Understanding rational numbers

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Abstract

Rational numbers are important as a foundation for later mathematics learning and particularly for learning algebra. Most researcher agree that students find rational numbers difficult. This article question the traditional use of partitioning as the starting point for the teaching of fractions. It seeks the origin of children's understanding of rational numbers in their understanding of division. A number of empirical studies are presented on children's use of action schemes for division, correspondence and partitioning. At last conclusions and implications for education are drawn.

Introduction

There may not be many things that mathematics educators and researchers agree on but most would certainly agree that students find rational numbers difficult (e.g. Behr, Lesh, Post, & Silver, 1983; Brousseau, Brousseau, & Warfield, 2004; Davis, Hunting, & Pearn,1993; Freudenthal, 1983; Kerslake, 1986; Kieren, 1988; Ohlsson, 1987, 1988; Pitkethly & Hunting, 1996; Post, Behr, & Lesh, 1986; Stafylidou & Vosniadou, 2004). Several misconceptions that children seem to hold about rational numbers have been described, a large variety of experimental teaching programmes developed, but teaching and learning about rational numbers remains a disheartening experience for many teachers and learners. It is, however, a topic of major importance in mathematics education. The recent report by the U.S.A. National Mathematics Panel reaffirms the importance of teaching students about rational numbers as a foundation for later mathematics learning, and in particular as a critical foundation for learning algebra (Fennell, Faulkner, Ma, Schmid, Stotsky, Wu et al., 2008).

This paper has as its starting point the simple idea that rational numbers are numbers in the domain of quotients (Brousseau, Brousseau, & Warfield, 2007; Kieren, 1993; Ohlsson, 1988). Although there are different subconstructs or meanings for rational numbers (see, for example, Behr, Harel, Post, & Lesh, 1992; Kieren, 1988), it seems reasonable to seek the origin of children's understanding of rational numbers in their understanding of division. Our hypothesis is that in division situations children can develop some insight into the equivalence and order of quantities that would normally be represented by fractions, even in the absence of knowledge of representations for fractions, either in written or oral form.

In the first section of this paper, the two schemes of action that children use in division situations are identified and the insights into aspects of rational numbers that each scheme can promote are explored. In the second and third sections, research about the development of each of the two schemes of action is considered. The last section summarizes the conclusions and implications for education.

Two Schemes of Action for Division

The mathematics education literature traditionally considers two types of division problems: partitive and quotative division. Fischbein, Deri, Nello, and Marino

(1985) define partitive division (which they also term sharing division) as a model for situations in which

> an object or collection of objects is divided into a number of equal fragments or subcollections. The dividend must be larger than the divisor; the divisor (operator) must be a whole number; the quotient must be smaller than the dividend (operand)… In quotative division or measurement division, one seeks to determine how many times a given quantity is contained in a larger quantity. In this case, the only constraint is that the dividend must be larger than the divisor. If the quotient is a whole number, the model can be seen as repeated subtraction (Fischbein, Deri, Nello, & Marino, 1985, p. 7).

This classification distinguishes two ways in which children use the *same scheme of action*, which will be referred to here as *partitioning*. In both types of division problems identified by Fischbein and his colleagues: (a) there is one whole (an object or a collection of objects) and the question is how it is divided into equal parts; (b) the dividend is larger than the divisor; and (c) the children's actions involve forming equal parts which exhaust the whole. The difference between the two types of problems is that in partitive division the children are told how many parts there should be but not the size of the parts, whereas in quotative division they are told the size of the parts and are asked to find out how many parts of this size fit into the whole.

When children use the scheme of partitioning, the insights that they gain about quantities can help them understand some principles that apply in the domain of rational numbers. They can, for example, reason that, the more parts they cut the whole into, the smaller the parts will be. This insight is relevant to quantities that are normally represented by fractions and could help them understand how fractions are ordered.

If children can achieve a higher level of precision in reasoning about partitioning, they could develop some understanding of the equivalence of fractions: they could come to understand that, if they have twice as many parts, each part would be halved in size. For example, you would eat the same amount of chocolate after cutting one chocolate bar into two parts and eating one part as after cutting it into four parts and eating two, because the number of parts and the size of the parts compensate for each other precisely.

It is an empirical question whether children attain these understandings in the domain of whole numbers and extend them to rational numbers.

Although partitioning is the scheme that is most often used to introduce children to fractions, it is not the only scheme of action relevant to division.

Children also use *correspondences in division* situations when the dividend is represented by one measure and the divisor is represented by another measure. The difference between partitioning and correspondence division is that in partitioning there is a single whole (or measure) and in correspondence there are two measures. An example of correspondence is when children share out chocolate bars to a number of recipients: here the dividend is in one domain of measures – the number of chocolate bars – and the divisor is in another domain – the number of children. In this case, and precisely because there are two domains of measures, the assumptions made about the relative size of the dividend and the divisor identified by Fischbein and colleagues do not apply: the dividend does not have to be larger than the divisor, and most children are ready to agree that it is perfectly possible to share one chocolate bar among three children – that is, it is perfectly possible to divide a smaller number by a larger number.

The difference between these two schemes of action may seem, at first glance, too subtle to be of interest when we are thinking of children's understanding of fractions, and research on children's understanding of fractions has not made this into a core distinction so far. However, it is argued in this paper that this is a crucial distinction both in terms of what insights each scheme of action affords and in terms of the empirical research results.

There are at least four differences between what children might learn about quantities which are usually represented by fractions from using the partitioning scheme or the scheme of correspondences.

The first difference is that there is no necessary relation between the size of the dividend and that of the divisor when children set two measures in correspondence. In contrast, as pointed out by Fischbein and his colleagues, the dividend is assumed to be larger than the divisor in partitioning. Therefore, it may be easier for children to develop an understanding of improper fractions when they use correspondences between two fields of measures than when they use partitioning of a single whole. They might have no difficulty in understanding that 3 chocolate bars shared between 2 children means that each child could get one chocolate bar plus a half. In contrast, in partitioning situations children might be puzzled if they are told that someone ate 3 parts of a chocolate bar divided in 2 parts.

