New mixed finite element formulations for transient and steady state creep problems

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We apply the mixed Petrov–Galerkin formulation to construct finite element approximations for transient and steady-state creep problems. With the new approach we recover stability, convergence, and accuracy of some Galerkin unstable approximations. We also present the main results on the numerical analysis and error estimates of the proposed finite element approximation for the steady problem, and discuss the asymptotic behavior of the continuum and discrete transient problems.

1. INTRODUCTION

This paper is an expanded version of the communication "A new finite element method for nonlinear creeping flow" (Loula and Guerreiro, 1989a) presented at the I Pan American Congress of Applied Mechanics, in which we apply the mixed Petrov–Galerkin formulation to construct finite element approximations for steady state creep problems. We now extend this methodology to transient or elastocreep problems.

Let $\Omega \subset R^n$, $1 \leq n \leq 3$, with smooth boundary Γ, be the domain occupied by an inelastic body subjected to volume force $\mathbf{f}$. The steady state creep problem we consider here consists in finding the Cauchy stress tensor σ^0: $\Omega \to R^n \times R^n$, and the velocity field $\mathbf{u}^0$: $\Omega \to R^n$, satisfying

$$\operatorname{div} \sigma^0 + \mathbf{f} = \mathbf{0} \quad \text{in } \Omega, \tag{1}$$

$$A(\sigma^0) = B\mathbf{u}^0 \quad \text{in } \Omega, \tag{2}$$

with homogeneous Dirichlet boundary conditions

$$\mathbf{u}^0(\mathbf{x}) = \mathbf{0} \quad \text{on } \Gamma, \tag{3}$$

where $B\mathbf{u} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ is the creep strain rate and $A(\sigma)$ is a nonlinear function of the deviatoric part of the stress tensor,

$$\mathbf{S} = \sigma - \frac{1}{n} \operatorname{tr} \sigma \mathbf{I} = \sigma - \mathbf{p}\mathbf{I} \tag{4}$$

with $\mathbf{p}$ being the hydrostatic pressure. The operator A is subjected to the internal constraint $\operatorname{tr} A = A : \mathbf{I} = 0$, implying that the creep strain rate must satisfy the incompressibility condition $\operatorname{div} \mathbf{u} = 0$ in Ω, which is the cause of the main difficulties in constructing finite element approximations for this class of problems.

We will also consider a transient or elastocreep problem consisting in finding, for each time $t \in [t_0, t_1]$, the Cauchy stress tensor σ: $\Omega \to R^n \times R^n$, and the velocity field $\mathbf{u}$: $\Omega \to R^n$, satisfying

$$\operatorname{div} \sigma + \mathbf{f} = 0 \text{in } \Omega \times [t_0, t_1], \tag{5}$$

$$C\dot{\sigma} + A(\sigma) = B\mathbf{u} \quad \text{in} \quad \Omega \times [t_0, t_1], \tag{6}$$

with

$$\mathbf{u}(\mathbf{x}) = \mathbf{0} \quad \text{on } \Gamma \times [t_0, t_1], \tag{7}$$

and

$$\sigma(\mathbf{x}, t_0) = \sigma^e(\mathbf{x}) \quad \text{in } \Omega, \tag{8}$$

where C is the inverse of the elasticity tensor, the dot denotes time derivative, and σ^e is the stress field which satisfies the associated elastic problem.

