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Common Fixed Point Theorems in

Cone Metric Space using W-Distance

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Abstract

In this paper, we introduce cone metric space with w-distance on X. Then we prove some common fixed point theorems for a pair of weakly compatible mappings in cone metric space using w-distance on X.

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Keywords: Cone metric space, w-distance, Common fixed point, Compatible maps.

1 Introduction

Haung and Zhang [3] generalized the concept of metric Space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for mappings satisfying different contractive conditions. They introduced cone metric space without w-distance. The metric space with w-distance was introduced by O. Kada et al [2]. Abbas and Jungck [1] proved some common fixed point theorems for weakly compatible mappings in the setting of cone metric space. D. IIić and Rakoćević [4], Rezapour and Hamlbarani [5] also proved some common fixed point theorems on cone metric spaces. Our objective is to extend these concepts together to establish a common fixed point theorem for a pair of weakly compatible mappings in cone metric space using w-distance. Consequently, we improve and generalize various results existing in the literature.

In the present paper, we extend results of Abbas and Jungck [1], Haung and Zhang [3] to more general aspects for common fixed point theorems to more new contractive type mappings in cone metric space using w-distance.

2 Preliminaries

DEFINITION 2.1. Let (X,d) be complete metric space. Then the mapping p : X×X→$[0,\infty)$ is called a w-distance on X if the following conditions are satisfied.

i) $0 \le p$ (x, y) for all x, y∈X;

ii) p (x, z) ≤ p (x, y) + p (y, z) for all x, y ,z ∈X;

iii) p (x, .)→ $[0,\infty)$ is lower semi continuous for all x ∈X;

iv) For any 0≤ η, there exist 0≤ δ such that p (z, x) ≤ δ and p (z, y)≤ δ imply d (x, y) ≤ η for all η , δ ∈ $[0,\infty)$.

DEFINITION 2.2. Let E be a real Banach space. $P \subseteq E$, is called a cone if and only if,

i) P is closed, non empty and $P \neq \{0\}$;

ii) $a,b \in \Re$, $a,b \ge 0$, and $x,y \in P$ implies $ax+by \in P$.

iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\le$ with respect to P by $x \le y$, If and only if $y - x \in P$. A cone P is called normal if there is a number K >0 ∍, $\forall x,y \in E$, the inequality $0 \le x \le y$ implies $\|x\| \le K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \operatorname{Int} P$.

DEFINITION 2.3. [3]. Let X be a non empty set. A mapping $d : X \times X \to E$ is called cone metric on X, if it satisfies the following properties.

i) $0 \le d(x,y)$ $\forall x,y \in X$ and $d(x,y)=0$ if and only if, $x = y$;

ii) $d(x,y) = d(y,x) \forall x,y \in X$;

iii) $d(x,y) \le d(x,z) + d(z,y) \forall x,y,z \in X$

Then (X,d) is called cone metric space.

DEFINITION 2.4. Let (X,d) be a cone metric space. We say that $\{x_n\}$ is,

i) a cauchy sequence if for every $u \in E$ with $u \gg 0$, there is M such that, $\forall n,m > M$, we have

$d(x_n,\ x_m) \ll u.$

ii) a convergent sequence if for every $u \in E$ with $u \gg 0$, there is M such that, $\forall \; n > M$, we have
$d(x_n, \; x) \ll u$ for some fixed $x \in X$.

DEFINITION 2.5. Let S and T be self mapping of a set X. If $u = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T and u is called a point of coincidence of S and T.

DEFINITION 2.6. [6] Two self mappings S and T of a set X are said to be weakly compatible if they commute at their coincidence point. i.e , if $Su = Tu$ for some $u \in X$, then $STu = TSu$

PROPOSITION 2.7. [1] Let S and T be weakly compatible self mappings of a set X. If S and T have a unique point of coincidence, i.e, $u = Sx = Tx$, then u is the unique common fixed point of S and T.

3. Results

Theorem 3.1. Let (X, d) be a complete cone metric space with $w-$distance p. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S, T : X \to X$ satisfy the contractive condition,

$$p(Sx, Sy) \leq r[p(Sx, Ty) + p(Sy, Tx) + p(Sx, Tx) + p(Sy, Ty)$$

Where, $r \in [0, 1/4)$ is a constant.

