m_\eta^2 = 4f_K^2 m_K^2$ of GMO-formula $m_\pi^2 + 3m_\eta^2 = 4m_K^2$ for pseudoscalar (PS) mesons, with $f_\pi^2 + 3f_\eta^2 = 4f_K^2$ imposed, and our q-deformed analog

$$m_\pi^2 + [3]_q^{-1}(2\ [2]_q - [3]_q)m_{\eta_8}^2 = 2\ [2]_q(2\ [2]_q - [3]_q)^{-1}m_K^2, \qquad [3]_q = [2]_q^2 - 1, \tag{6}$$

(found in [2]) of PS-mesonic GMO, with duly fixed $q = q_{\rm PS}$ and *physical* η-meson put in place of η_8 (thus there is no need for explicit mixing). We find, using $\xi_{\pi,K} \equiv (4f_K^2/f_\pi^2)^{-1}$, that

$$f_K^2/f_\pi^2 \ \leftrightarrow\ \frac{1}{2}[2]/(2[2]-[3]), \qquad [2]_\pm = 1-\xi_{\pi,K} \pm ((1-\xi_{\pi,K})^2+1)^{1/2}.$$

Since the ratio f_K/f_π is expressible through the Cabibbo angle (see e.g., [12]), we infer: the deformation parameter q_{PS} is directly connected with the Cabibbo angle.

In another way of reasoning, we use the result of [13] where the Lagrangian for quantum-group valued gauge field analog of the Weinberg–Salam (WS) model was constructed and the relation

$\tan\theta = h(q) \equiv (1-q^2)/(1+q^2)$ found. It provides proper mixing of the $U(1)$-component B_μ and the non-Abelian component A^3_μ. Physical photon $\tilde{A}_\mu$ and Z-boson of WS model appear through the Weinberg angle $\theta_{\rm W} = \theta = \arctan h(q)$. At $\theta = 0$ the potentials B_μ and A^3_μ get unmixed, but the q-deformation with $\theta \neq 0$ provides mixing inherent in the WS model. So, the mixing of (electro)weak gauge fields is adequately modelled by the q-deformation. Due to the latter, the weak angle and the q-parameter are explicitly linked. The relation [14] $\theta_{\rm W} = 2(\theta_{12} + \theta_{23} + \theta_{13})$, on the other hand, connects $\theta_{\rm W}$ with the Cabibbo angle $\theta_{12} \equiv \theta_{\rm C}$ (we neglect the 3rd family's θ_{13}, θ_{23}). Thus, the apparently different mixing angles, in the *bosonic* (interaction) and in the *fermionic* (matter) sectors of the electroweak model, are related.

We conclude that *the Cabibbo angle is linked with q-parameter* of a quantum-group (or q-algebra) based symmetry structure applied in the fermion sector. The explicit connection is remarkably simple: $\theta_{\mathbf{10}} = \theta_{\rm C}$, $\theta_{\mathbf{8}} = 2\,\theta_{\rm C}$. With $\theta_{\mathbf{8}} = \frac{\pi}{7}$ this suggests the exact value $\frac{\pi}{14}$ for $\theta_{\rm C}$.

2.3 Nonpolynomiality in $SU(3)$-breaking and Michel's statement

The universality of q-analog (5) concerns all admissible irreps of the '*compact*' dynamical $U_q(su_5)$, too. Say, calculation in the dynamical irrep $\{4000\}$ of $U_q(su_5)$ yields $M_\Delta = M_{\mathbf{10}} + \beta$, $M_{\Sigma^*} = M_{\mathbf{10}} + [2]\beta + \alpha$, $M_{\Xi^*} = M_{\mathbf{10}} + [3]\beta + [2]\alpha$, $M_\Omega = M_{\mathbf{10}} + [4]\beta + [3]\alpha$, from which (5) stems. With hypercharge Y, all the decuplet masses $M_{D_i} \equiv M(Y(D_i))$ are comprised by single formula

$$M_{D_i} = M_{\mathbf{10}} + \alpha[1-Y(D_i)]_q + \beta[2-Y(D_i)]_q \tag{7}$$

of explicit Y-dependence. The limit $q \to 1$ reduces it to $M_{D_i} = \tilde{M}_{\mathbf{10}} + aY(D_i)$ with $a = -\alpha - \beta$, $\tilde{M}_{\mathbf{10}} = M_{\mathbf{10}} + \alpha + 2\beta$, i.e., to linear dependence on hypercharge (or strangeness $S = Y - 1$).