2. VARIATIONAL FORMULATIONS

To present the variational formulations for the proposed problems, we need to introduce some definitions in function spaces. Let

$$L^p(\Omega) = \left\{ f, f \text{ measurable}, \int_\Omega |f(\mathbf{x})|^p \, d\Omega < +\infty \right\}, \quad 1 \leq p < \infty, \tag{9}$$

be the Banach space of functions whose absolute values are p-integrable in Ω, with norm

$$\|f\|_{0,p} = \left(\int_\Omega |f|^p \, d\Omega \right)^{1/p} \quad \forall f \in L^p(\Omega), \tag{10}$$

and dual $L^q(\Omega)$, with the duality pairing between $L^p(\Omega)$

and $L^q(\Omega)$ denoted by $(\cdot,\cdot)$ and defined as

$$(f,g)=\int_\Omega fg\,d\Omega \quad \forall f\in L^p(\Omega), \forall g\in L^q(\Omega),$$

$$\text{with } 1/p+1/q=1. \tag{11}$$

Let $\partial^\alpha f$ denotes the derivative of f in the distributional sense,

$$\partial^\alpha f=\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, \tag{12}$$

with α_i integer natural and $|\alpha|=\alpha_1+\cdots+\alpha_n$. Let $W^{m,p}(\Omega)$ be the Sobolev space

$$W^{m,p}(\Omega)=\{f\in L^p(\Omega);\forall\alpha,|\alpha|\le m,\partial^\alpha f\in L^p(\Omega)\}, \tag{13}$$

with its usual norm

$$\|f\|_{m,p}=\left(\sum_{|\alpha|\le m}\int_\Omega|\partial^\alpha f|^p\,d\Omega\right)^{1/p} \quad \forall f\in W^{m,p}(\Omega). \tag{14}$$

We now define

$$U=\{\tau=[\tau_{ij}],\ \tau_{ij}=\tau_{ji}\in L^p(\Omega);\ i,j=1,\ldots,n\} \tag{15}$$

as the space of stress tensors with norm

$$\|\tau\|_U=\left(\int_\Omega|\tau|^p\,d\Omega\right)^{1/p} \quad \forall\tau\in U, \tag{16}$$

and

$$V=\{\mathbf{v}=\{v_i\},\ v_i\in W_0^{1,q}(\Omega),\ i=1,\ldots,n\} \tag{17}$$

as the space of velocity vectors with norm

$$\|\mathbf{v}\|_V=\left(\int_\Omega\left(|\mathbf{v}|^q+|\nabla\mathbf{v}|^q\right)d\Omega\right)^{1/q} \quad \forall\mathbf{v}\in V, \tag{18}$$

where

$$W_0^{1,q}(\Omega)=\{f\in W^{1,q}(\Omega),\ f=0 \text{ on } \Gamma\}. \tag{19}$$

We shall also use the subspace $L_0^p(\Omega)=\{g\in L^p(\Omega);(g,1)=0\}$ of zero average pressure fields, the subspace $U_0=\{\tau\in U;(\operatorname{tr}\tau,1)=0\}$ of stress tensors with zero average trace and the subspace $U_T=\{\mathbf{T}\in U;\operatorname{tr}\mathbf{T}=0\}$ of deviatoric stress tensors. The variational formulation for the steady state creep problem we consider here consists in

Problem M^0: Given $\mathbf{f}\in V^*$, dual space of $\mathbf{V}$, find $(\sigma^0,\mathbf{u}^0)\in U_0\times V$ such that

$$(A(\sigma^0),\tau)=b(\tau,\mathbf{u}^0) \quad \forall\tau\in U_0, \tag{20}$$

$$b(\sigma^0,\mathbf{v})=f(\mathbf{v}) \qquad \forall\mathbf{v}\in V, \tag{21}$$

with

$$(A(\sigma),\tau)=\int_\Omega A(\mathbf{S}):\tau\,d\Omega \quad \forall\sigma,\tau\in U_0, \tag{22}$$

$$b(\tau,\mathbf{v})=\int_\Omega\tau:B\mathbf{v}\,d\Omega \qquad \forall\tau\in U_0,\forall\mathbf{v}\in V, \tag{23}$$

$$f(\mathbf{v})=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\Omega \qquad \forall\mathbf{v}\in V,\mathbf{f}\in V^*. \tag{24}$$

