Conservative Flux Recovery from the Q1 Conforming Finite Element Method on Quadrilateral Grids

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Compared with standard Galerkin finite element methods, mixed methods for second-order elliptic problems give readily available flux approximation, but in general at the expense of having to deal with a more complicated discrete system. This is especially true when conforming elements are involved. Hence it is advantageous to consider a direct method when finding fluxes is just a small part of the overall modeling processes. The purpose of this article is to introduce a direct method combining the standard Galerkin $Q1$ conforming method with a cheap local flux recovery formula. The approximate flux resides in the lowest order Raviart-Thomas space and retains local conservation property at the cluster level. A cluster is made up of at most four quadrilaterals. © 2004 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 20: 104–127, 2004

1. INTRODUCTION

Let Ω be a domain in $\mathbf{R}^2$ with boundary $\partial\Omega$ and consider the second-order elliptic boundary value problem

$$\begin{cases} -\operatorname{div}(\mathcal{K}\nabla p) = f & \text{in } \Omega, \\ \qquad\qquad p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{K} = \mathcal{K}(x)$ is a symmetric and uniformly positive definite matrix, i.e., there exist two positive constants c_1 and c_2 such that

$$c_1\xi^T\xi \le \xi^T\mathcal{K}(x)\xi \le c_2\xi^T\xi, \qquad \forall\xi \in \mathbf{R}^2, \qquad \forall x \in \bar{\Omega}.$$

In applications, the variable p can be interpreted, for example, as the temperature distribution in a heat conduction problem or as pressure in a porous medium problem. The vector variable $\mathbf{u} = -\mathcal{K}\nabla p$ (e.g., heat flux or Darcy velocity) is usually of considerable interest. Although the standard Galerkin finite element method applied to the variational formulation of (1.1) results in easy-to-solve symmetric positive definite finite element systems, it does not provide accurate flux automatically and is nonconservative at the element level. On the other hand, a mixed method, which approximates $\mathbf{u}$ and p simultaneously, can provide accurate flux and is locally conservative [1–3]. However, the tradeoff is an indefinite symmetric algebraic system that may be harder to solve iteratively [1–3]. In a contaminant transport problem, the above pressure equation is coupled with another temporal concentration equation in which an accurate flux is needed. It would be quite advantageous to have a way to evaluate flux quickly and cheaply to reduce the overall cost. It is therefore natural to pose the following question. Can one compute the approximate pressure and Darcy velocity to the same order of accuracy in two simple stages? First, an approximate pressure p_h is obtained via a standard conforming or nonconforming Galerkin finite element method applied to the second order elliptic problem (1.1). Then an approximate flux $\mathbf{u}_h$ to the exact flux $\mathbf{u} \in H(\mathrm{div}; \Omega) = \{\mathbf{w} : \mathbf{w} \in \mathbf{L}^2(\Omega), \nabla\cdot\mathbf{w} \in L^2(\Omega)\}$ is recovered by a physically intuitive and computationally efficient *formula* over each element K or at worst over each cluster of elements. For the nonconforming case, Chou and Tang [4] have shown that the above local recovery can be done in a very effective way: upon obtaining the $P1$ pressure, recover the velocity $\mathbf{u}_h$ in the lowest Raviart-Thomas space one element K at a time by the formula:

$$\mathbf{u}_h = -\bar{\mathcal{K}}\nabla p_h + \frac{f_K}{2}\begin{pmatrix} x - x_B \\ y - y_B \end{pmatrix} + \mathbf{C}_K, \qquad x \in K, \tag{1.2}$$

where $\bar{\mathcal{K}} = 1/K \int_K \mathcal{K}dx$ is the constant average of the tensor $\mathcal{K}$ over K; $f_K = 1/|K| \int_K fdx$, the average of f over K; (x_B, y_B), the barycenter of K; and $\mathbf{C}_K$ is a constant vector on K, which is determined by the continuity condition in the normal component and which can be computed by a very simple formula. No linear systems need be solved for this C_K. Formula (1.2) resembles the original flux definition and upon taking divergence is seen to conserve mass over each element, i.e., div $\mathbf{u}_h = f_K$ on K. In a subsequent article, Chou, Kwak, and Kim [5] generalized this technique to more general mixed finite element spaces on triangular and quadrilateral grids, such as BDM, BDFM spaces [3], etc. Some of the related articles using lowest order Raviart-Thomas space are Chen [6–8], Marini [9], and Courbet and Croisille [10]. This last article implicitly covered the local recovery flux issue.

Although Chou et al. are successful in recovering flux from nonconforming elements and obtaining conservative schemes, their conforming case is less satisfactory. In [4], the conforming case was discussed: the approximate velocity conserves mass but does not have continuous normal components across interelements. In a recent article [11] Destuynder and Métivet, while addressing an a posteriori estimate problem for the Poisson equation ($\mathcal{K} = I$), implicitly touched upon the above issue of mixed methods versus standard Galerkin methods. The common theme in both approaches [4, 11] for the conforming element is the use of weak (local) residuals

associated with clusters or spokes of triangles. [See Eq. (2.2) below.] Furthermore, the velocity is recovered over one cluster at a time, hence not elementwise. From a physical viewpoint, this might be unavoidable for conforming elements (see [12] and references therein).

The objective of this article is to develop a locally conservative flux recovery on lower order conforming elements, comparable to the nonconforming case [4, 5, 13], if one insists on using conforming elements. In addition to showing how to handle the nontrivial tensor coefficient case $\mathcal{K} \neq \mathcal{I}$, the identity matrix, we also demonstrate a general flux recovery procedure for the $Q1$ finite elements. We emphasize that the techniques used in the present conforming case are different from the nonconforming ones in [4, 5]. Rather, they are closer to those used in [11]. As a starting point we will focus on conforming bilinear finite elements on rectangular domains in the next two sections. Some of the techniques and ideas can be better explained in this setting. In section four we extend these techniques to quadrilateral grids.

II. CONSTRUCTION OF THE FLUX FORMULA FOR RECTANGULAR GRIDS

In this section we assume that the domain Ω admits a regular partition $\mathcal{T}_h$ of rectangles with sides less than or equal to h. We denote by $Q1$ the space of all polynomials whose degree ≤ 1 with respect to each of the two variables x and y. Define the lowest order Raviart-Thomas space

$$\mathbf{V}_h = \{\mathbf{u}_h \in H(\text{div}; \Omega) : \mathbf{u}_h|_K \in RT_0(K) \ \forall K \in \mathcal{T}_h\},$$

where $RT_0(K) = \{\mathbf{u} = (u^1, u^2) : u^1 = a + bx, u^2 = c + dy \text{ in } K\}$ and the standard $Q1$ conforming finite element space

$$X_h = \{p_h \in H_0^1(\Omega) : p_h|_K \in Q1\}.$$

Let us consider an arbitrary vertex (x_i, y_j) of the partition $\mathcal{T}_h$. We denote by C_{ij}^h the set of elements K of $\mathcal{T}_h$ sharing (x_i, y_j) as a common vertex. We also allow (x_i, y_j) to be a point on the boundary of Ω. A typical cluster C_{ij}^h at an interior vertex point (x_i, y_j) is shown in Fig. 1.

Let λ_{ij} be the piecewise bilinear global basis function associated with the vertex (x_i, y_j), so that λ_{ij} is one at (x_i, y_j) and zero at other nodes. The support of λ_{ij} is the cluster C_{ij}^h. Consider the $Q1$ conforming finite element method for solving problem (1.1): Find the approximate solution $p_h \in X_h$ to p such that

$$\int_\Omega \mathcal{K} \nabla p_h \cdot \nabla q_h dx = \int_\Omega f q_h dx \qquad \forall q_h \in X_h. \tag{2.1}$$

With the conforming finite element solution p_h on hand, our goal is to construct a conservative approximate flux $\mathbf{u}_h$ by piecing together some locally supported fluxes. To this end, let us first define a subspace of $\mathbf{V}_h$ as follows:

$$\mathbf{V}_h(C_{ij}^h) = \{\mathbf{w}_h \in \mathbf{V}_h, \text{support}(\mathbf{w}_h) = C_{ij}^h \text{ and } \mathbf{w}_h \cdot \mathbf{n} = 0 \text{ on } \partial C_{ij}^h\},$$

where $\mathbf{n}$ is the unit outward normal to the boundary ∂C_{ij}^h. It is necessary to point out that the definition of the boundary ∂C_{ij}^h above does not include the sides lying on the boundary $\partial \Omega$ of
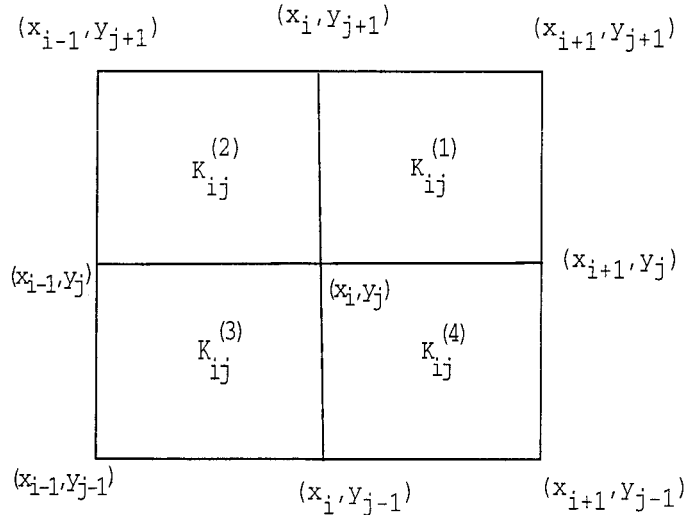
FIG. 1.   A cluster at (x_i, y_j).