Formula (7) involves *highly nonlinear dependence* of mass on hypercharge: for decuplet, Y alone causes $SU(3)$-breaking. Since for any N its q-number is $[N]_q = (q^N - q^{-N})/(q - q^{-1}) = q^{N-1} + q^{N-3} + \cdots + q^{-N+3} + q^{-N+1}$ (N terms), this shows exponential Y-dependence of masses. Such high nonlinearity makes (5) and (7) crucially different from the result (4) of traditional treatment accounting linear and quadratic effects in Y.

For octet baryon masses, *nonpolynomiality* in $SU(3)$-breaking effectively accounted [8] by the model is embodied in the expressions for isoplet masses, with explicit dependence on Y as well as isospin I, through $I(I+1)$. Matrix elements contributing to octet baryon masses contain e.g., the terms $([Y/2]_q[Y/2+1]_q - [I]_q[I+1]_q)$ or $([Y/2-1]_q[Y/2-2]_q - [I]_q[I+1]_q)$, with multipliers depending on irrep labels m_{15}, m_{55}, that show explicit dependence on hypercharge and on the q-analog $[I]_q[I+1]_q$ of $SU(2)$ Casimir. The q-bracket $[n]_q$ means $[n]_q = \frac{\sin(nh)}{\sin(h)}$, $q = \exp(\mathrm{i}h)$, so we see that octet baryon masses depend on hypercharge Y and isospin I (hence, on $SU(3)$-breaking effects) also in highly nonlinear – *nonpolynomial* – fashion.

We note finally that the conclusion made in the preceding subsection about the ability to find, due to the link $q \leftrightarrow \theta_C$, the exact value of Cabibbo angle is in accord with Michel's statement [15] that only account of higher-order breaking effects enables gaining of such result.

3 Cabibbo angle and the anyonic statistics parameter

From N sorts of lattice fermions $c_i(\boldsymbol{x})$, $c_i^\dagger(\boldsymbol{x})$, $i = 1, \ldots, N$, with usual lattice anticommutation relations (ACRs), using the lattice angle functions [16] $\theta_\gamma(\boldsymbol{x}, \boldsymbol{y})$ and $\theta_\delta(\boldsymbol{x}, \boldsymbol{y})$ for two opposite (γ- and δ-) types of cuts, the related ordering on the lattice ($\boldsymbol{x} > \boldsymbol{y}$ or $\boldsymbol{y} > \boldsymbol{x}$), and the two types of statistical operators $K_i(\boldsymbol{x}_\gamma)$ and $K_i(\boldsymbol{x}_\delta)$,

$$K_j(\boldsymbol{x}_\gamma) = \exp\big(\mathrm{i}\nu \sum_{\boldsymbol{y}\neq\boldsymbol{x}} \theta_\gamma(\boldsymbol{x}, \boldsymbol{y}) c_j^\dagger(\boldsymbol{y}) c_j(\boldsymbol{y})\big), \quad K_j(\boldsymbol{x}_\delta) = \exp\big(\mathrm{i}\nu \sum_{\boldsymbol{y}\neq\boldsymbol{x}} \theta_\delta(\boldsymbol{x}, \boldsymbol{y}) c_j^\dagger(\boldsymbol{y}) c_j(\boldsymbol{y})\big), \tag{8}$$

one defines [16] the anyonic oscillators $a_i(\boldsymbol{x}_\gamma) = K_i(\boldsymbol{x}_\gamma)c_i(\boldsymbol{x})$ and $a_i(\boldsymbol{x}_\delta) = K_i(\boldsymbol{x}_\delta)c_i(\boldsymbol{x})$ involving *the anyonic statistics parameter* ν. From this definition and ACR's for lattice fermions, the relations of permutation for anyonic oscillators then follow. Some of them, e.g.