A complete characterization of **Problem M^0** will depend on the creep constitutive equation defined by the operator $A(\cdot)$. In particular, we consider the Odqvist–Norton power law governing incompressible creeping flow of metal or non-Newtonian fluid, given by

$$A(\sigma)=\mu|\mathbf{S}|^{p-2}\mathbf{S}=B\mathbf{u}, \tag{25}$$

in which μ is a constant. When $p=2$ in (25), M^0 becomes formally identical to the linear incompressible elasticity problem with U and V being Hilbert spaces obtained by products of $L^2(\Omega)$ and $H_0^1(\Omega)=W_0^{1,2}$, respectively. The classical results by Brezzi (1974) on mixed methods could be applied to the analysis of this particular case of M^0. An analysis of M^0 in Banach spaces can be done using results by Scheurer (1977), where generalizations of Brezzi's theorems are presented for nonlinear variational equations or inequations subjected to linear constraints. In Loula and Guerreiro (1989b) the following theorem on existence and uniqueness of solution of M^0 in Hilbert spaces is proved.

Theorem 2.1. Let U and V be Hilbert spaces, and A: $U\to U$ a nonlinear operator strongly elliptic in U_T and Lipschitz continuous on bounded sets of U, ie, there exist constants $\alpha>0$ and $M(r)$ such that

$$(A(\mathbf{T}_2)-A(\mathbf{T}_1),\mathbf{T}_2-\mathbf{T}_1)\ge\alpha\|\mathbf{T}_2-\mathbf{T}_1\|_U^2 \quad \forall\mathbf{T}_1,\mathbf{T}_2\in U_T, \tag{26}$$

$$(A(\tau_2)-A(\tau_1),\tau_3)\le M(r)\|\tau_2-\tau_1\|_U\|\tau_3\|_U \quad \forall\tau_1,\tau_2\in Y(r),\forall\tau_3\in U, \tag{27}$$

where $Y(r)=\{\tau\in U;\|\tau\|_U<r\}$, is any open ball with finite radius r. Then **Problem M^0** has a unique solution $(\sigma^0,\mathbf{u}^0)\in U_0\times V$.

For the transient creep problem we have the following variational formulation.

Problem M: Find $(\sigma(t),\mathbf{u}(t))\in U\times V$ satisfying, for each time $t\in[t_0,t_1]$,

$$c(\dot\sigma,\tau)+(A(\sigma),\tau)=b(\tau,\mathbf{u}) \quad \forall\tau\in U, \tag{28}$$

$$b(\sigma,\mathbf{v})=f(\mathbf{v}) \qquad \forall\mathbf{v}\in V, \tag{29}$$

with

$$(\sigma(t_0),\tau)=(\sigma^e,\tau) \quad \forall\tau\in U. \tag{30}$$

The forms $(A(\sigma), \tau)$, $b(\tau, \mathbf{v})$, and $f(\mathbf{v})$ were already defined in (22), (23), and (24), respectively, and $c(\dot{\sigma}, \tau)$ given by

$$c(\dot{\sigma}, \tau) = \int_\Omega C\dot{\sigma} : \tau \, d\Omega \quad \forall\, \dot{\sigma}, \tau \in U \tag{31}$$

is such that

$$c(\tau, \tau) \geq \gamma \|\tau\|_U \quad \forall\, \tau \in U, \tag{32}$$

with $\gamma > 0$, independent of τ. For homogeneous and isotropic materials,

$$c(\dot{\sigma}, \tau) = \frac{1+\nu}{E} \int_\Omega \left(\dot{\sigma} : \tau - \frac{\nu}{1+\nu} \operatorname{tr} \dot{\sigma} \operatorname{tr} \tau \right) d\Omega \quad \forall\, \dot{\sigma}, \tau \in U, \tag{33}$$

where E is the Young modulus and ν is the Poisson ratio.