If the range of T contains the range of S, and $T(X)$ is complete subspace of X, then S and T have a unique coincidence point in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof: Let x_0 be arbitrary point in X. Then, Since $S(X) \subset T(X)$, we choose a point x_1 in X such that, $S(x_0) = T(x_1)$, Continuing this process, we choose x_n and x_{n+1} in X such that, $S(x_n) = T(x_{n+1})$. Then,

$$\begin{aligned} p(Tx_{n+1}, Tx_n) &= p(Sx_n, Sx_{n-1}) \\ &\leq r[p(Sx_n, Tx_{n-1}) + p(Sx_{n-1}, Tx_n) + p(Sx_n, Tx_n) + p(Sx_{n-1}, Tx_{n-1})] \\ &\leq 2r[p(Tx_{n+1}, Tx_n) + p(Tx_n, Tx_{n-1})], \end{aligned}$$

So, we have

$$p(Tx_{n+1}, Tx_n) \leq g \quad p(Tx_n, Tx_{n-1}) \text{ with } g = 2r/_{1-2r}.$$

Now for $n > m$, we get

$$p(Tx_n, Tx_m) \leq p(Tx_n, Tx_{n-1}) + p(Tx_{n-1}, Tx_{n-2}) + \cdots\cdots + p(Tx_{m+1}, Tx_m)$$

$$\leq (g^{n-1} + g^{n-2} + \ldots\ldots + g^m) p(Tx_1, Tx_0)$$

$$\leq \frac{g^m}{(1-g)} p(Tx_1, Tx_0)$$

Now by using the normality of cone P, we get

$$\|p(Tx_n, Tx_m)\| \leq \frac{g^m}{(1-g)} \ K \|p(Tx_1, Tx_0)\|$$

Then $p(Tx_n, Tx_m) \to 0$ as $n, m \to \infty$. So $\{Tx_n\}$ is a Cauchy sequence in X. Since $T(X)$ is complete subspace of X, so there exist q in $T(X)$ such that, $Tx_n \to q$, as $n \to \infty$. Consequently, we can find h in X such that, $T(h) = q$. Thus,

$p(Tx_n, Sh) = p(Sx_{n-1}, Sh)$

$\leq r[p(Sx_{n-1}, Th) + p(Sh, Tx_{n-1}) + p(Sx_{n-1}, Tx_{n-1}) + p(Sh, Th).$

By using the normality of cone, implies that,

$\|p(Tx_n, Sh)\| \leq K \ g \|p(Tx_{n-1}, Th)\| = 0$ as $n \to \infty$.

Hence,

$$p(Tx_n, Sh) \to 0 \text{ as } n \to \infty.$$

Also we have,

$$p(Tx_n, Th) \to 0 \text{ as } n \to \infty.$$

The uniqueness of a limit in cone metric space implies that $S(h) = T(h)$.
So h is a coincidence point of S and T. Again we show the uniqueness.
For this, suppose that there exist another point u in X such that, $S(u) = T(u)$.
So we have,

$$p(Tu, Th) = P(Su, Sh)$$
$$\leq r[p(Su, Th) + p(Sh, Tu) + p(Su, Tu) + p(Sh, Th)$$

By using the normality of cone , we get

$\|p(Tu, Th)\| = 0$, we get

$Tu = Th.$

Finally using proposition (2.7) we conclude that S and T have unique common fixed point. This completes the proof of theorem 3.1.

Theorem 3.2 . Let (X, d) be a complete cone metric space with $w-$distance p. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S, T : X \to X$ satisfies the contractive condition,

$$p(Sx, Sy) \leq r[p(Tx, Ty) + p(Sx, Tx) + p(Sy, Ty) + p(Tx, Sy) + p(Sx, Ty)], \quad \forall x, y \in X.$$

$r \in [0, 1/5)$ is constant.

If the range of T contains the range of S and $T(X)$ is complete subspace of X, then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof: Let x_0 be arbitrary point in X . Then, Since $S(X) \subset T(X)$, we choose a point x_1 in X such that, $S(x_0) = T(x_1)$, Continuing this process, we choose x_n and x_{n+1} in X such that, $S(x_n) = T(x_{n+1})$. Then,

$$p(Tx_{n+1}, Tx_n) = p(Sx_n, Sx_{n-1}) \leq$$
$$r[p(Tx_n, Tx_{n-1}) + p(Sx_n, Tx_n) + p(Sx_{n-1}, Tx_{n-1}) + p(Tx_n, Sx_{n-1}) + p(Sx_n, Tx_{n-1})$$
$$\leq 2r[p(Tx_{n+1}, Tx_n) + p(Tx_n, Tx_{n-1})]$$