Ω. (The reason will be clear once we look at the following problem.) Consider the problem: Find $\mathbf{u}_{ij}^h \in \mathbf{V}_h(C_{ij}^h)$ such that for all $K \in C_{ij}^h$

$$\int_K \operatorname{div} \mathbf{u}_{ij}^h dx = -\int_K \mathcal{K}\nabla p_h \cdot \nabla \lambda_{ij} dx + \int_K f\lambda_{ij} dx. \qquad (2.2)$$

The weakly local residual on the right side has been a common theme for many local flux recovery techniques [4, 5, 9, 11] (to name a few), for the construction of finite volume methods [13] or for an engineering interpretation of the finite element method derivation [12]. The residual is already available in the construction of the conforming finite element solution, and (2.2) is a small system whose solutions can be explicitly written down as we show next. So the construction of $\mathbf{u}_{ij}^h$ is cheap and effective, if one insists on using conforming schemes.

Theorem 2.1.   *A solution to (2.2) exists and can be written in the form*

$$\mathbf{u}_{ij}^h = \mathbf{u}_{ij}^{h,*} + \alpha_{ij}\mathbf{curl}\lambda_{ij}, \qquad \alpha_{ij} \in \mathbf{R}, \qquad (2.3)$$

where $\mathbf{u}_{ij}^{h,*}$ *is a particular solution of (2.2) and*

$$\mathbf{curl}\phi = \left(\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}\right)^t.$$

   Proof.   We begin with the case in which (x_i, y_j) is an internal vertex of the partition $\mathcal{T}_h$. Denote the $K_{ij}^{(l)}$, $l = 1, 2, 3, 4$ the four elements of C_{ij}^h (see Fig. 1) and by $I_{ij}^{(l)}$, the integrals

$$-\int_{K_{ij}^{(l)}} \mathcal{K}\nabla p_h \cdot \nabla \lambda_{ij} dx + \int_{K_{ij}^{(l)}} f\lambda_{ij} dx, \qquad l = 1, \ldots, 4.$$
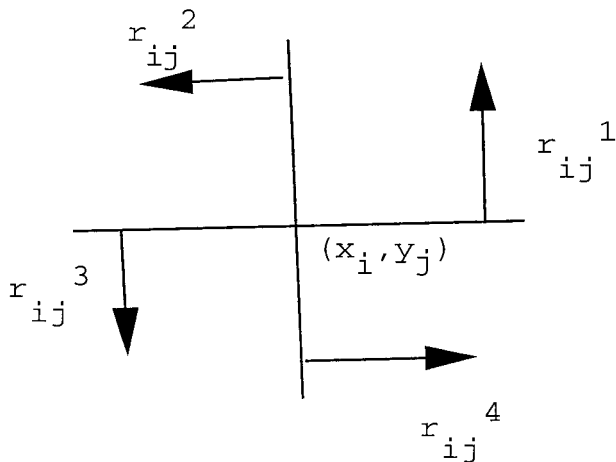
FIG. 2.   Preassigned directions.

Let $K^{(m)} = K^{(m)}_{ij}$ (drop the subscript for simplicity) and denote by $e_1 = K^{(1)} \cap K^{(4)}$, $e_2 = K^{(1)}$ $\cap K^{(2)}$, $e_3 = K^{(2)} \cap K^{(3)}$, $e_4 = K^{(3)} \cap K^{(4)}$, the four edges connected to (x_i, y_j). We preassign four unit normals ν_i as shown in Fig. 2. Then obviously the space $\mathbf{V}_h(C^h_{ij})$ has dimension 4 and is spanned by ϕ_k with the flux conditions $\int_{e_m} \phi_k \cdot \nu_m ds = \delta_{km}$, $k, m = 1, \ldots, 4$. Since $\mathbf{u}^h_{ij} \in \mathbf{V}_h(C^h_{ij})$, we can write it as

$$\alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 + \alpha_4 \phi_4.$$

Substituting this into (2.2) and noticing that each ϕ_k is supported in only two rectangles, we have the linear system

$$-\alpha_1 + \alpha_2 = I^{(1)}$$

$$-\alpha_2 + \alpha_3 = I^{(2)}$$

$$-\alpha_3 + \alpha_4 = I^{(3)}$$

$$-\alpha_4 + \alpha_1 = I^{(4)}. \tag{2.4}$$

The kernel of the coefficient matrix in (2.4) is easily seen to be one dimensional and is spanned by (1, 1, 1, 1). Equivalently, this means $\int_{K^{(m)}} \text{div } \mathbf{u}^h_{ij} dx = 0$ for every $K^{(m)}$. Being a constant over $K^{(m)}$, div $\mathbf{u}^h_{ij} = 0$ on $K^{(m)}$. Hence there exists a linear function λ_m such that $\mathbf{curl}\lambda_m = \mathbf{u}^h_{ij}$ there. By the boundary condition on $\partial K^{(m)} \cap \partial C^h_{ij}$:

$$\frac{\partial \lambda_m}{\partial \tau} = \mathbf{u}^h_{ij} \cdot \nu = 0,$$

where τ is the unit tangent vector so that (ν, τ) forms a right hand system. We can then choose $\lambda_m = 0$ on this boundary. Doing this for every $K^{(m)}$ and using the continuity condition of the normal component across the four inner edges of C^h_{ij}, we see that the assembled function over

the whole C_{ij}^h must be a multiple of λ_{ij}. Hence the divergence free vectors can be generated by $\mathbf{curl}\lambda_{ij}$. Simple calculation shows that this vector also spans the cokernel, and the compatibility condition is

$$\sum_{l=1}^{4} I_{ij}^{(l)} = 0,$$

which is nothing but one of the equations characterizing p_h.

Next we turn to the case (x_i, y_j) on the boundary of Ω. Everything we showed previously regarding the kernel is still true. In addition, there is one more unknown than equations in (2.2) and hence the coefficient matrix is full rank. Thus, there is no compatibility requirement. This completes the proof. ∎

We note that intuitively since the boundary of C_{ij}^h admits no flow, the statement $\int_{K^{(m)}} \text{div } \mathbf{w} \, dx = 0$, $m = 1, 2, 3, 4$ (mass conservation) suggests that an arrangement of fluxes of $\alpha = (1, 1, 1, 1)$ across the four edges emanating from (x_i, y_j) gives a divergence free field (see Fig. 2). The above theorem says it is the only way to generate a divergence free field and it can be generated by a curl as suggested in the figure.

Remark 2.1. A particular solution $\mathbf{u}_{ij}^{h,*}$ is easily obtainable from (2.4).

Remark 2.2. It should be pointed out that mass conservation and divergence free are in general two different statements. That is,

$$\int_{K^{(m)}} \text{div } \mathbf{w} \, dx = 0 \qquad \text{and} \qquad \text{div } \mathbf{w} \, dx = 0 \quad \text{on } K^{(m)}$$

are not the same unless div $\mathbf{w}$ is a constant on $K^{(m)}$. This is certainly true for the rectangular case, but not so in the quadrilateral case, which we will consider in a later section. Note that the proof for the existence of λ_{ij} in the previous theorem relies on existence of divergence free vectors.

Let $\mathcal{N}^h$ be the set of all interior vertices of $\mathcal{T}_h$ and let $\bar{\mathcal{N}}^h$ be the set of all vertices partition $\mathcal{T}_h$ (vertices on boundary added). Associated with an $\mathbf{u}_{ij}^h$ as defined in Theorem 2.1, we let

$$\mathbf{u}^h = \sum_{(x_i, y_j) \in \bar{\mathcal{N}}^h} \mathbf{u}_{ij}^h. \tag{2.5}$$

From the definition of $\mathbf{u}_{ij}^h$, $\mathbf{u}_{ij}^h$ vanishes outside C_{ij}^h, and on each element K one has $\sum_{(x_i, y_j) \in \mathcal{N}^h} \lambda_{ij} = 1$. Hence, let I_K be the set of all vertices of K, then we have over each $K \in \mathcal{T}_h$ that

$$\int_K \text{div } \mathbf{u}^h = \sum_{(x_i, y_j) \in \mathcal{N}^h} \int_K \text{div } \mathbf{u}_{ij}^h \tag{2.6}$$

$$= \sum_{(x_i, y_j) \in I_K} \left\{ -\int_K \mathcal{H} \nabla p_h \cdot \nabla \lambda_{ij} dx + \int_K f\lambda_{ij} dx \right\} \tag{2.7}$$

$$= \int_K f dx. \tag{2.8}$$

This shows that $\mathbf{u}^h$ is locally conservative. Of course, in this case one can also say div $\mathbf{u}^h = f_K$ on K.

