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The Singularity at the Apex of a Bonded Wedge-Shaped Stamp

The problem of determining the singularity at the apex of a rigid wedge bonded to an elastic half space is formulated by considerations of Green's functions for the loaded half space. The eigenvalue problem is reduced to finding the solution of a coupled pair of singular integral equations. A numerical solution for small wedge angles is given.

Introduction

In two recent papers the authors [1, 2] have studied the singular behavior of stresses in the neighborhood of the corner of a wedge-shaped crack under normal and shear loading, respectively. The results of [1, 2] apply also to the stamp problems in which wedge-shaped stamps on half spaces are under normal loading [1] and shear loading [2]. When a wedge-shaped stamp is bonded to the half space and loaded arbitrarily, the problem becomes far more difficult mathematically. For example, the two cases just mentioned will have singularity exponents of $-1/2$ along the wedge edges while for the bonded stamp case the singularity exponents will be complex.

In the present paper the eigenvalue problem for the strength of the singularity at the apex of a rigid wedge-shaped stamp, bonded to an elastic half space and loaded arbitrarily, is reduced to that of solving a coupled pair of singular integral equations. In theory, there are some numerical methods which may be used to solve the integral equations previously mentioned; but at least for the type of kernels that arise in the present analysis, these methods seem to be highly complicated in practice. In view of this, the general numerical treatment of the singular integral equations is postponed. Nevertheless, when the wedge angle is close to zero, an approximate analytical expression for the eignevalue problem is derived. Some numerical results are obtained from the approximate expression which are believed to be reasonably good and illustrate the solution behavior. Thus numerical results are given when the wedge angles are less than about 50°. Such a determination of the approximate results may be of considerable aid in applying a numerical scheme for the general case.

Reduction to Integral Equations

Let the half space $z > 0$ be occupied by an elastic material with rigidity modulus μ and Poisson's ratio ν. The displacement field in the half space due to a point force acting at the origin is well known (see [3, pp. 142-148]). If there is a continuously distributed system of point forces acting over a region D of the plane boundary $z = 0$, one can obtain the corresponding displacements by integration of those due to individual point forces over D (cf. [4, p. 185]). Thus the displacement components on $z = 0$ due to a continuously distributed system of point forces over a region D of $z = 0$ may be written

$$u_x(x, y) = \iint_D \left\{F_x(\xi, \eta)\left[\frac{1}{R} + \frac{b(x-\xi)^2}{R^3}\right] + F_y(\xi, \eta)\left[\frac{b(x-\xi)(y-\eta)}{R^3}\right]\right\} d\xi d\eta - \frac{1}{2}h \iint_D F_z(\xi, \eta)\left[\frac{(x-\xi)}{R^2}\right] d\xi d\eta, \quad (1)$$

$$u_y(x, y) = \iint_D \left\{F_x(\xi, \eta)\left[\frac{b(x-\xi)(y-\eta)}{R^3}\right] + F_y(\xi, \eta)\left[\frac{1}{R} + \frac{b(y-\eta)^2}{R^3}\right]\right\} d\xi d\eta - \frac{1}{2}h \iint_D F_z(\xi, \eta)\left[\frac{(y-\eta)}{R^2}\right] d\xi d\eta, \quad (2)$$

$$u_z(x, y) = \frac{1}{2}h \left\{\iint_D F_x(\xi, \eta)\left[\frac{(x-\xi)}{R^2}\right] d\xi d\eta + \iint_D F_y(\xi, \eta)\left[\frac{(y-\eta)}{R^2}\right] d\xi d\eta\right\} + \iint_D F_z(\xi, \eta)\left[\frac{1}{R}\right] d\xi d\eta, \quad (3)$$

where F_x, F_y, F_z are proportional to the point force densities in x, y, z directions, respectively, and