We stress that **Problem M** is formulated in the whole space U, and not in U_0 like **Problem $\mathbf{M^0}$**: In the transient creep problem the hydrostatic pressure is determined by the initial condition of the stress field since we admitted that the elastic strains are compressible ($\nu < 0.5$). Therefore, if $(\sigma(t), \mathbf{u}(t))$ and $(\sigma^0, \mathbf{u}^0)$ are solutions of $\mathbf{M}$ and $\mathbf{M^0}$, respectively, then we can prove that (Guerreiro, 1988):

$$\lim_{t \to \infty} \sigma(t) = \sigma^0 + c\mathbf{I}, \tag{34}$$

$$\lim_{t \to \infty} \mathbf{u}(t) = \mathbf{u}^0, \tag{35}$$

with $c \in R$ representing a constant hydrostatic pressure. Due to the strong nonlinearity of the operator $A(\cdot)$, and considering (34) and (35) it is usual to approximate the solution of the steady state creep problem by the solution of the transient problem for sufficiently large time t. We will discuss later some aspects of this methodology when applied to discrete finite element approximations

3. FINITE ELEMENT APPROXIMATIONS

For simplicity we restrict this presentation to bidimensional creep problems and assume that $\Omega \subset R^2$ is a polygonal domain discretized by a uniform mesh of Ne elements such that

$$\bar{\Omega} = \bigcup_{e=1}^{Ne} \bar{\Omega}^e \quad \text{and} \quad \Omega^e \cap \Omega^f = \varnothing, \quad e \neq f \tag{36}$$

where Ω^e denotes the interior of the e^{th} element and $\bar{\Omega}^e$ its closure. Let $Q_h^l(\Omega) \subset L^2(\Omega)$ be the space of C^{-1} piecewise polynomial finite element interpolations of degree l, and let $S_h^k(\Omega) = Q_h^k(\Omega) \cap H_0^1(\Omega)$ be the space of C^0 piecewise polynomial finite element interpolations of degree k, with zero value on the boundary. The mesh parameter h is defined as $h = \max h_e$, $e = 1, 2, \ldots, Ne$, with h_e being the diameter of element e. We then define finite element approximations for U and V as $U_h^l = (Q_h^l)^3 \subset U$ and $V_h^k = (S_h^k)^2 \subset V$, respectively, and introduce the subspaces: $Q_{0h}^l = Q_h^l \cap L_0^2(\Omega)$, $U_{0h}^l = U_h^l \cap U_0$ and $U_{Th}^l = U_h^l \cap U_T$. The Galerkin approximation for **Problem $\mathbf{M^0}$** in the space U_{0h}^l and V_h^k is given by

Problem M_h^0: Find $(\sigma_h^0, \mathbf{u}_h^0) \in U_{0h}^l \times V_h^k$ such that

$$\left(A(\sigma_h^0), \tau_h\right) = b\left(\tau_h, \mathbf{u}_h^0\right) \quad \forall\, \tau_h \in U_{0h}^l, \tag{37}$$

$$b\left(\sigma_h^0, \mathbf{v}_h\right) = f(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h^k. \tag{38}$$

An analysis of this method can be performed using Scheurer (1977), where generalizations of Brezzi's theorem on the discrete problem are also presented. Scheurer's analysis is based on the following hypotheses:

(H1) ($K_h(f)$-ellipticity of A): There exists a constant $\alpha_h > 0$, independent of h, such that

$$\left(A(\sigma_h) - A(\chi_h), \sigma_h - \chi_h\right) \geq \alpha_h \|\sigma_h - \chi_h\|_U^p \quad \forall\, \sigma_h, \chi_h \in K_h(f),\ p \geq 2, \tag{39}$$

where

$$K_h(f) = \left\{ \tau_h \in U_{0h}^l,\ b(\tau_h, \mathbf{v}_h) = f(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h^k \right\}. \tag{40}$$