III. ERROR ESTIMATES

In Theorem 2.1, the coefficient α_{ij} was left undetermined. In the next theorem we choose this coefficient so that the resulting approximate flux is close to the exact solution to first order. The error analysis below borrows an iterated error estimation trick from [11] and extends it from triangular grids to quadrilateral grids and from isotropic $\mathcal{K} = I$ to anisotropic case. We use the usual notation $W^{m,p}(D)$ and its associated norm and semi-norm $\|w\|_{m,p,D}$, $|w|_{m,p,D}$ for the L^p based Sobolev space on domain D. When $p = 2$ and $D = \Omega$, we write instead H^m, $\|w\|_m$, and $|w|_m$.

Theorem 3.1.   *Let $\mathcal{K} \in W^{1,\infty}(\Omega)$ and $f \in L^2(\Omega)$. Let the partition $\mathcal{T}_h$ be regular. Let $\mathbf{u}_h$ be defined as*

$$\mathbf{u}^h = \sum_{(x_i, y_j) \in \mathcal{N}^h} \mathbf{u}^h_{ij},$$

where $\mathbf{u}^h_{ij}$ is defined in (2.3) and the coefficient

$$\alpha_{ij} = -\int_{\gamma^1_{ij}} \left(\mathbf{u}^{h,*}_{ij} \cdot \nu + \frac{1}{2} \bar{\mathcal{K}} \overline{\nabla p_h} \cdot \nu \right) ds,$$

where $\bar{w}$ is the average of w over K, i.e., $\bar{w} = 1/|K| \int_K w dx$, and γ^1_{ij} is an edge incident from (x_i, y_j) and ν its associated unit normal as shown in Fig. 3. Then there exists a constant $C > 0$, independent of h, f, p, and $\mathbf{u}$ such that

$$\|\mathbf{u} - \mathbf{u}^h\|_0 \le Ch(|\mathbf{u}|_1 + \|f\|_0 + \|p\|_2).$$

Proof.   We first introduce $\tilde{\mathbf{u}} \in \mathbf{V}_h$, the usual interpolant of $\mathbf{u}$ based on fluxes, i.e., on each $K \in \mathcal{T}_h$:

$$\tilde{\mathbf{u}} \cdot \nu|_\gamma = \frac{1}{|\gamma|} \int_\gamma \mathbf{u} \cdot \nu ds \qquad \forall \gamma \in \partial K,$$

where γ is a side of K and $\mathbf{u}$ is the solution of (1.1). A basic well-known error estimate we shall need is

$$\|\tilde{\mathbf{u}} - \mathbf{u}\|_{0,K} \le Ch|\mathbf{u}|_{1,K}. \tag{3.1}$$

One the one hand, one has

$$\int_K \text{div } \tilde{\mathbf{u}} dx = \int_{\partial K} \tilde{\mathbf{u}} \cdot \nu ds = \int_{\partial K} \mathbf{u} \cdot \nu ds = \int_K \nabla \cdot \mathbf{u} dx = \int_K f dx.$$

On the other hand, div $\tilde{\mathbf{u}}$ is a constant on K, so we have

$$\text{div } \tilde{\mathbf{u}} = \frac{1}{|K|} \int_K f dx. \tag{3.2}$$

Setting $\mathbf{e}_K = \mathbf{u}^h - \tilde{\mathbf{u}}$, we have div $\mathbf{e}_K = $ div $\mathbf{u}^h - $ div $\tilde{\mathbf{u}} = (1/|K|) \int_K f dx - (1/|K|) \int_K f dx = 0$, and $\mathbf{e}_K \cdot \nu = \mathbf{u}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu$. Thus there exists a function φ_K such that

$$\begin{cases} \mathbf{e}_K = \nabla \varphi_K \quad \text{and} \quad \int_K \varphi_K dx = 0, \\ \varphi_K \in H^1(K). \end{cases}$$

In fact, φ_K is a solution of

$$\begin{cases} -\Delta \varphi_K = 0, \quad \text{in } K, \\ \int_K \varphi_K dx = 0, \\ \dfrac{\partial \varphi_K}{\partial \nu} = \mathbf{u}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu \quad \text{on } \partial K, \quad \varphi_K \in H^1(K). \end{cases} \tag{3.3}$$

From (3.3) and the fact that

$$\int_K -\Delta \varphi_K \cdot \varphi_K dx = -\int_{\partial K} (\nabla \varphi_K \cdot \nu) \varphi_K ds + \int_K \nabla \varphi_K \cdot \nabla \varphi_K dx,$$

we get

$$\|\mathbf{e}_K\|_{0,K}^2 = |\varphi_K|_{1,K}^2 = \int_{\partial K} (\nabla \varphi_K \cdot \nu) \varphi_K ds = \int_{\partial K} (\mathbf{u}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu) \varphi_K ds. \tag{3.4}$$

Now we let the four vertices of K be $A = (x_i, y_j)$, $B = (x_{i+1}, y_j)$, $C = (x_{i+1}, j_{j+1})$, and $D = (x_i, y_{j+1})$. Let us locally label the segments AB, BC, CD, DA as γ_1, γ_2, γ_3, γ_4, respectively. By the definition of $\mathbf{u}^h$

$$\|\mathbf{e}_K\|_{0,K}^2 = \int_{\partial K} (\mathbf{u}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)\varphi_K ds$$

$$= \int_{\partial K} \mathbf{u}_{ij}^h \cdot \nu - \varphi_K ds + \int_{\partial K} \mathbf{u}_{i+1,j}^h \cdot \nu - \varphi_K ds + \int_{\partial K} \mathbf{u}_{i+1,j+1}^h \cdot \nu\varphi_K ds$$

$$+ \int_{\partial K} \mathbf{u}_{i,j+1}^h \cdot \nu - \varphi_K ds - \int_{\partial K} - \tilde{\mathbf{u}} \cdot \nu)\varphi_K ds$$

$$= I_1 + I_2 + I_3 + I_4 - \int_{\partial K} - \tilde{\mathbf{u}} \cdot \nu)\varphi_K ds. \tag{3.5}$$

Observe that for each I_i term, a line integral around ∂K, only the two terms over sides adjacent to (x_r, y_s) are nonzero by the definitions of $\mathbf{u}_{rs}^h$. For instance,

$$I_1 = \int_{\partial K} (\mathbf{u}_{ij}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)\varphi_K ds$$

$$= \int_{\gamma_1} \mathbf{u}_{ij}^h \cdot \nu\varphi_K ds + \int_{\gamma_4} \mathbf{u}_{ij}^h \cdot \nu\varphi_K ds - \int_{\partial K} \tilde{\mathbf{u}} \cdot \nu\varphi_K ds.$$

Similarly,

$$I_2 = \int_{\gamma_1} \mathbf{u}_{i+1,j}^h \cdot \nu\varphi_K ds + \int_{\gamma_2} \mathbf{u}_{i+1,j}^h \cdot \nu\varphi_K ds - \int_{\partial K} \tilde{\mathbf{u}} \cdot \nu\varphi_K ds,$$

$$I_3 = \int_{\gamma_2} \mathbf{u}_{i+1,j+1}^h \cdot \nu\varphi_K ds + \int_{\gamma_3} \mathbf{u}_{i+1,j+1}^h \cdot \nu\varphi_K ds - \int_{\partial K} \tilde{\mathbf{u}} \cdot \nu\varphi_K ds,$$

$$I_4 = \int_{\gamma_3} \mathbf{u}_{i,j+1}^h \cdot \nu\varphi_K ds + \int_{\gamma_4} \mathbf{u}_{i,j+1}^h \cdot \nu\varphi_K - \int_{\partial K} \tilde{\mathbf{u}} \cdot \nu\varphi_K ds.$$