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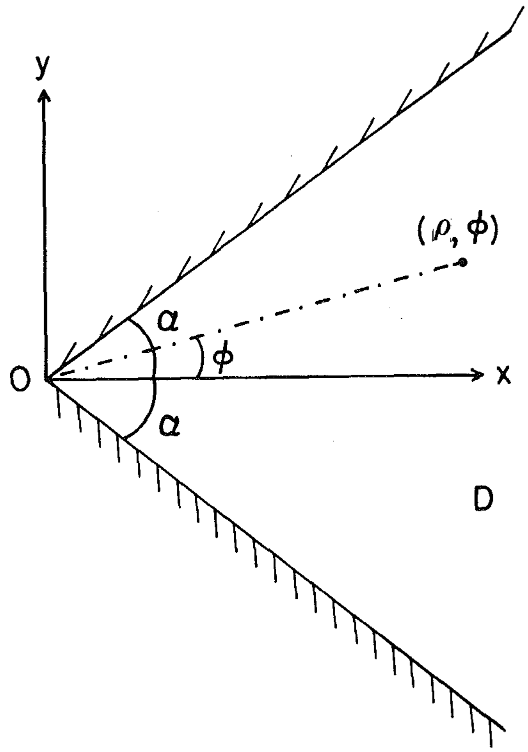

Fig. 1 The boundary $z = 0$ (x, y-plane) of the half space $z \geq 0$. The wedge-shaped region D of the boundary is in contact with the stamp. The remaining portion of the boundary is free from tractions.

$$R = [(x-\xi)^2 + (y-\eta)^2]^{1/2}, \tag{4}$$

$$b = \nu/(1-\nu), \quad h = 1 - b. \tag{5}$$

If $\nu = 1/2$, then $h = 0$ and the displacements (1)–(2) reduce to the same form as in the stamp problem with shear loading [2] while (3) becomes the displacement for the stamp problem with normal loading [1]. Naturally, the ensuing details are similar to those encountered in the previous analysis [1, 2].

Let D be the wedge-shaped region defined in polar coordinates by $-\alpha < \phi < \alpha$, $0 < \rho < \infty$ (see Fig. 1). Introduce the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta \tag{6}$$

$$\xi = \rho\cos\phi, \quad \eta = \rho\sin\phi \tag{7}$$

and use the polar transformations for the displacements

$$u_r = u_x\cos\theta + u_y\sin\theta \tag{8}$$

$$u_\theta = u_y\cos\theta - u_x\sin\theta \tag{9}$$

and for the force densities

$$R_r(\rho,\phi) = F_x(\rho\cos\phi, \rho\sin\phi)\cos\phi + F_y(\rho\cos\phi, \rho\sin\phi)\sin\phi, \tag{10}$$

$$R_\theta(\rho,\phi) = F_y(\rho\cos\phi, \rho\sin\phi)\cos\phi - F_x(\rho\cos\phi, \rho\sin\phi)\sin\phi, \tag{11}$$

$$R_z(\rho,\phi) = F_z(\rho\cos\phi, \rho\sin\phi). \tag{12}$$

Let the prescribed displacements on the stamp be independent of r (there is no loss of generality for the strength of singularity at the wedge apex is all that is of concern here). That is,

$$(\partial/\partial r)u_r(r,\theta) = 0, \quad (\partial/\partial r)u_\theta(r,\theta) = 0, \quad (\partial/\partial r)u_z(r,\theta) = 0, \quad (r,\theta)\in D. \tag{13}$$

We assume that the half space is not subjected to loads antisymmetric about the xz-plane, so that the functions $R_\rho(\rho,\phi)$ and $R_z(\rho,\phi)$ can be assumed to be even functions of ϕ while $R_\theta(\rho,\phi)$ is an odd function of ϕ. As in [1, 2], we set

$$R_r(\rho,\phi) = \rho^{\gamma-1}A(\phi), \quad R_\theta(\rho,\phi) = \rho^{\gamma-1}B(\phi), \quad R_z(\rho,\phi) = \rho^{\gamma-1}G(\phi), \quad 0 < \mathrm{Re}\,(\gamma) < 1, \tag{14}$$

where $B(\phi)$ is an odd while $A(\phi)$, $G(\phi)$ are even functions of ϕ. The representations (14) are supposed to be accurate in a small neighborhood of the wedge apex where the parameter γ determines the strength of singularity.