$$a_i(\boldsymbol{x}_\gamma)a_i(\boldsymbol{y}_\gamma) + q^{-\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})}a_i(\boldsymbol{y}_\gamma)a_i(\boldsymbol{x}_\gamma) = 0, \qquad a_i(\boldsymbol{x}_\gamma)a_i^\dagger(\boldsymbol{y}_\gamma) + q^{\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})}a_i^\dagger(\boldsymbol{y}_\gamma)a_i(\boldsymbol{x}_\gamma) = 0,$$

involve the deformation parameter q connected with the statistics parameter ν of (8) as $q = \exp(\mathrm{i}\pi\nu)$). The generating elements $A_{j,j+1}$, $A_{j+1,j}$ and H_j of the quantum algebra $U_q(su_N)$ are realized bilinearly through anyonic oscillators $a_i(\boldsymbol{x}_\gamma)$, $a_i^\dagger(\boldsymbol{y}_\gamma)$ and were shown to satisfy [16] the defining relations [3] of the quantum algebra $U_q(su_N)$. Dual realization in terms of $a_i(\boldsymbol{x}_\delta)$, $a_i^\dagger(\boldsymbol{y}_\delta)$ is also valid. Then, in anyonic realization of $U_q(su_N)$, both hadron mass operator $\hat{M}$ and basis vectors for hadronic irreps are explicitly constructed [17]. Say, for the irrep $\{4000\}$ of 'dynamical' $U_q(su_5)$, in accordance with the chain of q-algebras $U_q(su_3) \subset U_q(su_4) \subset U_q(su_5)$ and respective chain of irreps $[30] \subset [300] \subset \{4000\}$, all basis state vectors $|n_1n_2n_3n_4\rangle \equiv a_{n_1}^\dagger(\boldsymbol{x}_{1\gamma})a_{n_2}^\dagger(\boldsymbol{x}_{2\gamma})a_{n_3}^\dagger(\boldsymbol{x}_{3\gamma})a_{n_4}^\dagger(\boldsymbol{x}_{4\gamma})|0\rangle$ of baryons $\frac{3}{2}^+$ are constructed by acting with lowering generators upon the highest weight vector. E.g., for isoquartet baryon $|\Delta^{++}\rangle$ we get $|\Delta^{++}\rangle = [4]^{-1/2}(|5111\rangle + q^{-1}|1511\rangle + q^{-2}|1151\rangle + q^{-3}|1115\rangle)$, and similarly for $|\Sigma^*\rangle$, $|\Xi^*\rangle$, $|\Omega^-\rangle$. The dual basis vectors $\widetilde{|\Delta^{++}\rangle}$ etc., are also given. Masses M_{D_i} of baryons D_i in the dynamical $U_q(su_5)$-irrep $\{4000\}$ are calculated with mass operator $\hat{M}$ as $M_{D_i} = \widetilde{\langle D_i|}\hat{M}|D_i\rangle$ to yield: $M_\Delta = M_{\mathbf{10}}+\beta$, $M_{\Sigma^*} = M_{\mathbf{10}} + [2]_q\alpha + [2]_q\beta$, $M_{\Xi^*} = M_{\mathbf{10}} + [2]_q^2\alpha + [3]_q\beta$, and $M_{\Omega^*} = M_{\mathbf{10}} + [2]_q[3]_q\alpha + [4]_q\beta$, from which the relation (5) follows. This proves applicability [17] of quantum algebras and their irreps for deriving hadron mass relations by *using their anyonic realization*.

Since $\theta_{\mathbf{10}} = \theta_C$, *we have the connection: Cabibbo angle* $\leftrightarrow$ *anyonic statistics parameter* ν.

4 Diquark-quark model parameters and the Cabibbo angle

The LTK diquark-quark model [5] uses, besides the $SU(3)$ invariant masses m_t, m_s and m_q of the $SU(3)$ diquark triplet, diquark sextet, and 3rd quark triplet (so the subscripts "t", "s", "q"), also the mass parameters δ_t, δ_s and δ_q reflecting $SU(3)$ violation in the respective multiplets.

The improved form of GMO obtained in the LTK model is

$$\tfrac{3}{2}m_\Lambda + \tfrac{1}{2}m_\Sigma - m_N - m_\Xi = \mathcal{C}_{\mathrm{LTK}} \equiv \mu_s\big(3\xi_{ts}^2 + 18\xi_{ts} - 13\big), \tag{9}$$

where $\xi_{ts} = \frac{\delta_t-\delta_q}{\delta_s-\delta_q}$, $\mu_s = V_0\gamma_s^2$, $\gamma_s = \frac{(\delta_s-\delta_q)}{6V_0}$. The μ_s must be positive being also involved [5] in the decuplet mass combination: $8\mu_s = 2m_{\Xi^*} - m_\Omega - m_{\Sigma^*} > 0$. As result, the LTK model gave the value $\xi_{ts}^{\mathrm{LTK}} = -3$, which implied: δ_t must be greater (smaller) than δ_q with the parameter δ_s being smaller (greater) than δ_q. However, the value $\xi_{ts}^{\mathrm{LTK}} = -3$ contradicts data as it renders the r.h.s. of (9) to be negative thus correcting the GMO *in wrong direction*.