Summing up, using the global indexing and noting that each edge integral $\int_{\gamma_i} \tilde{\mathbf{u}} \cdot \nu\varphi_K ds$ in the $\int_{\partial K} \tilde{\mathbf{u}} \cdot \nu\varphi_K ds$-terms is summed twice, we get

$$\|\mathbf{e}_K\|_{0,K}^2 = |\varphi_K|_{1,K}^2 = \sum_{(r,s)\in\mathcal{I}_K} \sum_{m=1}^2 \int_{\gamma_{(rs)}^{(m)}} \left(\mathbf{u}_{rs}^h \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu\right)\varphi_K ds, \tag{3.6}$$

where (r, s) runs through $\mathcal{I}_K = \{(i, j), (i + 1, j), (i + 1, j + 1), (i, j + 1)\}$ and for a vertex (x_r, y_s), $\gamma_{rs}^{(m)}$, $m = 1, 2$ are the two sides of K sharing that vertex as a common extremity. Hence, to bound $\|\mathbf{e}_K\|_{0,K}^2$ it suffices to estimate a typical term like $\int_{\gamma_{ij}^{(m)}} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)\varphi_K ds$, $m = 1, 2$. Note that for $m = 1, 2$,
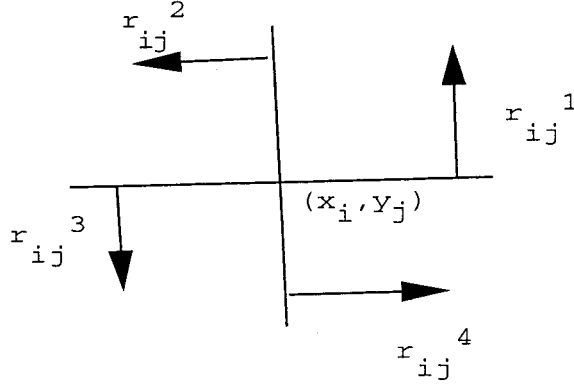
FIG. 3.   The orientation of a spoke at (x_i, y_j).

$$\left| \int_{\gamma_{ij}^{(m)}} \left(\mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\, \tilde{\mathbf{u}} \cdot \nu \right) \varphi_K ds \right| \le \left| \mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\, \tilde{\mathbf{u}} \cdot \nu \right|_{|\gamma_{ij}^{(m)}|} \sqrt{|\gamma_{ij}^{(m)}|}\, \|\varphi_K\|_{0,\gamma_{ij}^{(m)}}$$

$$\le \left(\left| \mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\, \tilde{\mathbf{u}} \cdot \nu \right|_{|\gamma_{ij}^{(m)}|} |\gamma_{ij}^{(m)}| \right) \frac{1}{\sqrt{|\gamma_{ij}^{(m)}|}}\, \|\varphi_K\|_{0,\gamma_{ij}^{(m)}}, \tag{3.7}$$

where $\big|_{\gamma_{ij}^{(m)}}$ stands for the restriction to $\gamma_{ij}^{(m)}$.

Observe that the first factor of the right side of (3.7) is the same for ν and $-\nu$. With this in mind we turn our attention to the cluster C_{ij}^h at (x_i, y_j). In reference to Fig. 3, we arrange the unit normals ν to the four sides γ_{ij}^k, $k = 1, 2, 3, 4$ of C_{ij} emanating from (x_i, y_j) counterclockwise as shown. (Note that the subscript has no parentheses.) Now on the one hand, for $k = 1, 2, 3, 4$

$$X_{ij}^k := \int_{\gamma_{ij}^k} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)ds = \int_{\gamma_{ij}^k} \left(\mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\, \tilde{\mathbf{u}} \cdot \nu \right)ds = |\gamma_{ij}^k| \left[\mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\, \tilde{\mathbf{u}} \cdot \nu \right]_{|\gamma_{ij}^k}.$$

On the other hand, referring to Fig. 1:

$$X_{ij}^{k+1} - X_{ij}^k = \int_{\gamma_{ij}^{k+1}} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)ds - \int_{\gamma_{ij}^k} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)ds$$

$$= \int_{K^k} \operatorname{div} \mathbf{u}_{ij}^h dx - \int_{K^k} (\operatorname{div} \tilde{\mathbf{u}})\lambda_{ij}dx - \int_{K^k} \tilde{\mathbf{u}} \cdot \nabla\lambda_{ij}dx$$

$$= \int_{K_k} (-\mathcal{K}\nabla p_h \cdot \nabla\lambda_{ij} + f\lambda_{ij})dx - \int_{K^k} (\operatorname{div} \tilde{\mathbf{u}})\lambda_{ij}dx - \int_{K^k} \tilde{\mathbf{u}} \cdot \nabla\lambda_{ij}dx$$

$$= \int_{K^k} (f - \operatorname{div} \tilde{\mathbf{u}})\lambda_{ij}dx - \int_{K^k} (\tilde{\mathbf{u}} + \mathcal{K}\nabla p_h) \cdot \nabla\lambda_{ij}dx.$$

Notice that

$$\|\lambda_{ij}\|_{0,K^k} \le Ch, \qquad |\lambda_{ij}|_{1,K^k} \le C, \qquad \text{and} \qquad \|\text{div } \tilde{\mathbf{u}}\|_{0,K^k} \le \|\text{div } \mathbf{u}\|_{0,K^k},$$

where the last inequality can be derived by the definition of $\tilde{\mathbf{u}}$. Then by the triangle inequality and (3.1), one gets, with $K^1 = K$,

$$
\begin{aligned}
|X_{ij}^2 - X_{ij}^1| &\le C(h\|f - \text{div } \tilde{\mathbf{u}}\|_{0,K} + \|\tilde{\mathbf{u}} - \mathbf{u}\|_{0,K} + \|\mathbf{u} + \mathcal{K}\nabla p_h\|_{0,K}) \\
&\le Ch(\|f\|_{0,K} + \|\text{div } \mathbf{u}\|_{0,K}) + \|\tilde{\mathbf{u}} - \mathbf{u}\|_{0,K} + |\mathcal{K}|_{\infty,K}\|\nabla p - \nabla p_h\|_{0,K} \\
&\le Ch(\|f\|_{0,K} + \|\text{div } \mathbf{u}\|_{0,K} + |\mathbf{u}|_{1,K}) + C|p - p_h|_{1,K} \\
&\le Ch(\|f\|_{0,K} + |\mathbf{u}|_{1,K}) + C|p - p_h|_{1,K}. \qquad (3.8)
\end{aligned}
$$

Now we turn to analyzing X_{ij}^1. Let

$$\bar{\mathcal{K}} = \frac{1}{|K|} \int_K \mathcal{K}\,dx, \qquad \overline{\nabla p_h} = \frac{1}{|K|} \int_K \nabla p_h\,dx.$$

Then

$$
\begin{aligned}
X_{ij}^1 &= \int_{\gamma_{ij}^1} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)\,ds \\
&= \int_{\gamma_{ij}^1} [(\alpha_{ij}\mathbf{curl}\lambda_{ij} + \mathbf{u}_{ij}^{h,*}) \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu]\,ds \\
&= \alpha_{ij} + \int_{\gamma_{ij}^1} \left(\mathbf{u}_{ij}^{h,*} \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu \right)\,ds \\
&= \alpha_{ij} + \int_{\gamma_{ij}^1} \left(\mathbf{u}_{ij}^{h,*} \cdot \nu + \frac{1}{2}\bar{\mathcal{K}}\overline{\nabla p_h} \cdot \nu \right)\,ds - \frac{1}{2}\int_{\gamma_{ij}^1} (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nu\,ds.
\end{aligned}
$$

Making a choice of $\alpha_{ij} = -\int_{\gamma_{ij}^1}(\mathbf{u}_{ij}^{h,*} \cdot \nu + (1/2)\bar{\mathcal{K}}\overline{\nabla p_h} \cdot \nu)\,ds$, one gets

$$X_{ij}^1 = -\frac{1}{2}\int_{\gamma_{ij}^1} (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nu\,ds.$$

Let

$$\xi = \frac{(x_{i+1} - x)(x - x_i)(y_{j+1} - y)}{(x_{i+1} - x_i)^2(y_{j+1} - y_j)},$$

a cubic polynomial vanishing on three sides of K and is 1/4 at the midpoint of the remaining side. Then by the Simpson's rule

$$\int_{\gamma_{ij}^1} \xi ds = \frac{1}{6} |\gamma_{ij}^1|, \qquad \|\xi\|_{0,K} \le Ch, \qquad |\xi|_{1,K} \le C.$$

Using the Gauss formula, we have

$$X_{ij}^1 = -3 \int_{\gamma_{ij}^1} (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nu \xi ds$$

$$= -3 \left\{ \int_K (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nabla \xi dx + \int_K \xi \operatorname{div} \tilde{\mathbf{u}} dx \right\}.$$

Hence

$$|X_{ij}^1| \le C(\|\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}\|_{0,K}|\xi|_{1,K} + \|\xi\|_{0,K}\|\operatorname{div} \tilde{\mathbf{u}}\|_{0,K})$$

$$\le C(\|\bar{\mathcal{K}}\overline{\nabla p_h} + \mathbf{u}\|_{0,K} + \|\tilde{\mathbf{u}} - \mathbf{u}\|_{0,K} + \|\xi\|_{0,K}\|\operatorname{div} \tilde{\mathbf{u}}\|_{0,K})$$

$$\le CJ_1 + Ch|\mathbf{u}|_{1,K} + Ch\|\operatorname{div} \mathbf{u}\|_{0,K}, \tag{3.9}$$

where

$$|J_1| = \|\mathbf{u} + \bar{\mathcal{K}}\overline{\nabla p_h}\|_{0,K} = \|-\mathcal{K}\nabla p + \bar{\mathcal{K}}\overline{\nabla p_h}\|_{0,K}$$

$$\le \|-\mathcal{K}\nabla p + \bar{\mathcal{K}}\nabla p\|_{0,K} + \|\bar{\mathcal{K}}\nabla p - \bar{\mathcal{K}}\overline{\nabla p}\|_{0,K} + \|\bar{\mathcal{K}}\|_{0,\infty}\|\overline{\nabla p} - \overline{\nabla p_h}\|_{0,K}$$

$$\le Ch|p|_{1,K} + Ch|p|_{2,K} + C|p - p_h|_{1,K}, \tag{3.10}$$

where the $|p|_{2,K}$ term is obtained by the Friedrichs' inequality [14] and the Bramble-Hilbert lemma.