Substituting R_r, R_θ, R_z from (14) into the implicitly written equations (13), interchanging the order of integrations, making use of the integral (see [5, p. 309])

$$\int_0^\infty \frac{x^{s-1}dx}{[1 + 2x\cos\theta + x^2]} = -\pi\,\mathrm{cosec}\,\pi s\,\mathrm{cosec}\,\theta\sin(s-1)\theta, \quad |\theta| < \pi, \quad 0 < \mathrm{Re}\,(s) < 2, \tag{15}$$

and carrying out the simplifications as in [1, 2] one finds

$$\int_{-\alpha}^{\alpha} A(\phi)[(1+b+b\gamma)\cos(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\} + b\gamma P_{\gamma-1}\{-\cos(\theta-\phi)\}]d\phi + (1-b\gamma)\int_{-\alpha}^{\alpha} B(\phi)\sin(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\}d\phi - \frac{1}{2}h\int_{-\alpha}^{\alpha} G(\phi)\cos(\gamma+1)\{\pi - |\theta-\phi|\}d\phi = 0, \quad -\alpha < \theta < \alpha, \tag{16}$$

$$(1+b+b\gamma)\int_{-\alpha}^{\alpha} A(\phi)\sin(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\}d\phi + \int_{-\alpha}^{\alpha} B(\phi)[-(1-b\gamma)\cos(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\} + b\gamma P_{\gamma-1}\{-\cos(\theta-\phi)\}]d\phi + \frac{1}{2}h\int_{-\alpha}^{\alpha} G(\phi)\,\mathrm{sgn}\,(\theta-\phi) \times \sin(\gamma+1)\{\pi - |\theta-\phi|\}d\phi = 0, \quad -\alpha < \theta < \alpha, \tag{17}$$

$$\frac{1}{2}h\int_{-\alpha}^{\alpha} [A(\phi)\cos\gamma\{\pi - |\theta-\phi|\} + B(\phi)\,\mathrm{sgn}\,(\theta-\phi)\sin\gamma\{\pi - |\theta-\phi|\}]d\phi - \int_{-\alpha}^{\alpha} G(\phi)P_\gamma\{-\cos(\theta-\phi)\}d\phi = 0, \quad -\alpha < \theta < \alpha, \tag{18}$$

where sgn denotes the signum function and the material constants b, h are given by (5). Given the wedge angle 2α, the simultaneous system (16)–(18) of integral equations represents an eigenvalue problem for the parameter γ. However, in their present form (16)–(18), these integral equations obscure the behavior of unknown functions A, B, G. Therefore, the next step is to put them into a standard form.

From the differentiation formulas for Legendre functions (see [6, p. 161]) it follows that

$$\frac{\partial}{\partial\theta}[\cos(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\}] = -\left[\frac{\partial}{\partial\theta}P_{\gamma-1}\{-\cos(\theta-\phi)\} + (\gamma+1)\sin(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\}\right], \tag{19}$$

$$\frac{\partial}{\partial\theta}[\sin(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\}] = (\gamma+1) \times \cos(\theta-\phi)P_\gamma\{-\cos(\theta-\phi)\} + \gamma P_{\gamma-1}\{-\cos(\theta-\phi)\}. \tag{20}$$

In view of these formulas, differentiating (16) with respect to θ and eliminating G using (17) one finds

$$\frac{\partial}{\partial\theta}\int_{-\alpha}^{\alpha} A(\phi)P_{\gamma-1}\{-\cos(\theta-\phi)\}d\phi \tag{21}$$

$$= \gamma \int_{-\alpha}^{\alpha} B(\phi) P_{\gamma-1}\{-\cos(\theta - \phi)\} d\phi. \tag{21 Cont.}$$

Set

$$W(\phi) = A(\phi) - \gamma V(\phi), \quad V(\phi) = \int_{-\alpha}^{\phi} B(t) dt, \tag{22}$$

where, since A and B are even and odd functions, respectively, the function W is even and $V(\pm\alpha) = 0$. Then, using integration by parts, equation (21) yields

$$\int_{-\alpha}^{\alpha} W(\phi) P_{\gamma-1}\{-\cos(\theta - \phi)\} d\phi = C, \quad -\alpha < \theta < \alpha, \tag{23}$$

where C is an arbitrary constant. The foregoing equation suggests (see [1]) that $W(\phi)$ and hence $A(\phi)$ have square root singularity at $\phi = \pm\alpha$.