On the other hand, our q-MR (3) for which $q_7 = \exp(\mathrm{i}\frac{\pi}{7})$, $\frac{\pi}{7}$=$2\theta_{\mathrm{C}}$, can be rewritten as

$$\frac{3}{2}m_\Lambda + \frac{1}{2}m_\Sigma - m_N - m_\Xi = \mathcal{C}_{q_7} \equiv \frac{2-[2]_7}{[2]_7-1}(m_\Xi - m_\Lambda) - \frac{1}{2}(m_\Sigma - m_\Lambda). \tag{10}$$

Comparison of the two improved versions (9), (10) leads to the relation [18]

$$([2]_7 - 1)^{-1} = \frac{9\tilde{\xi}_{ts}^2 - 6\tilde{\xi}_{ts} + 5}{4(\tilde{\xi}_{ts}^2 - 6\tilde{\xi}_{ts} + 5)}, \qquad \tilde{\xi}_{ts} = \frac{\tilde{\delta}_t - \tilde{\delta}_q}{\tilde{\delta}_s - \tilde{\delta}_q}, \tag{11}$$

which connects the value $q_7 = \exp\left(i\frac{\pi}{7}\right)$ of q-parameter in the q-GMO (10) with the ratio ξ_{ts} of the LTK model. Remark that the values of $\tilde{\delta}_t$, $\tilde{\delta}_s$ and $\tilde{\delta}_q$ in this relation are *understood as* *the optimized ones* reflecting, through the found connection, all-order account by (10) of $SU(3)$ symmetry breaking in octet baryon masses – just this fact is denoted by tildas over δ_t, δ_s, δ_q.

Since $[2]_7 = 2\cos\frac{\pi}{7} \approx 1.80194$, solving the relation (11) yields the values $\xi_{ts}^{(+)} \approx 0.741$ and $\xi_{ts}^{(-)} \approx -6.705$. By the very derivation, $\xi_{ts}^{(\pm)}$ should guarantee the validity of sum rule (9) to within 0.07 %. The both values differ essentially from the value $\xi_{ts}^{\rm LTK} = -3$ of LTK model, *providing proper positive correction to* GMO, see the above comment about $\xi_{ts}^{\rm LTK}$. Moreover, being positive, our $\xi_{ts}^{(+)}$ is not only well acceptable phenomenologically but also reflects *qualitatively* different, than $\xi_{ts}^{(-)}$ and $\xi_{ts}^{\rm LTK}$, physical situation. Namely, the mass parameter $\tilde{\delta}_t$ is greater (smaller) than $\tilde{\delta}_q$ when the mass parameter $\tilde{\delta}_s$ is greater (smaller) than $\tilde{\delta}_q$, since

$$\tilde{\delta}_t - \tilde{\delta}_q = 0.741(\tilde{\delta}_s - \tilde{\delta}_q) \qquad \text{i.e.,} \qquad \tilde{\delta}_t - \tilde{\delta}_s = 0.259(\tilde{\delta}_q - \tilde{\delta}_s).$$

Thus, the inequalities for the (optimized) parameters of LTK model should be

$$\text{either} \qquad \tilde{\delta}_q > \tilde{\delta}_t > \tilde{\delta}_s \qquad \text{or} \qquad \tilde{\delta}_s > \tilde{\delta}_t > \tilde{\delta}_q. \tag{12}$$

Since $\theta_{\mathbf{8}} = \frac{\pi}{7} = 2\ \theta_{\rm C}$, or $\theta_{\rm C} = \frac{\pi}{14}$, we finally arrive [18] at the formula yielding *direct link* of the Cabibbo angle to the (optimized) parameters of the LTK diquark-quark model:

$$\cos 2\theta_{\rm C} = \frac{13\tilde{\xi}_{ts}^2 - 30\tilde{\xi}_{ts} + 25}{2(9\tilde{\xi}_{ts}^2 - 6\tilde{\xi}_{ts} + 5)}, \qquad \tilde{\xi}_{ts} \equiv \frac{\tilde{\delta}_t - \tilde{\delta}_q}{\tilde{\delta}_s - \tilde{\delta}_q}. \tag{13}$$

This connection of the Cabibbo angle with the parameters of diquark-quark model, clearly, could not be found without equation (3) (or equation (10)) and the relation (11) deduced on its base.