A second possible difference between the two schemes of action may be that children could conclude when using correspondences that the way in which partitioning is carried out does not matter, as long as the correspondences between the two measures are "fair". They could, for example, understand that if you have 3 chocolate bars to be shared by 2 children, it is not necessary to divide all 3 chocolate bars in half, and then distribute the halves; giving a whole chocolate bar plus a half to each child would accomplish the same fairness in the correspondences between chocolate bars and children. This might be an important insight into understanding fractions. In the domain of natural numbers, a set with 3 elements is equivalent to all sets with 3 elements and only to sets with 3 elements; in the domain of rational numbers, two quantities described by different numbers can be equivalent (e.g. 1/2, 2/4, 3/6 etc. are represented by different numbers but are equivalent).

A third possible insight about quantities that can obtained from correspondences is related to ordering of quantities: the children may realize that, the more children sharing, the less each one will get. This can be described in the context of division as knowing that there is an inverse relation between the divisor and the quotient. It was hypothesized earlier that children might achieve a similar insight about this inverse relation through the scheme of partitioning. However, there is a difference between the principles children would need to abstract from each of the schemes: in partitioning, they need to establish a with-in-quantity relation (the more parts, the smaller the parts) whereas in correspondence they need to establish a between-quantity relation (the more children, the less chocolate). It is an empirical matter to find out whether it is easier to achieve one of these insights than the other, or whether they pose the same obstacles to children.

Finally, both partitioning and correspondences could help children understand something about the equivalence between quantities, but the reasoning required to achieve this understanding differs across the two schemes of action. When setting chocolate bars in correspondence with the recipients, the children might be able to reason that, if there were twice as many chocolate bars and twice as many children, the shares would be equivalent, even though the dividend and the divisor are different. This may be relatively simpler than the com-parable reasoning in partitioning. In partitioning, understanding equivalence is based on *inverse* proportional reasoning (twice as many pieces means that each piece is half the size) whereas in contexts where children use the correspondence scheme, the reasoning is based on a *direct* proportion (twice as many chocolate bars and twice as many children means that everyone still gets the same).

This exploratory analysis of how children might accomplish an understanding of equivalence and order of fractions when using partitioning or when using correspondences in division situations indicates that it may be fruitful to attend more to the difference between these schemes in division than hitherto. It is possible that the scheme of correspondences could afford in some sense a smoother transition from natural to rational numbers, at least as far as understanding equivalence and order of quantities is concerned.

In the second and third sections of this paper, a review of research about children's understanding of correspondence and of partitioning is presented. The literature on these schemes of action is vast but this paper focuses on research that sheds light on whether it is possible to find continuities between children's understanding of quantities that are represented by natural numbers and those that must be represented by rational numbers. For simplicity, the latter will be referred to simply as "fractional quantities".

Children's use of the correspondence scheme in making judgements about quantities

Piaget (1952) pioneered the study of how and when children use the correspondence scheme to draw conclusions about quantities. In one of his many studies about children's understanding of correspondences, Piaget (1952) asked the children to place one pink flower into each one of a set of vases; after removing the pink flowers, he asked the children to place a blue flower into each one of the same vases; then having set all the flowers aside, and leaving on the table only the vases, he asked the children to take from a box the exact number of straws required if they wanted to put one flower into each straw. Without counting, and only using correspondences, 5- and 6-year old children were able to make inferences about the equivalence between straws and flowers: by setting two straws in correspondence with each vase, they achieved an equivalent set. Piaget concluded that the children's judgements were based on "multiplicative equivalences" (p. 219) established by the use of the correspondence scheme: the children reasoned that, if there is a 2-to-1 correspondence between flowers and vases and a 2-to-1 correspondence between straws and vases, the number of flowers and straws must be the same.

In Piaget's study, the scheme of correspondence was used in a situation that involved ratio but not division. More recently, Frydman and Bryant (1988) carried out a series of studies where children established correspondences between sets in a division situation. The children were given a set of cubes, which were pretend sweets, to be shared fairly among different dolls. Children aged 4 could often carry out this sharing efficiently and fairly by using a one-for-you one-for-me type of procedure. After distributing the sweets, the children were confident that they did this sharing fairly and that both dolls had the same amount of sweets to eat. Frydman and Bryant asked the children to count the number of sweets that one doll had and then deduce the number of sweets that the other doll had. About 40% of the 4-year-olds were able to make the inference that the second doll had the same number of sweets as the first one; this proportion increased with age. This result extends Piaget's observations that children can make equivalence judgements not only in multiplication but also in division problems by using correspondences.

These studies show that children can use the scheme of correspondences to make equivalence judgements in the domain of natural numbers, even if they have not counted the sets and are not using number labels as mediators of this equivalence judgement. Their findings were replicated in a number of studies by Davis and his colleagues (Davis, 1990; Davis & Hunting, 1990; Davis & Pepper, 1992; Pitkethly & Hunting, 1996), who refer to this scheme of action as "dealing". These authors have argued that this scheme is basic to children's understanding of fractions (Davis & Pepper, 1992). They used a variety of situations, including redistribution when a new recipient comes, to study children's ability to use correspondences to make inferences about equality. In redistribution situations many children thought that it was better to count after having carried out the distribution, in order to make sure that the amounts were the same (Davis & Pitkethly, 1990). Nevertheless, they concluded that children do use correspondences in order to establish equivalences between quantities generated through a division.

Correa, Nunes, and Bryant (1998) extended these studies by showing that children can make inferences about quantities resulting from a division not only when the divisors are the same but also when they are different. In order to circumvent the possibility that children feel the need to count the sets after division because they think that they could have made a mistake in sharing, Bryant and his colleagues did not ask the children to do the sharing: the sweets were shared by the experimenter, outside the children's view, after the children had seen that the number of sweets to be shared was the same.

There were two conditions in this study: same dividend and same divisor versus same dividend and different divisors. In the same dividend and same divisor condition, the children should be able to conclude for the equivalence between the sets; in the same dividend and different divisor condition, the children should conclude that the more recipients there are, the fewer sweets they receive.

About two thirds of the 5-year-olds, the vast majority of the 6-year-olds, and all the 7-year-olds concluded that the recipients had equivalent shares when the dividend and the divisor were the same. Equivalence was easier than the inverse relation between divisor and quotient: 34%, 53% and 81% of the children in these three age levels, respectively, were able to conclude that the more recipients there are, the smaller each one's share will be. Correa (1994) also found that children's success in making these inferences improved if they solved these problems after practising sharing sweets between dolls; this indicates that thinking about how to establish correspondences improves their ability to make inferences about the relations between the quantities resulting from sharing.