So altogether, we deduce that

$$|X_{ij}^1| \le Ch(|p|_{1,K} + |p|_{2,K} + |\mathbf{u}|_{1,K} + \|f\|_{0,K}) + C|p - p_h|_{1,K}. \tag{3.11}$$

From (3.8) and (3.9), we obtain

$$|X_{ij}^l| \le Ch(|p|_{1,C_{ij}^h} + |p|_{2,C_{ij}^h} + |\mathbf{u}|_{1,C_{ij}^h} + \|f\|_{0,C_{ij}^h}) + C|p - p_h|_{1,C_{ij}^h}, \qquad l = 1, 2, 3, 4. \tag{3.12}$$

Now using (3.6), (3.7), (3.12), the fact that

$$\|\varphi_K\|_{0,\gamma_{ij}^m} \leq C\sqrt{h}\,|\varphi_K|_{1,K},$$

and summing over K, we have

$$\|\mathbf{u}^h - \tilde{\mathbf{u}}\|_0 \leq Ch(\|p\|_2 + |\mathbf{u}|_1 + \|f\|_0) + C|p - p_h|_1.$$

Let $\mathcal{K} \in W^{1,\infty}(\Omega)$ and $p \in H^2(\Omega)$, then there exists a constant C independent of h such that

$$\|p - p_h\|_0 + h|p - p_h|_1 \leq Ch^2\|p\|_2. \tag{3.13}$$

So we have

$$\|\mathbf{u}^h - \tilde{\mathbf{u}}\|_0 \leq Ch(\|p\|_2 + |\mathbf{u}|_1 + \|f\|_0).$$

Finally,

$$\|\mathbf{u}^h - \mathbf{u}\|_0 \leq \|\mathbf{u}^h - \tilde{\mathbf{u}}\|_0 + \|\tilde{\mathbf{u}} - \mathbf{u}\|_0$$
$$\leq Ch(|\mathbf{u}|_1 + \|f\|_0 + \|p\|_2).$$

This completes the proof. ∎

IV. QUADRILATERAL GRIDS

Let $\mathcal{Q}_h$ be a partition of Ω into convex quadrilaterals with diameters less than or equal to h. The partition is logically rectangular in the sense that each quadrilateral has unique eastern, western, northern, and southern adjacent neighbors if they exist. Hence one can write $\mathcal{Q}_h = \{Q_{i,j}\}$, indexed by two indices. Here we deviate from the usual cell-center convention and use the lower left corner (x_i, y_j) to index a quadrilateral Q. In Fig. 1, distort those K's and call them Q's. For Q_{ij}, the left, right, bottom, and top edges of $Q_{i,j}$ are, respectively, denoted by

$$e_l = e_{i,j+1/2} = \partial Q_{i-1,j} \cap \partial Q_{i,j}, \qquad e_r = e_{i+1,j+1/2} = \partial Q_{i,j} \cap \partial Q_{i+1,j},$$

and

$$e_b = e_{i+1/2,j} = \partial Q_{i,j-1} \cap \partial Q_{i,j}, \qquad e_t = e_{i+1/2,j+1} = \partial Q_{i,j} \cap \partial Q_{i+1,j},$$

where we used the midpoint of an edge to index it. Let $\hat{\mathbf{x}} = (\hat{x}, \hat{y})$ and $\mathbf{x} = (x, y)$. We take the unit square $\hat{Q} = [0, 1] \times [0, 1]$ as the reference element (cf. Fig. 4) in the $\hat{x}\hat{y}$-plane with vertices denoted by

$$\hat{\mathbf{x}}_1 = (0, 0), \quad \hat{\mathbf{x}}_2 = (1, 0), \quad \hat{\mathbf{x}}_3 = (1, 1), \quad \hat{\mathbf{x}}_4 = (0, 1).$$

Let Q be a convex quadrilateral with vertices $\mathbf{x}_i$ arranged in counterclockwise order. Then there exists a unique invertible bilinear transformation F_Q which maps $\hat{Q}$ onto Q and satisfies
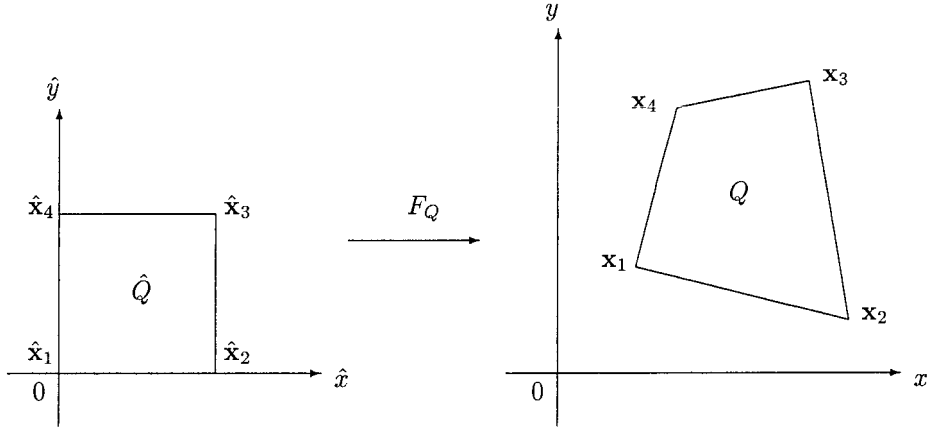
FIG. 4.   The bilinear mapping $F_Q : \hat{Q} \to Q$.

$$\mathbf{x}_i = F_Q(\hat{\mathbf{x}}_i), \qquad i = 1, 2, 3, 4.$$

In fact, it is given by

$$\mathbf{x} = F_Q(\hat{\mathbf{x}}) = \mathbf{x}_1 + \mathbf{x}_{21}\hat{x} + \mathbf{x}_{41}\hat{y} + \mathbf{g}\hat{x}\hat{y}, \tag{4.1}$$

where we set

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \qquad \mathbf{g} = \mathbf{x}_{12} + \mathbf{x}_{34}.$$

By a simple calculation it is easy to see that the Jacobian matrix $\mathcal{J}_Q$ of F_Q is given by

$$\mathcal{J}_Q = \begin{pmatrix} \dfrac{\partial x}{\partial \hat{x}} & \dfrac{\partial x}{\partial \hat{y}} \\ \dfrac{\partial y}{\partial \hat{x}} & \dfrac{\partial y}{\partial \hat{y}} \end{pmatrix} = (\mathbf{x}_{21} + \mathbf{g}\hat{y}, \mathbf{x}_{41} + \mathbf{g}\hat{x}). \tag{4.2}$$

Denote the S_i the subtriangle of Q with vertices $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, and $\mathbf{x}_{i+1}$ ($\mathbf{x}_0 = \mathbf{x}_4$). Let h_Q be the diameter of Q and $\rho_Q = 2 \min_{1 \le i \le 4}$ {diameter of a circle inscribed in S_i}. Throughout the article we assume a *regular* family of partitions $\mathcal{Q} = \{\mathcal{Q}_h\}$, i.e., there exists a positive constant σ, independent of h, such that

$$\frac{h_Q}{\rho_Q} \le \sigma \qquad \forall Q \in \mathcal{Q}_h, \forall \mathcal{Q}_h \in \mathcal{Q}. \tag{4.3}$$

The following upper bounds can be found, e.g., in [15]:

$$|\mathcal{J}_Q|_{\infty,\hat{Q}} \le Ch_Q, \qquad |\mathcal{J}_Q^{-1}|_{\infty,Q} \le Ch_Q^{-1}, \tag{4.4}$$

where $\|M\|_{\infty,K} := \sup_{x \in K}\|M(x)\|$, the supremum of the spectral norm of the matrix function M. Hereafter C will denote a generic positive constant that is independent of h. It may have different values in different places, especially when used in proof.