5 Cabibbo mixing in multiparticle correlations?

The model of ideal gas of q-bosons based on the algebra of q-deformed oscillators of Arik–Coon (AC) or Biedenharn–Macfarlane (BM) type [19], may be used to describe [20] unusual behavior of 2-particle correlations of identical pions or kaons produced in relativistic heavy ion collisions. The approach yields explicit expressions and clear predictions [21] for the intercept λ (dependent on the temperature, particle mass, pair mean momentum, *and the deformation parameter* q).

Physical observables are evaluated as thermal averages $\langle A\rangle = {\rm Sp}(A\rho)/{\rm Sp}(\rho)$, $\rho = e^{-\beta H}$, where the Hamiltonian is $H = \sum \omega_i N_i$ and $\beta = 1/T$. With $b_i^\dagger b_i = [N_i]_q$ and $[2]_q = 2\cos\theta$, the q-deformed distribution function for the BM-type q-bosons results (see e.g. [20]) as

$$\langle b_i^\dagger b_i\rangle = (e^{\beta\omega_i} - 1)/(e^{2\beta\omega_i} - 2\cos\theta\ e^{\beta\omega_i} + 1). \tag{14}$$

At θ=0 (or q=1), it yields Bose–Einstein (BE) distribution, as q=1 recovers usual bosonic commutation relations. Deviation of q-distribution (14) from the quantum BE distribution is seen to tend towards the Boltzmann one (reduced quantum statistical effects). For kaons, whose mass $m_K > 3m_\pi$, analogous curve gets closer (than pion's one) to the BE distribution. Note that for AC-type q-bosons, the q-distribution looks more simple: $\langle b_i^\dagger b_i\rangle = (e^{\beta\omega_i} - q)^{-1}$.

5.1 Intercept λ of two-pion (two-kaon) correlations and Cabibbo angle

To obtain explicitly the intercept λ of two-particle correlations one calculates the two-particle distribution $\langle b^\dagger b^\dagger bb\rangle$ and normalizes it by $\langle b^\dagger b\rangle^2$. The result, see [20], for AC-type q-bosons reads $\lambda = q - \frac{q(1-q^2)}{e^{\omega/T} - q^2}$, $-1 \le q \le 1$, and for BM-type q-bosons, with $\mathcal{F}(\beta\omega) \equiv \cosh(\beta\omega)$, it is

$$\lambda + 1 \equiv \langle b^\dagger b^\dagger bb\rangle/(\langle b^\dagger b\rangle)^2 = \frac{2\cos\theta(\mathcal{F}(\beta\omega) - \cos\theta)^2}{(\mathcal{F}(\beta\omega) - 1)(\mathcal{F}(\beta\omega) - 2\cos^2\theta + 1)}. \tag{15}$$

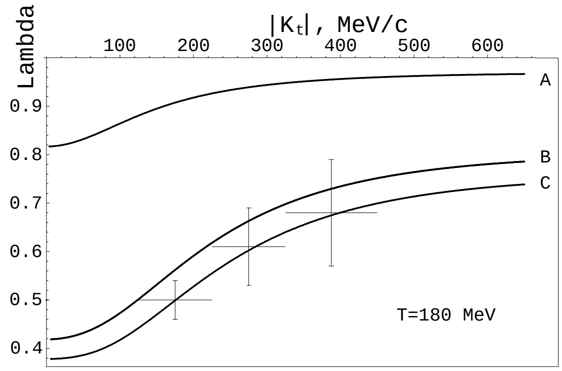

Figure 1. The intercept λ of two-pion correlation function versus transverse momentum $|\boldsymbol{K}_t|$ at fixed $T = 180$ MeV and fixed $q = \exp(i\theta)$: A) $\theta = 10°$, B) $\theta = 25.7°$ (i.e., $2\theta_{\rm C}$), C) $\theta = 28.5°$.