In all the previous studies, the dividend was composed of discrete quantities and was larger than the divisor. The next question to consider is whether children can make similar judgements about equivalence when the situations involve continuous quantities and the dividend is smaller than the divisor: that is, when children have to think about fractional quantities.

Kornilaki and Nunes (2005) investigated this possibility by comparing children's inferences in division situations that involved discrete quantities and dividends larger than the divisors, and also situations that involved continuous quantities and dividends smaller than the divisors. In the discrete quantities tasks, the children were shown one set of small toy fish to be distributed fairly among a group of white cats and another set of fish to be distributed to a group of brown cats; the number of fish was always greater than the number of cats. In the continuous quantities tasks, the dividend was made up of fish-cakes, to be distributed fairly among the cats: the number of cakes was always smaller than the number of cats, and varied between 1 and 3 cakes, whereas the number of cats to receive a portion in each group varied between 2 and 9. Following the paradigm devised by Correa, Nunes, and Bryant (1998), the children were neither asked to distribute the fish nor to partition the fish cakes. They were asked whether, after a fair distribution in each group, each cat in one group would receive the same amount to eat as each cat in the other group. Empson, Junk, Dominguez, & Turner (2005) have stressed that

> the depiction of equal shares of, for example, sevenths in a part–whole representation is not a necessary step to understanding the fraction 1/7 (for contrasting views, see Charles and Nason, 2000; Lamon, 1996; Pothier and Sawada, 1983). What is necessary, however, is understanding that 1/7 is the amount one gets when 1 is divided into 7 same-sized parts (Empson, Junk, Dominguez, & Turner, 2005).

In some trials, the number of fish (dividend) and cats (divisor) was the same; in other cases, the dividend was the same but the divisor was different. So in the first type of trials the children were asked about equivalence after sharing and in the other set the children were asked to order the quantities obtained after sharing.

There were 16 trials with discrete quantities and 24 trials with continuous quantities; this large number of trials allowed the researchers to establish whether the children were performing above chance.

The majority of the children succeeded in all the items where the dividend and the divisor were the same: 62% of the 5-year-olds, 84% of the 6-year-olds and all the 7-year-olds answered all the questions correctly. When the dividend was the same and the divisors differed, the rate of success was 31%, 50% and 81%, respectively, for the three age levels. There was no difference in the level of success attained by the children with discrete versus continuous quantities.

In almost all the items, the children explained their answers by referring to the type of relation between the dividends and the divisors: same divisor, same share or, with different divisors, the more cats receiving a share, the smaller their share. The use of numbers as an explanation for why the recipients' shares would be the same or not was observed in 6% of answers by the 7-year-olds when the quantities were discrete and less often than this by the younger children. Attempts to use numbers to speak about the shares in the continuous quantities trials were practically inexistent (3% of the 7-year-olds explanations). Thus the analysis of justifications supports the idea that the children were reasoning about relations rather than using counting when they made their judgments of equivalence or ordered the quantities that would be obtained after division.

This study replicated previous findings that young children can use correspondences to make inferences about equivalences and also added new evidence relevant to children's understanding of fractional quantities: many young children who have never been taught about fractions used correspondences to order fractional quantities. They did so successfully when the division would have resulted in unitary fractions and also when the dividend was greater than 1 and the results would not be a unitary fraction (e.g. 2 cakes to be shared by 3, 4 or 5 cats).

Young children are notoriously bad at partitioning continuous quantities into equal shares (see, for example, Hierbert & Tonnessen, 1978; Hunting & Sharpley, 1988a, 1988b; Miller, 1984; Piaget, Inhelder, & Szeminska, 1960), so Kornilaki and Nunes concluded that their inference making abilities must have been ahead of their procedural skills for partitioning and most likely stemmed from the knowledge about relations between the dividend and the divisor gained in the context of discrete quantities.

Recently Mamede (2006) replicated the results of children's ability to make inferences about asymmetrical relations in sharing situations in the context of fractional quantities. She worked with Portuguese children in their first year in school, who had received no school instruction about fractions. Their performance was only slightly weaker than that of English children: 55% of the 6-year-olds and 71% of the 7-year-olds were able to make the inference that the larger the divisor, the smaller the share that each recipient would receive.

These studies strongly suggest that children can learn principles about how the dividend and the divisor are related from experiences with sharing when they establish correspondences between the two domains of measures, the shared quantities and the recipients. They suggest that a relatively smooth transition from natural numbers to rational numbers is possible when children use correspondences to understand the relations between quantities. This argument is central to Streefland's (1987; 1993; 1997) hypothesis about what is the best starting point for teaching fractions to children and has been advanced by others also (Davis & Pepper, 1992; Kieren, 1983; Vergnaud, 1983).

These studies tell an encouraging story about children's understanding of the logic of division even when the dividend is smaller than the divisor, but there is one further point that should be considered in the transition between natural and rational numbers. In the domain of rational numbers there is an infinite set of equivalences (e.g. 1/2 = 2/4 = 3/6 etc.) and in the studies described previously the children were only asked to make equivalence judgements when the dividend and the divisor were the same. Can they still make the inference of equivalence in sharing situations when the dividend and the divisor are different across situations, but the dividend-divisor ratio is the same?

Nunes, Bryant, Pretzlik, Bell, Evans and Wade (2007) asked English children, aged between 7.5 and 10 years, who were in their 4^{th} and 5^{th} years in school, to make comparisons between the shares that would be received by children in sharing situations where the dividend and divisor were different but their ratio was the same. Previous research (see, for example, Behr, Harel, Post, & Lesh, 1992; Kerslake, 1986) shows that children in these age levels have difficulty with the equivalence of fractions. The children in this study had received some instruction on fractions: they had been taught about halves and quarters in problems about partitioning. They had only been taught about one pair of equivalent fractions: they were taught that one half is the same as two quarters. In the correspondence item in this study, the children were presented with two pictures: in the first, a group of 4 girls was going to share fairly one pie; in the second, a group of 8 boys was going to share fairly 2 pies that were exactly the same as the pie that the girls had. The question was whether each girl would receive the same share as each boy. The overall rate of correct responses was 73% (78% in Year 4 and 70% in Year 5; this difference was not significant). This is an encouraging result: the children had only been taught about halves and quarters; nevertheless, they were able to attain a high rate of correct responses for fractional quantities that could be represented as 1/4 and 2/8.