Simple calculation shows that the determinant $J_Q = \det \mathcal{J}_Q$ is a linear function of $\hat{x}$ and $\hat{y}$:

$$J_Q(\hat{x}, \hat{y}) = \alpha + \beta\hat{x} + \gamma\hat{y}, \tag{4.5}$$

where

$$\alpha = \det(\mathbf{x}_{21}, \mathbf{x}_{41}), \quad \beta = \det(\mathbf{x}_{21}, \mathbf{g}), \quad \gamma = \det(\mathbf{g}, \mathbf{x}_{41}).$$

The following upper bounds for the L_∞-norm of the functions J_Q and J_Q^{-1} can also be found in [15]:

$$|J_Q|_{\infty,\hat{Q}} \leq Ch_Q^2, \qquad |J_Q^{-1}|_{\infty,Q} \leq Ch_Q^{-2}. \tag{4.6}$$

The Piola transformation $\mathcal{P}_Q$ transforms a vector-valued function on $\hat{Q}$ to one on Q by

$$\mathbf{v} = \mathcal{P}_Q\hat{\mathbf{v}} = \frac{1}{J}\mathcal{J}\hat{\mathbf{v}} \circ F^{-1}, \tag{4.7}$$

where we drop the subscript Q for brevity. This transformation preserves the $H(\mathrm{div})$ space on the reference element and has the following well-known properties (cf. [16–18]): If we let $\hat{p} = p \circ F$, then

$$\int_Q \nabla p \cdot \mathbf{v}\,dxdy = \int_{\hat{Q}} \hat{\nabla}\hat{p} \cdot \hat{\mathbf{v}}\,d\hat{x}d\hat{y}, \tag{4.8}$$

$$\int_Q \mathrm{div}\,\mathbf{v}\,dxdy = \int_{\hat{Q}} \mathrm{div}\,\hat{\mathbf{v}}\,d\hat{x}d\hat{y}, \tag{4.9}$$

$$\mathrm{div}\,\mathbf{v} = \frac{1}{J}\mathrm{div}\,\hat{\mathbf{v}}. \tag{4.10}$$

The following lemma can be shown easily by (4.4) and (4.6).

Lemma 4.1. *Let* $\mathbf{v}$ *and* $\hat{\mathbf{v}}$ *be related by* (4.7). *For regular partitions, there exist positive constants* C_1 *and* C_2 *such that for every* $\mathbf{v} \in (L^2(Q))^2$, *we have*

$$C_1\|\mathbf{v}\|_{0,Q} \leq \|\hat{\mathbf{v}}\|_{0,\hat{Q}} \leq C_2\|\mathbf{v}\|_{0,Q}. \tag{4.11}$$

A. Pressure and Velocity Spaces on Quadrilaterals

The approximate velocity space $\mathbf{V}_h$ we choose is the lowest-order Raviart-Thomas space, which is defined as follows:

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_Q = \mathcal{P}_Q\hat{\mathbf{v}} \ \forall \hat{\mathbf{v}} \in \mathbf{V}_h(\hat{Q}) \text{ and } \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad (4.12)$$

where $\mathbf{V}_h(\hat{Q})$ denotes the local space on $\hat{Q}$,

$$\mathbf{V}_h(\hat{Q}) = \{\hat{\mathbf{v}} : \hat{\mathbf{v}} = (a + b\hat{x}, c + d\hat{y}), a, b, c, d \in \mathbf{R}\}.$$

For further properties of these spaces, see [16–18].

Now if $\mathbf{n}_i$ denotes the unit outward normal to the edge e_i of Q, then for $\hat{\mathbf{v}} \in \mathbf{V}_h(\hat{Q})$,

$$|e_i|\mathbf{v} \cdot \mathbf{n}_i = \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}_i, \qquad i = 1, 2, 3, 4, \quad (4.13)$$

where $\hat{\mathbf{n}}_i$ is the unit exterior normal to $\hat{e}_i$. Due to (4.13) every $\mathbf{v} \in \mathbf{V}_h$ has constant normal components on the edges, which can be used as degrees of freedom. We remind the reader that $\mathbf{v}$ is no longer a polynomial on Q unless $\mathbf{Q}$ is a parallelogram and that its divergence is given by

$$\text{div } \mathbf{v}|_Q = \frac{1}{J}\int_Q \text{div } \mathbf{v} \, dxdy, \quad (4.14)$$

which *is not* a constant. Denote the edge-based basis for $\mathbf{V}_h(\hat{Q})$ by

$$\hat{\phi}_{x,0} = \begin{pmatrix} 1 - \hat{x} \\ 0 \end{pmatrix}, \quad \hat{\phi}_{x,1} = \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix}, \quad \hat{\phi}_{y,0} = \begin{pmatrix} 0 \\ 1 - \hat{y} \end{pmatrix}, \quad \hat{\phi}_{y,1} = \begin{pmatrix} 0 \\ \hat{y} \end{pmatrix}. \quad (4.15)$$

Remark 4.1. We note that $\hat{\phi}_{x,0}$ is a horizontal flow, linearly decreasing from 1 to 0, $\hat{\phi}_{y,0}$ is a vertical flow, linear decreasing from 1 to 0, and so on.

Thus we can easily glue together different pieces to get the basis of $\mathbf{V}_h$. For a "vertical" edge $e_{i,j+1/2}$, we associate with it a basis function (representing a "rightward horizontal flow" confined in two boxes $Q_{i,j}$, $Q_{i-1,j}$):

$$\phi_{i,j+1/2} = \begin{cases} \mathcal{P}_{Q_{i,j}}\hat{\phi}_{x,0} & \text{on } Q_{i,j}, \\ \mathcal{P}_{Q_{i-1,j}}\hat{\phi}_{x,1} & \text{on } Q_{i-1,j}, \\ 0 & \text{elsewhere.} \end{cases} \quad (4.16)$$

Similarly, we associate a "horizontal" edge $e_{i+1/2,j}$ a basis function (representing an "upward vertical flow" confined in two boxes):

$$\phi_{i+1/2,j} = \begin{cases} \mathcal{P}_{Q_{i,j}}\hat{\phi}_{y,0} & \text{on } Q_{i,j}, \\ \mathcal{P}_{Q_{i,j-1}}\hat{\phi}_{y,1} & \text{on } Q_{i,j-1}, \\ 0 & \text{elsewhere.} \end{cases} \quad (4.17)$$

More precisely, $\phi_{i,j+1/2}$ has *unit* flux through the edge $e_{i,j+1/2}$ and has *zero* flux through all the other edges, and similarly for $\phi_{i+1/2,j}$.

Our pressure space X_h will be the standard isoparametric $Q1$ conforming finite element space on quadrilaterals:

$$X_h = \{p \in H_0^1(\Omega) : p|_Q = F_Q(\hat{p}) \; \forall \hat{p} \in X_h(\hat{Q})\},$$

where $X_h(\hat{Q})$ is the local space on $\hat{Q}$,

$$X_h(\hat{Q}) = \{\hat{p} : \hat{p} \in Q1\}.$$

Consider the problem of finding the approximate solution $p_h \in X_h$ to p such that

$$\int_\Omega \mathcal{K}\nabla p_h \cdot \nabla q_h dx = \int_\Omega fq_h dx, \qquad \forall q_h \in X_h. \tag{4.18}$$

Having obtained p_h by (4.18), we turn to the construction of the approximate flux $\mathbf{u}_h$. As before the cluster at an arbitrary vertex (x_i, y_j) of the partition $\mathcal{D}_h$ is the set C_{ij}^h made up of those quadrilaterals Q in $\mathcal{D}_h$ sharing (x_i, y_j) as the common vertex. A typical C_{ij}^h is still like one shown in Fig. 1 with rectangles replaced by quadrilaterals. Of course, clusters can be at boundary nodes. First we define a subspace of $\mathbf{V}_h$ as follows:

$$\mathbf{V}_h(C_{ij}^h) = \{\mathbf{w}_h \in \mathbf{V}_h, \text{support}(\mathbf{w}_h) = C_{ij}^h \text{ and } \mathbf{w}_h \cdot \mathbf{n} = 0 \text{ on } \partial C_{ij}^h\},$$

where $\mathbf{n}$ is the unit outward normal to the boundary ∂C_{ij}^h. It is understood that the symbol ∂C_{ij}^h excludes those sides that are on the boundary of Ω.

Then we introduce the following problem: Find $\mathbf{u}_{ij}^h \in \mathbf{V}_h(C_{ij}^h)$ such that for all $Q \in C_{ij}^h$

$$\int_Q \mathbf{div}\, \mathbf{u}_{ij}^h dxdy = -\int_Q \mathcal{K}\nabla p_h \cdot \nabla\lambda_{ij}dx + \int_Q f\lambda_{ij}dxdy. \tag{4.19}$$

It can be easily checked that the Piola transformation in (4.7) preserves curl, i.e.,

$$\mathbf{curl}\lambda = \mathcal{P}_Q\mathbf{curl}\hat{\lambda}, \tag{4.20}$$

where the curl operator on the right side is on $\hat{Q}$ and the left one is on Q. In particular, we shall need this result for λ_{ij}, i.e., the unique function in X_h that is one at (x_i, y_j) and zero at all other vertices.

Theorem 4.2.   *A solution to (4.19) exists and has the form*

$$\mathbf{u}_{ij}^h = \mathbf{u}_{ij}^{h,*} + \alpha_{ij}\mathbf{curl}\lambda_{ij}, \qquad \alpha_{ij} \in \mathbf{R}, \tag{4.21}$$

where $\mathbf{u}_{ij}^{h,*}$ is a particular solution of (4.19).