With $\omega = (m^2 + \boldsymbol{K}^2)^{1/2}$, asymptotically at large mean momentum of pion (kaon) pair and fixed temperature, the intercept λ tends to the constant $\lambda^{\rm AC}_{\rm asymp} = q$ for the AC-type q-bosons (q real), and for the BM-type q-bosons (q is a phase factor) to the constant

$$\lambda^{\rm BM}_{\rm asymp} = 2\cos\theta - 1, \qquad \theta = -{\rm i}\ln q. \tag{16}$$

As suggested in [21,22], correlations of pions and kaons are characterized by the same value of q (a kind of universality). Then, experimentally there should be a tendency of merging $\lambda(\pi)$ and $\lambda(K)$ at large enough mean momenta: $\lambda_{\rm asymp}(\pi) = \lambda_{\rm asymp}(K)$. Recent RHIC/STAR experiment gives [22] the values $\lambda_1(\pi^-)$, $\lambda_2(\pi^-)$ and $\lambda_3(\pi^-)$ for π^--intercept, as averaged over three intervals of transverse momenta $\boldsymbol{K}_t$ and integrated over rapidity y, $-0.5 \le y \le 0.5$.

In Fig. 1, we show three curves for the intercept λ which correspond to fixing in (15) certain values of the deformation angle θ, along with the data $\lambda_j(\pi^-)$, j=1, 2, 3 (with error bars). The temperature is $T = 180$ MeV for all curves. We find remarkable agreement with data for the curve C (i.e., $\theta = 28.5°$). Another interesting case is the curve B at $\theta = \frac{\pi}{7} = 2\theta_{\rm C} \simeq 25.7°$. At $T = 180$ MeV, the curve B agrees, within error bars, with the points $\lambda_2(\pi)$ and $\lambda_3(\pi)$. However, slightly higher effective temperature $T \simeq 198$ MeV makes the curve marked by $\theta = 2\theta_{\rm C}$ respecting *all the three error bars*. Among various mixing angles known for hadrons, only the angle $2\theta_{\rm C}$ seems to be relevant to the issue of intercept $\lambda(\pi)$. Hence, it is tempting to suggest that just the angle $2\theta_{\rm C}$ embodies the assumed universality, to be seen in 2-particle correlations:

$$\lambda^{\rm BM}_{\rm asymp}(\pi)|_{\theta=\pi/7} = \lambda^{\rm BM}_{\rm asymp}(K)|_{\theta=\pi/7} = 2\cos\frac{\pi}{7} - 1 \approx 0.80194. \tag{17}$$

Insisting on the asymptotical coincidence $\lambda_{\rm asymp}(\pi) = \lambda_{\rm asymp}(K)$, we predict for kaons: *the intercept of 2-kaon correlations at any transverse momenta should obey:* $\lambda(K) \lesssim 0.80194$.

5.2 Three-particle correlations of pions (viewed as q-bosons) and $\theta_{\rm C}$

Now consider higher order (or multi-particle) correlations, first in the case of AC-type q-oscillators. It is not difficult to derive [23] the following 3-particle monomode correlation function

$$\langle a_i^\dagger a_i^\dagger a_i^\dagger a_i a_i a_i\rangle = \frac{(1+q)(1+q+q^2)}{(e^{\eta_i}-q)(e^{\eta_i}-q^2)(e^{\eta_i}-q^3)} \tag{18}$$

with $\eta_i \equiv \beta\omega_i$. From this, and 1-particle distribution $\langle a_i^\dagger a_i\rangle = \frac{1}{e^{\eta_i}-q}$, the intercept of 3-particle correlations for AC-type q-bosons in monomode case, dropping the subscript "i", reads:

$$\lambda^{(3),{\rm AC}} + 1 = \frac{\langle (a^\dagger)^3 a^3\rangle}{\langle a^\dagger a\rangle^3} = \frac{(1+q)(1+q+q^2)(e^{\eta_i}-q)^2}{(e^{\eta_i}-q^2)(e^{\eta_i}-q^3)}, \qquad -1 \le q \le 1. \tag{19}$$