The studies reviewed so far asked the children about quantities resulting from division and always included two domains of measures, thus engaging the children's correspondence reasoning. However, they did not involve asking the children to represent these quantities through fractions. The final study reviewed here is a brief teaching study (Nunes, Bryant, Pretzlik, Evans, Wade, & Bell, 2008), where the children were taught to represent fractions in the context of two domains of measures, shared quantities and recipients, and were asked about the equivalence between fractions. The types of arguments that children produced to justify the equivalence of fractions were then analyzed and compared to the insights that we hypothesized would emerge in the context of sharing from the use of the correspondence scheme. Brief teaching studies are of great value in research because they allow the researchers to know what under-standings children can construct if they are given a specific type of guidance in the interaction with an adult (Cooney, Grouws, & Jones, 1988; Steffe & Tzur, 1994; Tzur, 1999; Yackel, Cobb, Wood, Wheatley, & Merkel, 1990) and because of their ecological validity: children spend much of their time in school trying to use what they have been taught to solve mathematics problems. Because this study has only been published in a summary form (Nunes, Bryant, Pretzlik, & Hurry, 2006), some detail is presented here.

The children (N=62) were in the age range from 7.5 to 10 years, in the 4^{th} or 5^{th} year in school. As those in the previous study, they had only been taught about half and quarters and the equivalence between half and two quarters. They worked with a researcher outside the classroom in small groups (12 groups of between 4 and 6 children, depending on the class size) and were asked to solve each problem first individually, and then to discuss their answers in the group. The sessions were audio- and video-recorded. The children's arguments were transcribed *verbatim;* the information from the video-tapes was later coordinated with the transcripts in order to help the researchers understand the children's arguments.

This study used tasks from Streefland (1991). The children solved two of his sharing tasks on the first day and an equivalence task on the second day of the teaching study. The tasks were presented in booklets with pictures, where the children also wrote their answers. The tasks used on the first day were:

1. Six girls are going to share a packet of biscuits. The packet is closed; we don't know how many biscuits are in the packet. (a) If each girl received one biscuit and there were no biscuits left, how many biscuits were in the packet? (b) If each girl received a half biscuit and there were no biscuits left, how many biscuits were in the packet? (c) If some more girls join the group, what will happen when the biscuits are shared? Do the girls now receive more or less each than the six girls did?
2. Four children will be sharing 3 chocolates. (a) Will each one be able to get one bar of chocolate? (b) Will each one be able to get at least a half bar of chocolate? (c) How would you share the chocolate? (The booklets contained a picture with three chocolate bars and four children and the children were asked to show how they would share the chocolate bars.) Write what fraction each one gets.

After these tasks had been completed, the researcher told the children that they were going to practice writing fractions that they had not yet learned in school. The children were asked to write "half" with numerical symbols; this they knew already. The researcher then asked the children to explain why there was a number 1 above the line, a number 2 below the line, and a line between the numbers (for a discussion of children's interpretation of fraction symbols in this situation, see Charles & Nason, 2000, and Empson, Junk, Dominguez, & Turner, 2005). Working from the children's responses, the researcher guided them to the realization this symbol can be interpreted as "1 chocolate bar divided by 2 children" and that the line indicates a division. The children were then asked: if there is 1 chocolate bar to be shared among 4 children, what fraction will they receive? This was a known notation but we wanted the children to reinterpret it as "1 divided by 4", and not just think about the symbol as meaning "1 piece out of 4", which was their initial interpretation, based on the instruction that they received in the context of partitioning. Then, without showing the children any pictures, the researcher asked them to write what fraction of a chocolate bar children would receive if they had:

1. 1 chocolate bar shared by 3 children;
2. 1 chocolate bar shared by 5 children;
3. 2 chocolate bars shared by 5 children.

The equivalence task, presented on the second day was:

1. Six children went to a pizzeria and ordered 2 pizzas to share between them. The waiter brought one first and said they could start on it because it would take time for the next one to come. (a) How much will each one get from the first pizza that the waiter brought? Write the fraction that shows this. (b) How much will each one get from the second pizza? Write your answer. (c) If you add the two pieces together, what fraction of a pizza will each one get? You can write a plus sign between the first fraction and the second fraction, and write the answer for the share each one gets in the end. (d) If the two pizzas came at the same time, how could they share it differently? (e) Are these fractions (the ones that the children wrote for answers c and d) equivalent?

According to the hypotheses presented in the previous section, we would expect children to develop some insights into rational numbers by thinking about different ways of sharing the same amount. It was expected that they might realize: (1) that it is possible to divide a smaller number by a larger number; (2) that different fractions might represent the same amount; (3) that twice as many things to be divided and twice as many recipients would result in equivalent amounts; and that (4) the larger the divisor, the smaller the quotient. This latter idea can not be explored in the context of a problem about equivalent fractions.

The children's explanations for why they thought that the fractions were or were not equivalent provided evidence for all the three insights that we anticipated, and more, as described below.

It is possible to divide a smaller number by a larger number.

There was no difficulty among the students in attempting to divide 1 pizza among 6 children. In response to part *a* of the equivalence problem, all children wrote at least one fraction correctly (some children wrote more than one fraction for the same answer, always correctly).

In response to part *c*, when the children were asked how they could share the 2 pizzas if both pizzas came at the same time and what fraction would each one receive, some children answered 1/3 and others answered 2/12 from each pizza, giving a total share of 4/12. The latter children, instead of sharing 1 pizza among 3 girls, decided to cut each pizza in 12 parts: i.e. they cut the sixths in half. This led to a discussion of the different answers even before the researcher actually presented them with the question regarding the equivalence of the different fractions.

Different fractions can represent the same amount.

This insight was expressed in all groups. For example, one child said that "They're the same amount of people, the same amount of pizzas, and that means the same amount of fractions. It doesn't matter how you cut it." Another child said: "Because it wouldn't really matter when they shared it, they'd get that [3 girls would get 1 pizza], and then they'd get that [3 girls would get the other pizza], and then it would be the same." Another child said: "It's the same amount of pizza. They might be different fractions but the same amount [this child had offered 4/12 as an alternative to 2/6]." Another child said: "Erm, well basically just the time doesn't make much difference, the main thing is the number of things."

A dividend twice as large and a divisor twice as large result in equivalent amounts.