Differentiating (17) with respect to θ and using (16), (19)–(20) and (23) gives

$$\begin{aligned} h \sin \pi\gamma G(\theta) &+ \frac{\partial}{\partial\theta} \int_{-\alpha}^{\alpha} B(\phi) P_\gamma\{-\cos(\theta - \phi)\} d\phi \\ &- \gamma \int_{-\alpha}^{\alpha} B(\phi) \left\{ \tan\left(\frac{\theta - \phi}{2}\right) [P_{\gamma-1}\{-\cos(\theta - \phi)\} \right. \\ &\left. - P_\gamma\{-\cos(\theta - \phi)\}] + T_{\gamma-1}(\theta - \phi) \right\} d\phi \\ &= \gamma C, \quad -\alpha < \theta < \alpha, \end{aligned} \tag{24}$$

where C is the arbitrary constant and

$$\begin{aligned} T_\nu(y) &= \int_0^y P_\nu\{\cos(\pi - |t|)\} dt, \quad y > 0 \\ T_\nu(y) &= -T_\nu(-y), \qquad\qquad y < 0. \end{aligned} \tag{25}$$

Since (23) is derived from (16)–(17), equation (24) is essentially a result of eliminating $A(\phi)$ from (16)–(17). Also, the differentiation of (18) with respect to θ yields

$$\begin{aligned} h \sin \pi\gamma B(\theta) &- \frac{\partial}{\partial\theta} \int_{-\alpha}^{\alpha} G(\phi) P_\gamma\{-\cos(\theta - \phi)\} d\phi \\ &= \lambda_0(\theta), \quad -\alpha < \theta < \alpha, \end{aligned} \tag{26}$$

with

$$\lambda_0(\theta) = -\frac{1}{2} h\gamma \int_{-\alpha}^{\alpha} W(\phi) \operatorname{sgn}(\theta - \phi) \sin \gamma\{\pi - |\theta - \phi|\} d\phi. \tag{27}$$

where W is given by (22). Since W satisfies the integral equation (23), it may be regarded as known. Thus the unknown functions G, B satisfy the coupled pair of integral equations (24) and (26). Since G and B are, respectively, even and odd functions, if one sets

$$f(\theta) = G(\theta) + iB(\theta) \tag{28}$$

then

$$f(\theta) + f(-\theta) = 2G(\theta), \quad f(\theta) - f(-\theta) = 2iB(\theta). \tag{29}$$

Now, multiplying (26) by i and adding to (24) one can write equations (24) and (26) in the combined form

$$\begin{aligned} h \sin \pi\gamma f(\theta) &+ i \int_{-\alpha}^{\alpha} f(\phi) K(\theta, \phi) d\phi \\ &= \gamma C + i\lambda_0(\theta), \quad -\alpha < \theta < \alpha, \end{aligned} \tag{30}$$

where λ_0 is given by (27) and where

$$K(\theta, \phi) = -\frac{\partial}{\partial\theta} P_\gamma\{-\cos(\theta - \phi)\} + \frac{1}{2}\gamma[L(\theta - \phi) - L(\theta + \phi)], \tag{31}$$

$$L(x) = \tan\left(\frac{x}{2}\right)[P_{\gamma-1}(-\cos x) - P_\gamma(-\cos x)] + \gamma T_{\gamma-1}(x). \tag{32}$$

It might be noted that the kernel K of the integral equation (30) has Cauchy-type singularity (cf. [1, 2]), so that the solution of (30) will involve an arbitrary constant in addition to C. Thus the integral equations (23) and (30) are equivalent to (16)–(18) only if supplemented by two additional conditions on the unknown functions f and W. This situation has arisen due to the differentiation of (16)–(18) in deriving (23) and (30); and one can obtain the additional conditions from (16)–(18) by assigning fixed values to the free variable θ. A suitable choice in this case is $\theta = 0$ and with some manipulations it can be shown that

$$\int_{-\alpha}^{\alpha} f(\phi) f_j(\phi) d\phi = \int_{-\alpha}^{\alpha} W(\phi) W_j(\phi) d\phi, \quad j = 1, 2 \tag{33}$$

where

$$f_1(\phi) = P_\gamma(-\cos\phi) - \frac{1}{2} ih \sin \pi\gamma \operatorname{sgn}(\phi), \tag{34}$$