Proof.   We only show the case in which (x_i, y_j) is an internal vertex of the partition $\mathcal{T}_h$. The rest of proof is like in the rectangular case and is omitted it.

We adopt the old notation. In Fig. 1, distort those rectangles a little and call them $Q_{ij}^{(m)}$ instead of $K_{ij}^{(m)}$, $m = 1, \ldots, 4$. Denote by $I_{ij}^{(l)}$, the integrals

$$-\int_{Q_{ij}^{(l)}} \mathcal{H}\nabla p_h \cdot \nabla \lambda_{ij} dx + \int_{Q_{ij}^{(l)}} f\lambda_{ij} dx, \qquad l = 1, \ldots, 4.$$

Let $K^{(m)} = K_{ij}^{(m)}$ (drop the subscript for simplicity) and let the four edges connected to (x_i, y_j) be defined as e_i, $i = 1, \ldots, 4$, starting with $e_1 = Q^{(1)} \cap K^{(4)}$ and then counterclockwise. We preassign four unit normals ν_i as shown in Fig. 2. Then obviously the space $\mathbf{V}_h(C_{ij}^h)$ has dimension 4 and is spanned by ϕ_k with the flux conditions $\int_{e_m} \phi_k \cdot \nu_m ds = \delta_{km}$, $k, m = 1, \ldots, 4$.

It is not necessary to know the specific form of these basis functions. We mention in passing that in terms of the basis functions in (4.16) and (4.17), $\phi_1 = \phi_{i+1/2,j}$ (upwards), $\phi_2 = -\phi_{i,j+1/2}$ (leftwards), $\phi_3 = -\phi_{i-1/2,j}$ (downwards), $\phi_4 = \phi_{i,j-1/2}$ (rightwards). This helps to visualize the situation.

Now since $\mathbf{u}_{ij}^h \in \mathbf{V}_h(C_{ij}^h)$, we can write it as

$$\alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4.$$

Substituting this into (4.19) and noticing that each ϕ_k is supported in only two quadrilaterals, we have the linear system

$$-\alpha_1 + \alpha_2 = I^{(1)}$$

$$-\alpha_2 + \alpha_3 = I^{(2)}$$

$$-\alpha_3 + \alpha_4 = I^{(3)}$$

$$-\alpha_4 + \alpha_1 = I^{(4)}. \tag{4.22}$$

The kernel is spanned by $(1, 1, 1, 1)$. Let $\mathbf{w}$ be a vector field such that its edge fluxes α_i are 1, 1, 1, 1. Then $\int_{Q^{(m)}} \text{div } \mathbf{w} dx dy = 0$ for every $Q^{(m)}$. (So far we see that the proof has proceeded exactly as in the rectangular case.)

Now let us look at $\int_{Q^{(m)}} \text{div } \mathbf{w} dx dy = 0$. Due to (4.10), div $\mathbf{w}$ is not a constant, because J is linear in $\hat{x}$ and $\hat{y}$. However, by (4.9) we have

$$0 = \int_{\hat{Q}} \text{div } \hat{\mathbf{w}} d\hat{x} d\hat{y}$$

and div $\hat{\mathbf{w}} = 0$ on $\hat{Q}$, being a constant. Now argue as in the rectangular case: for each $Q^{(m)}$, there exists a bilinear function $\hat{\lambda}_m$ vanishing on some two adjacent boundary edges of $\partial\hat{Q}$ and satisfying $\text{curl}\hat{\lambda}_m = \hat{\mathbf{w}}$. Furthermore, by (4.13)

$$1 = \mathbf{w} \cdot \nu_i |e_i| = \hat{\mathbf{w}} \cdot \hat{\nu}_i = \text{curl}\hat{\lambda}_m \cdot \hat{\nu}_i = \hat{\nabla}\hat{\lambda}_m \cdot \hat{\tau}_i = \frac{\partial \hat{\lambda}_m}{\partial \hat{\tau}_i},$$

where τ_i is the unit tangent vector along e_i that points to (x_i, y_j). We see that $\hat{\lambda}_m$ is one at the origin. Hence there exists a continuous piecewise "bilinear" function $\lambda_{ij} \in X_h(C_{ij}^h)$ with the value one at (x_i, y_j) and zero on the boundary ∂C_{ij}^h. For each m, $\hat{\lambda}_m$ is bilinear and $\hat{w}$ is divergence free on $\hat{Q}$. Finally, the curl statement in (4.21) comes from (4.20). ∎

Note that Remarks 2.1 and 2.2 hold for the quadrilateral case as well.

B. Error Estimates

We now choose an α_{ij} in Theorem 4.2 so that the error in the conservative velocity is first order. First let us recall that the Raviart-Thomas interpolant $\Pi_h : H^1(\Omega)^2 \to \mathbf{V}_h$ is defined as follows [18]: define $\hat{\Pi}$ on $\hat{Q}$ via the following degrees of freedom:

$$\int_{\hat{e}} \hat{\Pi}\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}ds = \int_{\hat{e}} \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}ds \qquad \forall \text{ edges } \hat{e} \text{ of } \hat{Q},$$

and then set

$$\Pi_Q \mathbf{v} = \mathcal{P}_Q(\hat{\Pi}\widehat{\mathbf{v}}) \qquad \forall \mathbf{v} \in (H^1(Q))^2,$$

where $\mathcal{P}_Q \hat{\mathbf{v}} = \mathbf{v}$. Finally, we define

$$\Pi_h \mathbf{v}|_Q = \Pi_Q \mathbf{v}. \tag{4.23}$$

Theorem 4.3. *Assume that $\mathcal{K} \in W^{1,\infty}(\Omega)$ and $f \in L^2(\Omega)$. Let the partition $\mathcal{D}_h$ of the domain be regular. Let $\mathbf{u}_h$ be defined as*

$$\mathbf{u}^h = \sum_{(x_i,y_j) \in \bar{\mathcal{N}}^h} \mathbf{u}_{ij}^h,$$

where $\mathbf{u}_{ij}^h$ is defined in (4.21) and the coefficient

$$\alpha_{ij} = -\int_{\gamma_{ij}^1} \left(\mathbf{u}_{ij}^{h,*} \cdot \nu + \frac{1}{2} \bar{\mathcal{K}} \overline{\nabla p_h} \cdot \nu \right) ds,$$

where γ_{ij}^1 is an edge incident from (x_i, y_j) and ν its associated unit normal as shown in Fig. 3. Then there exists a constant $C > 0$, independent of h, f, p, and $\mathbf{u}$ such that

$$\|\mathbf{u} - \mathbf{u}^h\|_0 \leq Ch(|\mathbf{u}|_1 + \|f\|_0 + \|p\|_2).$$

Proof. For ease of presentation, we will use the symbols $\tilde{\mathbf{u}} \in \mathbf{V}_h$ and $\Pi_h \mathbf{v}$ exchangeably to stand for the same Raviart–Thomas interpolant of $\mathbf{u}$ throughout the proof. Recall the following error estimates [3, 15] for quadrilateral grids

$$\|\tilde{\mathbf{u}} - \mathbf{u}\|_{0,Q} \leq Ch|\mathbf{u}|_{1,Q}. \tag{4.24}$$

Define on Q the error $\mathbf{e}_Q = \mathbf{u}^h - \tilde{\mathbf{u}}$, we have

$$\int_Q \operatorname{div} \mathbf{e}_Q dxdy = \int_Q \operatorname{div} \mathbf{u}^h dxdy - \int_Q \operatorname{div} \tilde{\mathbf{u}} dxdy = 0,$$

where we have used the conservation property of $\mathbf{u}^h$ and the interpolation degrees of freedom of $\tilde{\mathbf{u}}$. Consequently, by (4.9)

$$0 = \int_Q \operatorname{div} \mathbf{e}_Q dxdy = \int_{\hat{Q}} \operatorname{div} \hat{\mathbf{e}}_Q d\hat{x} d\hat{y}.$$

Since $\operatorname{div} \hat{\mathbf{e}}_Q$ is a constant on $\hat{Q}$ we have

$$\operatorname{div} \hat{\mathbf{e}}_Q = 0.$$

Also by (4.13) and with $|s|$ denoting the length of the side s normal to ν, we have

$$\hat{\mathbf{e}}_Q \cdot \hat{\nu} = (\mathbf{e}_Q \cdot \nu)|s| = (\mathbf{u}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)|s|$$

$$= \hat{\mathbf{u}}^h \cdot \hat{\nu} - \hat{\Pi}\hat{\mathbf{u}} \cdot \hat{\nu}.$$

Thus there exists a function $\hat{\varphi}_Q$ such that

$$\begin{cases} \hat{\mathbf{e}}_Q = \nabla \hat{\varphi}_Q \quad \text{and} \quad \int_{\hat{Q}} \hat{\varphi}_Q d\hat{x} d\hat{y} = 0, \\ \hat{\varphi}_Q \in H^1(\hat{Q}). \end{cases}$$