This principle was expressed in 11 of the 12 groups. For example, one child said: "It's half the girls and half the pizzas; three is a half of six and one is a half of two." Another child said: "If they have two pizzas, then they could give the first pizza to three girls and then the next one to an-other three girls. (…) If they all get one piece of that each, and they get the same amount, they all get the same amount".

So all three ideas we thought that could appear in this context were expressed by the children. But two other principles, which we did not expect to observe in this correspondence problem, were also enunciated by the children.

The number of parts and size of parts are inversely proportional.

This principle was enunciated in eight of the 12 groups. For example, one child who cut the pizzas the second time around in 12 parts each said: "Because it's double the one of that [total number of pieces] and it's double the one of that [number of pieces for each], they cut it twice and each is half the size; they will be the same". Another child said: "because 1 sixth and 1 sixth is actually a different way in fractions [from 1 third] and it doubled [the number of pieces] to make it [the size of the piece] littler, and halving [the number of pieces] makes it [the size of the piece] bigger, so I halved it and it became 1 third".

The fractions show the same part-whole relation.

This reasoning, which we had not expected to emerge from the use of the correspondence scheme, was enunciated in only one group (out of 12), initially by one child, but was then reiterated by a second child in her own terms. The first child said: "You need three two sixths to make six [he shows the 6 pieces marked on one pizza], and you need three one thirds to make three (shows the 3 pieces marked on one pizza). [He then wrote the computation presented in Figure 1 and said] "There's two sixths, add two sixths three times to make six sixths. With one third, you need to add one third three times to make three thirds." Note that he does not write "6/6" or "3/3" but he expresses this verbally. It is not his reasoning, but his notation, that is at fault. He does not seem to have any doubts that 6/6 and 3/3 are equivalent wholes: he assumes this in his argument.

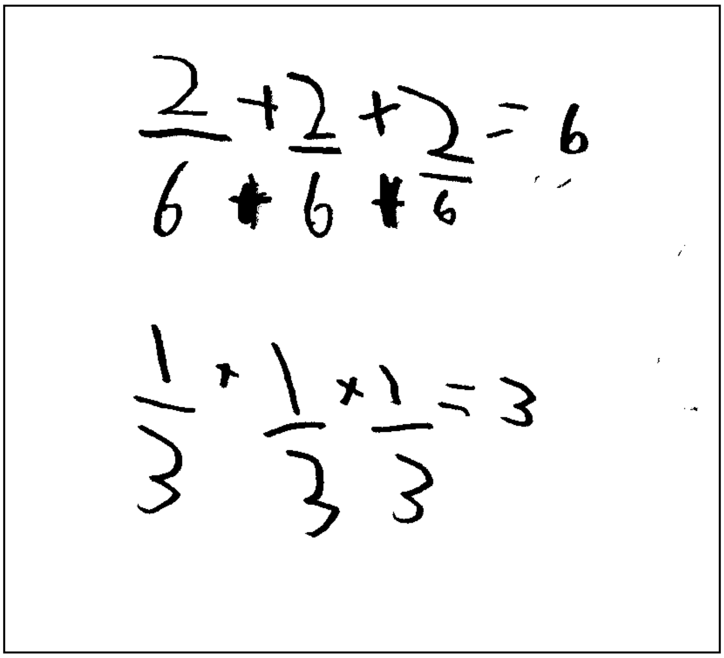

Figure 1. A child's written production to show that 2/6 three times makes.

To summarize: this brief teaching experiment was carried out to elicit discussions between the children in situations where they could use the correspondence scheme in division. The first set of problems, in which they are asked about sharing discrete quantities, created a background for the children to use this scheme of action. We then helped them to construct an interpretation for written fractions where the numerator is the dividend, the denominator is the divisor, and the line indicates the operation of division. This interpretation did not replace their original interpretation of number of parts taken from the whole; the two meanings coexisted and appeared in the children's arguments as they explained their answers. In the subsequent problems, where the quantity to be shared was continuous and the dividend was smaller than the divisor, the children had the opportunity to explore the different ways in which continuous quantities can be shared. They were not asked to actually partition the pizzas, and some made marks on the pizzas whereas others did not. Figure 2 presents an example of a drawing that contains some marks for possible partition but not all the marks: the most salient feature of the child's drawing is the use of correspondences. When the pizzas were divided into thirds, the dots inside the pizzas were used to represent the recipients. The scheme of correspondence played a major role in the children's reasoning. Sometimes the correspondences were carried out mentally and expressed verbally and sometimes the children used drawings and gestures which indicated the correspondences.

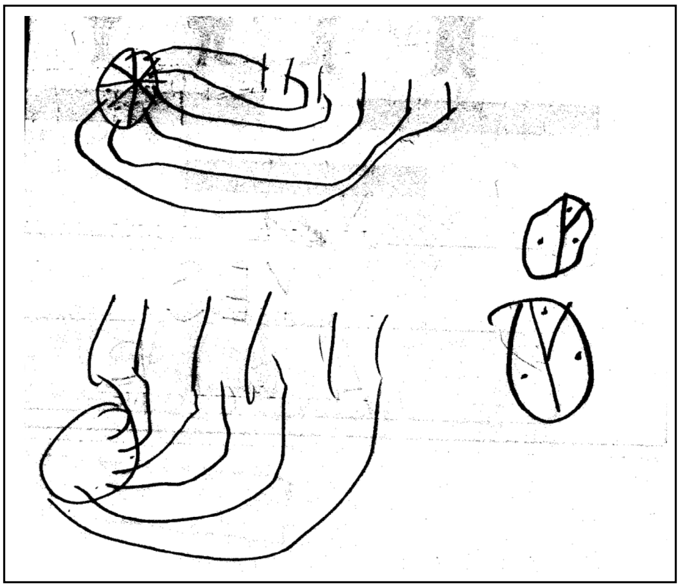

Figure 2. A child's drawing used to discuss the equivalence between 2/6 and 1/3.

Other researchers have identified children's use of correspondences to solve problems that involve fractions, although they did not necessarily use this label in describing the children's answers. Empson (1999), for example, presented the following problem to children aged about 6 to 7-years (first graders in the USA): 4 children got 3 pancakes to share; how many pancakes are needed for 12 children in order for the children to have the same amount of pancake as the first group? She observed that three children solved this problem by partitioning and three solved it by placing 3 pancakes in correspondence to each group of 4 children. Similar strategies were reported when children solved another problem that involved 2 candy bars shared among 3 children.