The n-particle generalization of (18) is obtained in the form (see [23])

$$\langle (a_i^\dagger)^n (a_i)^n \rangle = \frac{\lfloor n \rfloor_q!}{\prod_{r=1}^n (e^{\eta_i} - q^r)}, \qquad \lfloor m \rfloor_q \equiv \frac{1-q^m}{1-q} = 1 + q + q^2 + \cdots + q^{m-1}, \tag{20}$$

where $\lfloor n \rfloor_q! = \lfloor 1 \rfloor_q \lfloor 2 \rfloor_q \cdots \lfloor n \rfloor_q$. Moreover, this admits direct two-parameter extension [23] to the model of pq-Bose gas, based on the use of so-called pq-oscillators, in the form

$$\langle (A_i^\dagger)^n (A_i)^n \rangle = \frac{[\![n]\!]_{qp}!\ (e^{\eta_i} - 1)}{\prod_{r=o}^n (e^{\eta_i} - q^r p^{n-r})}, \qquad [\![m]\!]_{qp} \equiv \frac{q^m - p^m}{q - p}. \tag{21}$$

With the use of one-particle distribution $\langle A^\dagger A \rangle = \frac{e^w - 1}{(e^w - q)(e^w - p)}$ of qp-bosons (see also [24]) we obtain general formula [23] for the qp-deformed n-th order intercept $\lambda_{q,p}^{(n)}$:

$$\lambda_{q,p}^{(n)} \equiv -1 + \frac{\langle A^{\dagger n} A^n \rangle}{\langle A^\dagger A \rangle^n} = -1 + [n]_{qp}! \frac{(e^w - p)^n (e^w - q)^n}{(e^w - 1)^{n-1} \prod_{k=0}^n (e^w - q^{n-k} p^k)}. \tag{22}$$

From this, the result (19) for AC-type q-bosons follows if $p = 1$. Similarly, putting $p = q^{-1}$ reduces to the important case of BM-type q-bosons, which for the 3-particle intercept yields

$$\lambda^{(3),\mathrm{BM}} \equiv -1 + \frac{\langle a^{\dagger 3} a^3 \rangle}{\langle a^\dagger a \rangle^3} = -1 + \frac{[2]_q [3]_q (e^{2w} - 2\cos\theta e^w + 1)^2}{(e^w - 1)^2 (e^{2w} - 2\cos(3\theta) e^w + 1)} \tag{23}$$

(compare it with equation (19)). Again, like the intercept $\lambda_{\mathrm{asymp}}^{(2)}$ in (16), at large w (large momenta or low T) $\lambda_{\mathrm{asymp}}^{(3)}$ gets dependent solely on the deformation parameter $q = \exp(\mathrm{i}\theta)$, i.e.

$$\lambda_{\mathrm{asymp}}^{(3),\mathrm{BM}} = -1 + [2]_q [3]_q = -1 + 2\cos\theta (2\cos\theta - 1)(2\cos\theta + 1)\ . \tag{24}$$

To confront the results (22), (24) with experimental data, we consider the combination [25]

$$r_0 = \left(\lambda^{(3)} - 3\lambda^{(2)}\right)\left(\lambda^{(2)}\right)^{-3/2}. \tag{25}$$

Then, the r_0 that follows from (15) and (23) (or $r_{0,\mathrm{asymp}}$ stemming from (16) and (24)) is at fixed T a decreasing function of θ for $0 \le \theta < \pi/3$. What about the Cabibbo angle? If we take the value $\theta = 2\theta_{\mathrm{C}}$ of deformation angle as we did in the preceding subsection we get $r_0|_{\theta = 2\theta_{\mathrm{C}}} \simeq 0.8955$. The universality conjecture dictates now that *all posible values of* the quantity r_0 (composed of intercepts $\lambda^{(2)}$ and $\lambda^{(3)}$) for either pions or kaons should respect the value 0.8955. The presently available data [25] extracted in Pb-Pb and Au-Au collisions seems to be yet insufficient to prove or disprove this assertion.

6 Conclusion

The use of quantum groups (quantum algebras) in the context of baryon phenomenology implies important fact that the deformation parameter q is linked very simply to the basic (fermion) Cabibbo mixing angle θ_{C}. In turn, this leads to unexpected connections for θ_{C} and for the whole concept of mixing, e.g., with the parameters of diquark-quark model of baryons, with the statistics parameter of anyonic picture, with the experimentally testable intercept parameters of multi-pion (-kaon) correlation functions, etc. Of course, it would be highly desirable to (re)obtain the considered connections of θ_{C} in a more strict way, and we hope this will be realized in a not very distant future.

Acknowledgements

The author thanks L. Boya and J. Beckers for interesting discussions during the conference.

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