(H2) (Continuity of A): There exists a constant $M_h < \infty$, independent of h, such that

$$\|A(\sigma_h) - A(\tau_h)\|_U \leq M_h \left(\|\sigma_h\|_U + \|\tau_h\|_U\right)^{p-2} \|\sigma_h - \tau_h\|_U \quad \forall\, \sigma_h, \tau_h \in U_h^l,\ p \geq 2. \tag{41}$$

(H3) (Discrete LBB condition for the velocity field): There exists a constant $\beta_h > 0$, independent of h, such that

$$\sup_{\tau_h \in U_h^l} \frac{b(\tau_h, \mathbf{v}_h)}{\|\tau_h\|_U} \geq \beta_h \|\mathbf{v}_h\|_V \quad \forall\, \mathbf{v}_h \in V_h^k. \tag{42}$$

Due to the internal constraint $A : I = 0$, the operator A is not elliptic in the whole space U. For this reason hypothesis (H1) is generally very hard to fulfill. Normally it depends on the next hypothesis.

(H4) (Discrete LBB condition for the pressure): There exists a constant $\eta_h > 0$, independent of h, such that

$$\sup_{\mathbf{v}_h \in V_h^k} \frac{(\mathbf{q}_h, \operatorname{div} \mathbf{v}_h)}{\|\mathbf{v}_h\|_V} \geq \eta_h \|\mathbf{q}_h\|_{0,p} \quad \forall\, \mathbf{q}_h \in Q_{0h}. \tag{43}$$

As presented here $\mathbf{M}_h^0$ is always unstable. For $l \geq k$ hypotheses (H2) and (H3) are verified but not (H4) and consequently (H1). For $l < k$ we may verify hypotheses (H4) and (H1) but not (H3), which governs the stability of the Lagrange multiplier, in this case the velocity field. To construct stable Galerkin finite element approximations for this type of constrained problems, we have to segregate the pressure field from the deviatoric part of the stress tensor, and interpolate the pressure, the deviatoric stress, and the velocity fields independently in order to fulfill hypotheses (H1)–(H4). This fact, of course, causes serious limitations in approximating this type of problem using the classical approach. Violation of any of the hypotheses (H1)–(H4) may cause nonuniqueness of solution for Problem M_h^0. The most common loss of uniqueness is due to violation of the discrete LBB condition for the pressure field, (H4). We observe that if $(\sigma_h^0, \mathbf{u}_h^0)$ is one solution of $\mathbf{M}_h^0$, then any other pair $(\sigma_h^0 + \tilde{\mathbf{p}}_h^0 \mathbf{I}, \mathbf{u}_h^0)$, with $\tilde{\mathbf{p}}_h^0 \in Q_{0h}$, satisfying

$$\left(\tilde{\mathbf{p}}_h^0, \operatorname{div} \mathbf{v}_h\right) = 0 \quad \forall\, \mathbf{v}_h \in V_h^k, \tag{44}$$

is also solution of $\mathbf{M}_h^0$. Nontrivial pressure fields $\tilde{\mathbf{p}}_h^0$, verifying (44), are usually referred to as spurious pressure modes. For equal order discontinuous pressure and continuous velocity interpolations, Eq (44) has many nontrivial solutions especially with lower-order interpolations. These spurious modes cause the instability of Galerkin approximations. About spurious pressure modes in finite element approximations of Stokes problem, see Oden and Kikuchi (1982) and Oden and Jacquotte (1984).