Kieren (1993) also documented the use of correspondences as a basis for children's judgments when they compared fractions. In his problem, the fractions were not equivalent: there were 7 recipients and 4 items in Group A and 4 recipients and 2 items in Group B. The children were asked how much each recipient would get in each group and whether the recipients in both groups would get the same amount. Kieren presents a drawing by an 8-year-old, where the items are partitioned in half and the correspondences between the halves and the recipients are shown; in Group A, a line without a recipient shows that there is an extra half in that group and the child argues that for the amounts to be the same there should be one more person in Group A. Kieren termed this solution "corresponding or 'ratiolike' thinking" (p. 54).

Conclusion

Children can use the scheme of correspondences to:

- Establish equivalences between sets that have the same ratio to a reference set (Piaget, 1952).
- Re-distribute things after having carried out one distribution (Davis and colleagues).
- To reason about equivalences resulting from division both when the dividend is larger or smaller than the divisor (Bryant and colleagues; Empson, 1999; Nunes and colleagues).
- To order fractional quantities (Kieren, 1993; Kornilaki & Nunes, 2005; Mamede, 2007).

All these studies were carried out with children up to the age of 10 years and showed positive results. This stands in clear contrast to the literature on children's difficulties with fractions and prompts the question of whether the difficulties might stem from the use of partitioning as the starting point for the teaching of fractions (see also Lamon, 1996; Streefland, 1987). The next section examines the development of children's partitioning action and its connection with children's concepts of fractions.

Children's use of the scheme of partitioning in making judgements about quantities

The scheme of partitioning has been named also subdivision and dissection (Pothier & Sawada, 1983), and is consistently defined as the process of dividing a whole into parts. This process is not understood as the activity of cutting something into parts in any old way but as a process that must be guided from the outset by the aim of obtaining a predetermined number of equal parts.

Piaget, Inhelder and Szeminska (1960) pioneered the study of the connection between partitioning and fractions. They spelled out a number of ideas, which they thought were necessary for children to develop an understanding of fractions, and analysed them in partitioning tasks. The motivation for partitioning was sharing a cake between a number of recipients, but the task was itself one of partitioning. They suggested that "the notion of fraction depends on two fundamental relations: the relation of part to whole (…) and the relation of part to part" (p. 309). Piaget and colleagues identified a number of insights that children need to achieve in order to understand fractions:

1. The whole must be conceived as divisible, an idea that children under the age of about 2 seem not to attain.
2. The number of parts to be achieved is determined from the outset.
3. The parts must exhaust the whole (i.e. there should be no second round of partitioning and no remainders).
4. The number of cuts and the number of parts are related (e.g. if you want to divide something in 2 parts, you should use only 1 cut).
5. All the parts should be equal;
6. Each part can be seen as a whole in itself, nested into the whole but also susceptible to further division.
7. The whole remains invariant and is equal to the sum of the parts.

Piaget and colleagues observed that children rarely achieved correct partitioning before the age of about 6. A major strategy in carrying out successful partitioning was the use of successive divisions in two: so children are able to succeed in dividing a whole into fourths before they can succeed with thirds. Successive halving helped the children with some fractions: dividing something into 8ths is easier this way. However, it interfered with success in other fractions: some children attempting to divide a whole into fifths end up with sixths, by dividing the whole first in halves and then subdividing each half into three parts.

Piaget and colleagues also investigated whether children understood the seventh criterion for a true concept of fraction, i.e. the conservation of the whole. This conservation, they argued, would require the children to understand that each piece could not be counted simply as one piece, but had to be understood in its relation to the whole. Some children failed to understand this, and argued that if someone ate a cake cut into 1/2 + 2/4 and a second person ate a cake cut into 4/4, the second one would eat more because he had four parts and the first one only had three. Although these children would recognise that if the pieces were put together in each case they would form one whole cake, they still maintained that 4/4 was more than 1/2 + 2/4. Finally, they also observed that children did not have to achieve the highest level of development in the scheme of partitioning in order to understand the conservation of the whole.

Children's difficulties with partitioning continuous wholes into equal parts have been confirmed many times with pre-schoolers and children in their first years in school (e.g. Hiebert & Tonnessen, 1978, and Hunting & Sharpley, 1988b observed that children often did not anticipate the number of cuts and did not cut the whole extensively). These studies also extended our knowledge about children's expertise in partitioning. For example, Pothier & Sawada (1983) and Lamon (1996) proposed more detailed schemes for the analysis of the development of partitioning schemes and other researchers (Hiebert & Tonnessen, 1978; Hunting & Sharpley, 1988; Miller, 1984; Novillis, 1976) showed that the difficulty of partitioning discrete and continuous quantities is not the same, as hypothesized by Piaget. Children can use a procedure for partitioning discrete quantities that is not applicable to continuous quantities: they can "deal out" the discrete quantities but not the continuous ones. Thus they perform significantly better with the former than the latter, and the smooth transition from discrete to continuous quantities observed with the correspondence scheme is not replicated with the partitioning scheme.

These studies present a less positive picture of the insights into rational numbers afforded by the scheme of partitioning than by the correspondence scheme. However, the focus of the studies was on the scheme of partitioning *per se*; the question investigated here is whether partitioning can promote the understanding of equivalence and ordering of fractions, as hypothesized in the first section of this paper.

Many studies investigated children's understanding of equivalence of fractions in partitioning contexts (e.g. Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuth, Post, & Lesh, 1984; Larson, 1980; Kerslake, 1986), but differences in the methods used in these studies render the comparisons between partitioning and correspondence studies ambiguous. For example, if the studies start with a representation of the fractions, rather than a problem about quantities, they cannot be compared to the studies reviewed in the previous section, which asked the children to think about quantities without necessarily using fractional representation. Thus only studies that use methods comparable to those used in the correspondence investigations reported earlier on were selected for discussion.