So, we have

$$p(Tx_{n+1}, Tx_n) \leq g \quad p(Tx_n, Tx_{n-1}) \text{ with } g = {}^{2r}\!\!\diagup_{1-2r}.$$

Now for $n > m$, we get

$$p(Tx_n, Tx_m) \leq p(Tx_n, Tx_{n-1}) + p(Tx_{n-1}, Tx_{n-2}) + \cdots\cdots + p(Tx_{m+1}, Tx_m)$$
$$\leq (g^{n-1} + g^{n-2} + \ldots\ldots + g^m) p(Tx_1, Tx_0)$$
$$\leq \frac{g^m}{(1-g)} p(Tx_1, Tx_0)$$

Now by using the normality of cone, we get

$$\|p(Tx_n, Tx_m)\| \leq \frac{g^m}{(1-g)} \quad K \|p(Tx_1, Tx_0)\|$$

Then $p(Tx_n, Tx_m) \to 0$ as $n, m \to \infty$. So $\{Tx_n\}$ is a Cauchy sequence in X . Since $T(X)$ is complete subspace of X , so there exist q in $T(X)$ such that, $Tx_n \to q$, as $n \to \infty$. Consequently, we can find h in X such that, $T(h) = q$. Thus,

$$p(Tx_n, Sh) = p(Sx_{n-1}, Sh)$$
$$\leq r[p(Tx_{n-1}, Th) + p(Sx_{n-1}, Tx_{n-1}) + p(Sh, Th) + p(Tx_{n-1}, Sh) + p(Sx_{n-1}, Th)].$$
$$\leq r[p(Tx_{n-1}, Th) + p(Tx_n, Tx_{n-1}) + p(Sh, Th) + p(Tx_{n-1}, Sh) + p(Tx_n, Th)].$$

By using the normality of cone, implies that,

$$\|p(Tx_n, Sh)\| \leq K \quad g \|p(Tx_{n-1}, Th)\| = 0 \text{ as } n \to \infty.$$

Hence,

$$p(Tx_n, Sh) \to 0 \text{ as } n \to \infty.$$

Also we have,

$$p(Tx_n, Th) \to 0 \text{ as } n \to \infty.$$

The uniqueness of a limit in cone metric space implies that $S(h) = T(h)$.
So h is a coincidence point of S and T. Again we show the uniqueness.
For this, suppose that there exist another point u in X such that, $S(u) = T(u)$.
So we have,

$$p(Tu, Th) = P(Su, Sh)$$
$$\leq r[p(Tu, Th) + p(Su, Tu) + p(Sh, Th) + p(Tu, Sh) + p(Su, Th)].$$

By using the normality of cone, we get

$\|p(Tu,Th)\| = 0$, we get

$Tu = Th.$

In the same way by using proposition (2.7), the rest can be proved.
This completes the proof of theorem 3.2.

Remark 3.3 . In theorem 3.2, if $T = I_x$ is the identity map on X, and X is complete cone metric space, then as a consequences of theorem 3.2, we obtain the following result.

Corollary 3.4. Let (X,d) be a complete cone metric space with $w-$distance p. Suppose that the mapping $S : X \to X$ satisfies the condition ,

$p(Sx,Sy) \le r[p(x,y) + p(x,Sx) + p(y,Sy) + p(x,Sy) + p(y,Sx)] \quad \forall x, y \in X.$

Then S has a unique fixed point u in X, and for any $x_0 \in X$, the successive iterates, $x_n = Sx_{n-1}$ converges to u.

Corollary 3.5 . Let (X,d) be a complete cone metric space with $w-$distance p. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S,T : X \to X$ satisfies the contractive condition,

$p(Sx,Sy) \le r(p(Sx,Tx) + p(Sy,Ty)) \;\; \forall x, y \in X$

Where $r \in [0, 1/2)$ is constant. Also $T(X) \subset S(X)$, then S and T have a unique coincidence point in X.

Corollary 3.6 . Let (X,d) be a complete cone metric space with $w-$distance p. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S,T : X \to X$ satisfies the contractive condition,

$p(Sx,Sy) \le r(p(Sx,Ty) + p(Sy,Tx)) \;\; \forall x, y \in X$

Where $r \in [0, 1/2)$ is constant. Also $T(X) \subset S(X)$, then S and T have a unique coincidence point in X.

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