$$W_1(\phi) = \frac{1}{2} h \cos \gamma\{\pi - |\phi|\}, \tag{35}$$

$$\begin{aligned} f_2(\phi) &= \frac{1}{2} h(\gamma + 1) \cos(\gamma + 1)\{\pi - |\phi|\} \\ &- i[(2\gamma + 1) \sin\phi P_\gamma(-\cos\phi) - \gamma^2 T_{\gamma-1}(\phi)], \end{aligned} \tag{36}$$

$$\begin{aligned} W_2(\phi) &= (\gamma + 1)[(1 + b + b\gamma) \cos\phi P_\gamma(-\cos\phi) \\ &+ b\gamma P_{\gamma-1}(-\cos\phi)]. \end{aligned} \tag{37}$$

The complete eigenvalue problem for the complex variable γ is given by the equations (23), (27), (30)–(37), (25), and (5). The behavior of the unknown function $f(\phi)$ and hence that of $G(\phi)$ and $B(\phi)$ near $\phi = \pm\alpha$ can be determined by considering the dominant part of the singular integral equation (30). The details will appear in the next section devoted to an approximate solution. As pointed out earlier, the numerical solution of singular integral equation (30) in the general case seems to be extremely difficult by any of the known methods, and is therefore postponed.

Approximate Solution

When the corner angle 2α of the bond region is small, an approximate analytical expression can be derived for the eigenvalue problem. For small y an approximate expression [1, 2] for the Legendre function may be written

$$P_\gamma(-\cos y) \simeq (2/\pi) \sin \pi\gamma[\log|y| + k_0], \tag{38}$$

$$k_0 = \frac{1}{2} A_0 - \log 2, \quad A_0 = \psi(-\gamma) + \psi(1 + \gamma) - 2\psi(1), \tag{39}$$

where the ψ function is the logarithmic derivative of the gamma function. Then, following [1, 2] an approximate solution for the integral equation (23) is given by

$$W(\alpha t) = -\Delta/\{V_2\sqrt{1 - t^2}\}, \quad |t| < 1, \tag{40}$$

where

$$\Delta = \gamma C/(\pi\alpha), \quad V_2 = \gamma V_1 - (2/\pi) \sin \pi\gamma, \tag{41}$$

$$V_1 = \cos \pi\gamma + (2/\pi) \sin \pi\gamma[\gamma Q(\gamma) + \log(\alpha/4)], \tag{42}$$

$$Q(\gamma) = \sum_{n=1}^{\infty} \frac{1}{n(n + \gamma)}. \tag{43}$$

Using (38) and retaining the terms of order $\log\alpha$ and those independent of α, one can write the singular integral equation (30) in the form

$$\frac{2}{\pi i} \int_{-\alpha}^{\alpha} \frac{f(\phi) d\phi}{\phi - \theta} - hf(\theta) = 0, \quad -\alpha < \theta < \alpha. \tag{44}$$

This homogeneous integral equation has the nontrivial solution (see [7 p. 128])

$$f(\alpha x) = C_0(1 - x)^{-\delta}(1 + x)^{\delta - 1}, \tag{45}$$

where C_0 is an arbitrary constant and

Table 1 Singularity exponents for small wedge angles

Poisson's Ratio ν	Angle of Weld (2α/π)						
	0.04021	0.11610	0.12500	0.16670	0.22432	0.25000	0.28858
0.0	0.1269 + .0585i	0.1749 + .0743i	0.1797 + .0755i	0.2007 + .0803i	0.2277 + .0846i	0.2393 + .0858i	0.2561 + .0867i
0.25	0.1264 + .0340i	0.1725 + .0431i	0.1769 + .0438i	0.1966 + .0467i	0.2218 + .0494i	0.2325 + .0501i	0.2481 + .0509i
0.30	0.1265 + .0280i	0.1724 + .0353i	0.1768 + .0359i	0.1964 + .0381i	0.2214 + .0400i	0.2320 + .0405i	0.2475 + .0409i
0.35	0.1266 + .0215i	0.1725 + .0266i	0.1769 + .0270i	0.1965 + .0283i	0.2214 + .0291i	0.2319 + .0292i	0.2474 + .0288i
0.40	0.1269 + .0141i	0.1729 + .0162i	0.1774 + .0162i	0.1969 + .0160i	0.2219 + .0143i	0.2325 + .0130i	0.2479 + .0098i
0.425	0.1271 + .0098i	0.1733 + .0091i	0.1777 + .0087i	0.1974 + .0056i	0.2144	0.2221	0.2338
0.450	0.1274 + .0038i	0.1658	0.1695	0.1857	0.2061	0.2148	0.2274
0.475	0.1341	0.1614	0.1652	0.1818	0.2027	0.2114	0.2242