Kamii and Clark (1995) presented children with identical rectangles and cut them into fractions using different cuts. For example, one rectangle was cut horizontally in half and the second was cut across a diagonal. The children had the opportunity to verify that the rectangles were the same size and that the two parts from each rectangle were the same in size. They asked the children: if these were chocolate cakes, and the researcher ate a part cut from the first rectangle and the child ate a part cut from the second, would they eat the same amount? This method is highly comparable to the studies by Kornilaki and Nunes (2005) and by Mamede (2007), where the children do not have to carry out the actions, so their difficulty with partitioning does not influence their judgements. They also use similarly motivated contexts, ending in the question of whether recipients would eat the same amount. However, the question posed by Kamii and Clark draws on the child's understanding of partitioning and the relations between the parts of the two wholes because each whole corresponds to a single recipient.

The children in Kamii's study were considerably older than those in the correspondence studies: they were in the fifth or sixth year in school (approximately 11 and 12 years). Both groups of children had been taught about equivalent fractions. In spite of having received instruction, the children's rate of success was rather low: only 44% of the fifth graders and 51% of the sixth graders reasoned that they would eat the same amount of chocolate because these were halves of identical wholes.

Kamii and Clark then showed the children two identical wholes, cut one in fourths using a horizontal and a vertical cut, and the other in eighths, using horizontal cuts only. They discarded one fourth from the first "chocolate cake", leaving 3/4 to be eaten, and asked the children to take the same amount from the other cake, which had been cut into eighths, for themselves. The percentage of correct answers was this time even lower: 13% of the fifth graders and 32% of the sixth graders correctly identified the number of eighths required to take the same amount as 3/4.

Recently, we (Nunes & Bryant, 2005) included a similar question about halves in a survey of English children's knowledge of fractions. The children in our study were in their fourth and fifth year in school. The children were shown pictures of a boy and a girl and two identical rectangular areas, the "chocolate cakes". The boy cut his cake along the diagonal and the girl cut hers horizontally. The children were asked to indicate whether they ate the same amount of cake and, if not, to mark the child who ate more. Our results were more positive than Kamii and Clark's: 55% of the fourth graders and 80% of the fifth graders answered correctly. However, these results are weak by comparison to children's rate of correct responses when the problem draws on their understanding of correspondences. In the Kornilaki and Nunes study, 100% of the 7-year-olds (third graders) realized that same dividend and same divisor results in equivalent shares.

Mamede (2007) carried out a direct comparison between children's use of the correspondence and the partitioning scheme in solving equivalence and order problems with fractional quantities. In this well-controlled study, she used story problems involving chocolates and children, similar pictures and mathematically identical questions; the division scheme relevant to the situation was the only variable distinguishing the problems. In correspondence problems, for example, she asked the children: in this party, three girls are going to share fairly one chocolate cake; in this other party, six boys are going to share fairly two chocolate cakes. The children were asked to decide whether each boy would eat more than each girl, each girl would eat more than each boy, or whether they would have the same amount to eat. In the partitioning problems, she asked the children: this girl and this boy have identical chocolate cakes; the cakes are too big to eat at once so the girl cuts her cake into 3 identical parts and eats one and the boy cuts his cake into 6 identical parts and eats 2. The children were asked whether the girl and the boy ate the same amount or whether one ate more than the other. The children (age range 6 to 7) were Portuguese and in their first year in school; they had received no instruction about fractions.

In the correspondence questions, 35% of the 6-year-olds' and 49% of the 7-year-olds responses were correct; in the partitioning questions, 10% of the answers of children in both age levels were correct. These highly significant differences suggest that the use of correspondence reasoning supports children's understanding of equivalence between fractions whereas partitioning did not seem to afford the same insights.

Finally, it is important to compare students' arguments for the equivalence and order of quantities represented by fractions in teaching studies where partitioning is used as the basis for teaching. Many teaching studies that aim at promoting students' understanding of fractions through partitioning have been reported in the literature (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Brousseau, Brousseau, & Warfield, 2004; 2007; Empson, 1999; Kerslake, 1986; Olive & Steffe, 2002; Olive & Vomvoridi, 2006; Saenz-Ludlow, 1994; Steffe, 2002). In most of these, students' difficulties with partitioning are circumvented either by using pre-divided materials (e.g. Behr, Wachsmuth, Post, & Lesh, 1984) or by using computer tools where the computer carries out the division as instructed by the student (e.g. Olive & Steffe, 2002; Olive & Vomvoridi, 2006).

Many studies combine partitioning with correspondence during instruction, either because the researchers do not use this distinction (e.g. Saenz-Ludlow, 1994) or because they wish to construct instruction that combines both schemes in order to achieve a better instructional program (e.g. Brousseau, Brousseau, & Warfield, 2004; 2007).

Two studies which analyzed student's arguments focus the instruction on partitioning. The first was carried out by Berhr, Wachsmuth, Post, and Lesh (1984), who used manipulatives of different types during instruction but also taught the students how to use algorithms (division of the denominator by the numerator to find a ratio) to check on the equivalence of fractions. Behr et al. provided a detailed analysis of children's arguments regarding the ordering of fractions. In summary, they report the following insights after instruction.

- When ordering fractions with the same numerator and different denominators, students seem to be able to argue that there is an inverse relation between the number of parts into which the whole was cut and the size of the parts. This argument appears either with explicit reference to the numerator ("there are two pieces in each, but the pieces in two fifths are smaller." p. 328) or without it ("the bigger the number is, the smaller the pieces get." p. 328).
- A third fraction can be used as a reference point when two fractions are compared: three ninths is less than three sixths because "three ninths is … less than half and three sixths is one half" (p. 328). It is not clear how the students had learned that 3/6 and 1/2 are equivalent but they can use this knowledge to solve another comparison.

- Students used the ratio algorithm to verify whether the fractions were equivalent: 3/5 is not equivalent to 6/8 because "if they were equal, three goes into six, but five doesn't go into eight." (p. 331).
- Students learned to use the manipulative materials in order to carry out perceptual comparisons: 6/8 equals 3/4 because "I started with four parts. Then I didn't have to change the size of the paper at all. I just folded it, and then I got eight." (p. 331).

Behr et al. report that, after 18 weeks of instruction, a large proportion of the students (27%) continued to use the manipulatives in order to carry our perceptual comparisons; the same proportion (27%) used a third fraction as a reference point and a similar proportion (23%) used the ratio algorithm that they had been taught to compare fractions.

Finally, there is no evidence that the students were able to understand that the number of parts and size of parts could compensate for each other precisely in a proportional manner; for example, in the comparison between 6/8 and 3/4 the students could have argued that there are twice as many parts in when the whole was cut into 8 parts in comparison with cutting into 4 parts, so you need to take twice as many (6) in order to have the same amount.