In fact, $\hat{\varphi}_Q$ is a solution of

$$\begin{cases} -\Delta \hat{\varphi}_Q = 0, \quad \text{in } \hat{Q}, \\ \int_{\hat{Q}} \hat{\varphi}_Q dxdy = 0, \\ \dfrac{\partial \hat{\varphi}_Q}{\partial \hat{\nu}} = \hat{\mathbf{u}}^h \cdot \hat{\nu} - \hat{\Pi}\hat{\mathbf{u}} \cdot \hat{\nu} \quad \text{on } \partial \hat{Q}, \hat{\varphi}_Q \in H^1(\hat{Q}). \end{cases} \tag{4.25}$$

From (4.25), we have

$$0 = \int_{\hat{Q}} -\Delta \hat{\varphi}_Q \cdot \hat{\varphi}_Q d\hat{x} d\hat{y} = -\int_{\partial \hat{Q}} (\nabla \hat{\varphi}_Q \cdot \hat{\nu}) \hat{\varphi}_Q d\hat{s} + \int_{\hat{Q}} \nabla \hat{\varphi}_Q \cdot \nabla \hat{\varphi}_Q d\hat{x} d\hat{y}.$$

Now on the one hand, $\|\mathbf{e}_Q\|_{0,Q}^2 \leq C\|\hat{\mathbf{e}}_Q\|_{0,\hat{Q}}^2$ by (4.11), and on the other hand by the last equation we have

$$\|\hat{\mathbf{e}}_Q\|_{0\hat{Q}}^2 = |\hat{\varphi}_Q|_{1\hat{Q}}^2 = \int_{\partial\hat{Q}} (\nabla\hat{\varphi}_Q \cdot \hat{\nu})\hat{\varphi}_Q d\hat{s}$$

$$= \int_{\partial\hat{Q}} (\hat{\mathbf{u}}^h \cdot \hat{\nu} - \hat{\Pi}\hat{\mathbf{u}} \cdot \hat{\nu})\hat{\varphi}_Q d\hat{s}$$

$$= \int_{\partial Q} (\mathbf{u}^h \cdot \nu - \mathbf{u} \cdot \nu)\varphi_Q ds. \qquad (4.26)$$

(Note that this last expression played a central role in the rectangular case.) Thus it suffices to estimate the last term again. Now let the four vertices of Q be $A = (x_i, y_j)$, $B = (x_{i+1}, y_j)$, $C = (x_{i+1}, \hat{j}_{j+1})$, and $D = (x_i, y_{j+1})$. Let's locally label the segments AB, BC, CD, DA as γ_1, γ_2, γ_3, γ_4, respectively. As in the rectangular case,

$$\int_{\partial Q} (\mathbf{u}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)\varphi_Q ds = \int_{\partial Q} (\mathbf{u}_{ij}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)\varphi_Q ds + \int_{\partial Q} (\mathbf{u}_{i+1,j}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)\varphi_Q ds$$

$$+ \int_{\partial Q} (\mathbf{u}_{i+1,j+1}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)\varphi_Q ds + \int_{\partial Q} (\mathbf{u}_{i,j+1}^h \cdot \nu - \tilde{\mathbf{u}} \cdot \nu)\varphi_Q ds$$

$$= I_1 + I_2 + I_3 + I_4. \qquad (4.27)$$

These I terms are handled exactly like in the rectangular case and we have *our main inequality:*

$$C\|\mathbf{e}_Q\|_{0,Q}^2 \leq \sum_{(r,s)\in\mathcal{I}_Q} \sum_{m=1}^2 \int_{\gamma_{(rs)}^{(m)}} \left(\mathbf{u}_{rs}^h \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu\right)\varphi_Q ds, \qquad (4.28)$$

where (r, s) runs through $\mathcal{I}_Q = \{(i, j), (i + 1, j), (i + 1, j + 1), (i, j + 1)\}$ and for a vertex (x_r, y_s), $\gamma_{rs}^{(m)}$, $m = 1, 2$ are the two sides of Q sharing that vertex as a common extremity. Hence, to bound $\|\mathbf{e}_Q\|_{0,Q}^2$ it suffices to estimate a typical term like $\int_{\gamma_{ij}^{(m)}} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)\varphi_Q ds$, $m = 1, 2$. Note that for $m = 1, 2$,

$$\left|\int_{\gamma_{ij}^{(m)}} \left(\mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu\right)\varphi_Q ds\right| \leq \left|\mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu\right|_{|\gamma_{ij}^{(m)}} \sqrt{|\gamma_{ij}^{(m)}|} \|\varphi_Q\|_{0,\gamma_{ij}^{(m)}}, \qquad (4.29)$$

where $|_{\gamma_{ij}^{(m)}}$ stands for the restriction to $\gamma_{ij}^{(m)}$.

Using the same notation as in the rectangular case, we see that on the one hand, for $k = 1$, 2, 3, 4,

$$X_{ij}^k := \int_{\gamma_{ij}^k} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)ds = \int_{\gamma_{ij}^k} \left(\mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu\right)ds = |\gamma_{ij}^k|\left[\mathbf{u}_{ij}^h \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu\right]\Big|_{\gamma_{ij}^k}.$$

On the other hand, referring to Fig. 1, it still holds that

$$X_{ij}^{k+1} - X_{ij}^k = \int_{Q^k} (f - \operatorname{div}\tilde{\mathbf{u}})\lambda_{ij}dxdy - \int_{Q^k} (\tilde{\mathbf{u}} + \mathcal{K}\nabla p_h) \cdot \nabla\lambda_{ij}dxdy.$$

Then by the triangle inequality and (4.24), one gets, with $Q^1 = Q$,

$$|X_{ij}^2 - X_{ij}^1| \leq Ch(\|f\|_{0,Q} + |\mathbf{u}|_{1,Q}) + C|p - p_h|_{1,Q}. \tag{4.30}$$

Now we turn to analyzing X_{ij}^1. Let

$$\bar{\mathcal{K}} = \frac{1}{|Q|}\int_Q \mathcal{K}dxdy, \qquad \overline{\nabla p_h} = \frac{1}{|Q|}\int_Q \nabla p_h dxdy.$$

Then

$$X_{ij}^1 = \int_{\gamma_{ij}^1} (\mathbf{u}_{ij}^h \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu)ds$$

$$= \int_{\gamma_{ij}^1} [(\alpha_{ij}\mathbf{curl}\lambda_{ij} + \mathbf{u}_{ij}^{h,*}) \cdot \nu - \lambda_{ij}\tilde{\mathbf{u}} \cdot \nu]ds$$

$$= \alpha_{ij} + \int_{\gamma_{ij}^1} \left(\mathbf{u}_{ij}^{h,*} \cdot \nu - \frac{1}{2}\tilde{\mathbf{u}} \cdot \nu\right)ds$$

$$= \alpha_{ij} + \int_{\gamma_{ij}^1} \left(\mathbf{u}_{ij}^{h,*} \cdot \nu + \frac{1}{2}\bar{\mathcal{K}}\overline{\nabla p_h} \cdot \nu\right)ds - \frac{1}{2}\int_{\gamma_{ij}^1} (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nu ds.$$

Making a choice of $\alpha_{ij} = -\int_{\gamma_{ij}^1} (\mathbf{u}_{ij}^{h,*} \cdot \nu + (1/2)\bar{\mathcal{K}}\overline{\nabla p_h} \cdot \nu)ds$, one gets

$$X_{ij}^1 = -\frac{1}{2}\int_{\gamma_{ij}^1} (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nu ds.$$

Let ξ be such that $\hat{\xi} = \hat{x}(1 - \hat{x})(1 - \hat{y})$, a cubic polynomial vanishing on three sides of $\hat{Q}$ and is 1/4 at (1/2, 0). Then by the Simpson's rule

$$\int_{\gamma_{ij}^1} \xi ds = \frac{1}{6}|\gamma_{ij}^1|, \qquad \|\xi\|_{0,Q} \le Ch, \qquad |\xi|_{1,Q} \le C.$$

Using the Gauss formula, we have

$$X_{ij}^1 = -3\int_{\gamma_{ij}^1} (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nu\xi ds$$

$$= -3\left\{\int_Q (\bar{\mathcal{K}}\overline{\nabla p_h} + \tilde{\mathbf{u}}) \cdot \nabla\xi dxdy + \int_Q \xi\,\mathbf{div}\,\tilde{\mathbf{u}}dx.\right.$$

As before, we deduce that

$$|X_{ij}^1| \le Ch(|p|_{1,Q} + |p|_{2,Q} + |\mathbf{u}|_{1,Q} + \|f\|_{0,Q}) + C|p - p_h|_{1,Q}, \tag{4.31}$$

and an iterated argument leads to

$$|X_{ij}^l| \le Ch(|p|_{1,C_{ij}^h} + |p|_{2,C_{ij}^h} + |\mathbf{u}|_{1,C_{ij}^h} + \|f\|_{0,C_{ij}^h}) + C|p - p_h|_{1,C_{ij}^h}, \qquad l = 1, \ldots, 4. \tag{4.32}$$

The rest of the proof is just like what follows (3.11). This completes the proof. ∎

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