$$\delta = \frac{1}{2} + ic, \quad c = \frac{1}{2\pi} \log\left(\frac{2+h}{2-h}\right). \tag{46}$$

Equation (46) gives the usual singularity exponents that arise in plane elastostatics for a bonded rigid stamp (see e.g., reference [8]). It is interesting to note that Mode I and Mode II behavior produce complex-valued singularity exponents along the edge as shown in equation (46) while the Mode III will give a real-valued one. Thus correct behavior of solutions at the edges of the wedge is noted. Then, substituting W and f from (40) and (45) into the conditions (33), retaining terms of the order $\log \alpha$ and those independent of α, and eliminating the arbitrary constants C_0 and Δ yield the eigenvalue equation

$$[\pi V_1\sqrt{4-h^2} + (2/\pi)\sin \pi\gamma\{(\sqrt{4-h^2}-2)\pi \log 2 - 8L_0 + \pi h J_0\}]$$
$$\times [(1+b)\pi V_1 + 2b \sin \pi\gamma] + \frac{1}{4}\sqrt{4-h^2}(\pi h \cos \pi\gamma)^2 = 0, \tag{47}$$

where b, h, and V_1 are given by (5) and (42)–(43) and the integrals J_0, L_0 may be written

$$J_0 = \int_0^\infty \sin(2cy)\,\mathrm{sech}\, y\,dy, \tag{48}$$

$$L_0 = \int_0^\infty \log(\tanh y)\sin^2(cy)\,\mathrm{sech}\, y\,dy, \tag{49}$$

which may be evaluated by means of the Gauss-Laguerre quadrature formula.

It may be remarked here that if $\nu = 1/2$, then $h = 0$ and (47) degenerates into two eigenvalue equations

$$V_1 = 0, \pi V_1 + \sin \pi\gamma = 0, \tag{50}$$

which correspond to the singularities at the vertices of wedge-shaped stamps loaded normally [1] and tangentially [2], respectively.

A numerical root search procedure has been applied to equation (47). The results for several combinations of α and ν are given in Table 1. In order to afford comparison with the corresponding potential theory problem, the values of α have been taken from Morrison and Lewis [9]. It is observed that the imaginary part of γ is fairly small for small values of α.

Interfacial Crack

The problem of determining the strength of singularity at the corner of a wedge-shaped crack at the interface of two dissimilar half spaces can be formulated essentially on the same lines. Indeed, the eigenvalue equations (16)–(18) hold good for the interfacial crack where $2(\pi - \alpha)$ is the angle of the crack corner and

$$b = \frac{\nu_1\mu_2 + \nu_2\mu_1}{(1-\nu_1)\mu_2 + (1-\nu_2)\mu_1}, \quad h = \frac{(1-2\nu_1)\mu_2 - (1-2\nu_2)\mu_1}{(1-\nu_1)\mu_2 + (1-\nu_2)\mu_1}, \tag{51}$$

in which μ_1, μ_2 are the rigidity moduli and ν_1, ν_2 the Poisson ratios of the upper and lower half spaces. For $0 \le \nu_1, \nu_2 \le 1/2$ the bimaterial parameters b and h satisfy the inequalities

$$0 \le b \le 1, \quad 0 \le h \le 1 \tag{52}$$

where the negative values of h have been ignored because the eigenvalue problem is unaltered if h is substituted for $-h$. So that the parameters b and h remain within the same limits as in the bonded stamp problem. As in the stamp problem there will be oscillating singularities along the crack edges and at the apex. However, as pointed out by Comninou [10] for plane problems of interfacial cracks, there will be a region of interpenetration unless one assumes a local zone of contact near all crack edges.

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