In conclusion, students seemed to develop some insight into the inverse relation between the divisor and the quantity but this only helped them when the dividend was kept constant: they could not extend this understanding to other situations where the numerator and the denominator differed.

The second set of studies which focused on partitioning was carried out by Steffe and his colleagues (Olive & Steffe, 2002; Olive & Vomvoridi, 2006; Steffe, 2002). Because the aim of much of the instruction was to help the children learn to label fractions or compose fractions that would be appropriate for the label, it is not possible to extract from their reports the children's arguments for equivalence of fractions.

However, one of the protocols (Olive & Steffe, 2002) provides evidence for the student's difficulty with improper fractions, which we thought might result from the use of partitioning as the basis for the concept of fractions. The researcher asked Joe to make a stick 6/5 long. Joe said that he could not because there are only five of them. After prompting, Joe physically adds one more fifth to the five already used, but it is not clear whether this physical action convinces him that 6/5 is mathematically appropriate. In a subsequent example, where Joe labels a stick made with 9 sticks that had been defined as "one seventh" of an original stick 9/7, but according to the researchers "an important perturbation" remains. Joe later counts 8 of a stick that had been labeled "one seventh" but doesn't use the label "eight sevenths". When the researcher proposes this label, he questions it: "How can it be EIGHT sevenths?" (Olive & Steffe, 2002, p. 426). He later refused to make a stick that is 10/7, even though the procedure is physically possible. Subsequently, on another day, Joe's reaction to another improper fraction is: "I still don't understand how you could do it. *How can a fraction be bigger than itself*? (Olive & Steffe, 2002, p. 428).

According to the researchers, Joe only sees that improper fractions are acceptable when they presented a problem where pizzas were to be shared by people. When 12 friends ordered 2 slices each of pizzas cut into 8 slices, Joe realized immediately that more than one pizza would be required; the traditional partitioning situation, where one whole is divided into equal parts, was transformed into a less usual one, where two wholes are required but the size of the part remains fixed.

This example illustrates that students have difficulty with improper fractions in the context of partitioning but can overcome this by thinking of more than one whole.

Conclusion

Partitioning, defined as the action of cutting a whole into a predetermined number of equal parts, shows a slower developmental process than correspondence. In order for children to succeed, they need to anticipate the solution so that the right number of cuts produces the right number of equal parts and exhausts the whole. Its accomplishment, however, does not seem to produce immediate insights into equivalence and order of fractional quantities. Apparently, many children do not see it as necessary that halves from two identical wholes are equivalent, even if they have been taught about the equivalence of fractions in school.

In order to use this scheme of action as the basis for learning about fractions, teaching schemes and researchers rely on pre-cut wholes or computer tools to avoid the difficulties of accurate partitioning. Students can develop insight into the inverse relation between the number of parts and the size of the parts through the partitioning scheme but there is no evidence that they realize that if you cut a whole in twice as many parts each one will be half in size. Finally, improper fractions seem to cause uneasiness to students who have developed their conception of fractions in the context of partitioning; it is important to be aware of this uneasiness if this is the scheme chosen in order to teach fractions.

Final remarks and educational implications

The analysis presented in this paper starts from the assumption that children learn mathematical concepts from their schemes of action and the reflections about these that are afforded by the schemes themselves and by social interaction, with teachers or peers. Two types of action schemes that can be used in division situations were distinguished: partitioning, which involves dividing a whole into equal parts, and correspondence situations, where two quantities (or measures) are involved, a quantity to be shared and a number of recipients of the shares. The development of these action schemes differs: children as young as 5 or 6 years in age are quite good at establishing correspondences to produce equal shares whereas they experience much difficulty in partitioning continuous quantities.

The affordances of these two schemes of action were explored, and it was hypothesized that these differ. For example, in correspondence situations, children could achieve some insight into the equivalence of fractions where the dividend and the divisor differ by thinking that, if there are twice as many things to be shared and twice as many recipients, then each one's share is the same. In partitioning, children could achieve an understanding of equivalence by realizing that the number of parts and the size of the parts compensate for each other: if a whole is cut into twice as many parts, the size of each part will be halved. The evidence currently available from children's arguments and strategies when they are learning about fractions using either of these action schemes shows that children using the correspondence scheme can develop some understanding of equivalence in the way it was hypothesized but those learning to use partitioning did not produce the anticipated arguments.

Research reviewed here shows that it is possible for some children to reason about quantities that would be represented by fractions without knowing how to represent them. This was established unambiguously by Kornilaki and Nunes (2005) and by Mamede (2007), who demonstrated that children who had not yet been taught about fractions and could not represent fractional quantities numerically (see Mamede, 2007) could nevertheless establish the order and equivalence of quantities that would be generated when a certain number of cakes was shared between a larger number of recipients.

These findings have important implications for education. First, we know that children can reason about quantities that are represented by natural numbers without having to count the elements; now we also know that they can reason about quantities that would be represented by fractions without knowing fractional representation. This means that schools could be working towards developing the children's quantitative reasoning before, or at the same time as, they are taught fractional representations. Currently the most usual practice, in the U.K. at any rate, is to focus initially on representations and only later to promote the students' reasoning about order and equivalence of fractions.

Second, the practice of anchoring children's fraction concepts on partitioning must be reconsidered. This scheme develops more slowly than the scheme of correspondences and seems to afford less insight into relations between fractional quantities.

Third, teachers might profit from being aware of children's own arguments for the equivalence and order of fractions when they use these two action schemes. Educational practice seems to be to teach children algorithms that represent these insights without necessarily anchoring them in the children's understanding of quantities: for example, children might be asked to double the numerator and the denominator successively and thus construct a number of equivalent fractions. However, successful learning of this procedure is not the same as developing an insight into why it works. If a teacher knows the students' arguments for such equivalences in correspondence situation, the teacher can help the students express this insight numerically and perhaps arrive at the algorithm.

Finally, the analysis presented here opens the way for a fresh research agenda in the teaching and learning of fractions. The source for the new research questions is the idea that children can achieve insights into relations between fractional quantities before knowing how to represent them. It is possible to envisage a research agenda that would not be about children's misconceptions about fractions as much of the work in the past has been, but about children's possibilities of success with fractions when teaching starts from thinking about quantities rather than from learning fractional representations.

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