The Galerkin approximation for the transient creep problem is given by

Problem M_h: For each $t \in [t_0, t_1]$, find $(\sigma_h, \mathbf{u}_h) \in U_h^l \times V_h^k$, such that

$$c(\dot{\sigma}_h, \tau_h) + \left(A(\sigma_h), \tau_h\right) = b(\tau_h, \mathbf{u}_h) \quad \forall\, \tau_h \in U_h^l, \tag{45}$$

$$b(\sigma_h, \mathbf{v}_h) = f(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h^k, \tag{46}$$

with

$$\left(\sigma_h(t_0), \tau_h\right) = \left(\sigma^e, \tau_h\right) \quad \forall\, \tau_h \in U_h^l. \tag{47}$$

Contrary to **Problem $\mathbf{M}_h^0$**, even for $l = k$, **Problem $\mathbf{M}_h$** has a unique solution. But in the presence of spurious pressure modes in $\mathbf{M}_h^0$, we should expect the following asymptotic behavior for the discrete pressure:

$$\lim_{t \to \infty} \mathbf{p}_h(t) = \mathbf{p}_h^0 + \mathbf{p}_{sh}^e \tag{48}$$

where $\mathbf{p}_h^0$ is the pressure field satisfying M_h^0 with the spurious pressure modes filtered out, and $\mathbf{p}_{sh}^e$ is the projection of the spurious pressure modes on the pressure initial condition. Therefore, although M_h has a unique solution, it does not approximate well the real solution of the continuous problem. We also note that misbehaviors, like locking and spurious pressure oscillations typical of unstable approximations of steady state incompressible problems, are masked in the transient case. To overcome these limitations, we used the mixed Petrov–Galerkin method introduced in Loula, Hughes, Franca, and Miranda (1987) in the context of Timoshenko's beam problem. With this new formulation $\mathbf{M}_h^0$ is approximated by

Problem PG_h^0: Find $(\sigma_h^0, \mathbf{u}_h^0) \in U_{0h}^l \times V_h^k$, such that

$$\left(A_h(\sigma_h^0), \tau_h\right) = b\left(\tau_h, \mathbf{u}_h^0\right) - g_h(\tau_h) \quad \forall\, \tau_h \in U_{0h}^l, \tag{49}$$

$$b\left(\sigma_h^0, \mathbf{v}_h\right) = f(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h^k, \tag{50}$$

where

$$\left(\mathbf{A}_h(\sigma_h), \tau_h\right) = \left(\mathbf{A}(\sigma_h), \tau_h\right) + \frac{\delta h^2}{\vartheta}\left(\operatorname{div} \sigma_h, \operatorname{div} \tau_h\right)_h \quad \forall\, \sigma_h, \tau_h \in U_{0h}, \tag{51}$$

$$g_h(\tau_h) = \frac{\delta h^2}{\vartheta}\left(\mathbf{f}, \operatorname{div} \tau_h\right)_h \quad \forall\, \tau_h \in U_{0h}^l, \tag{52}$$

with δ being a positive scalar, ϑ a viscosity parameter, and $(\cdot, \cdot)_h$ defined as

$$(\mu_h, \phi_h)_h = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \mu_e \cdot \phi_e \, d\Omega \quad \forall\, \mu_h, \phi_h \in \left(Q_h^l\right)^2, \tag{53}$$

with

$$\|\mu_h\|_h^2 = (\mu_h, \mu_h)_h \quad \forall\, \mu_h \in \left(Q_h^l\right)^2, \tag{54}$$

where μ_e and ϕ_e are the restrictions of μ_h and ϕ_h to element e.

A detailed analysis of $\mathbf{PG}_h^0$ can be found in Loula and Guerreiro (1989b), where the following result on existence, uniqueness and convergence is proved.

Theorem 3.1. For $\delta > 0$ and $l \geq k \geq 2$, **Problem $\mathbf{PG}_h^0$** has a unique solution $(\sigma_h^0, \mathbf{u}_h^0) \in U_{0h}^l \times V_h^k$ and the following estimate holds

$$\|\sigma^0 - \sigma_h^0\|_{h,U} + \|\mathbf{u}^0 - \mathbf{u}_h^0\|_V \leq C\left(\|\sigma^0 - \tau_h\|_{h,U} + \|\mathbf{u}^0 - \mathbf{v}_h\|_V\right) \quad \forall\, \tau_h \in U_{0h}^l \quad \forall\, \mathbf{v}_h \in V_h^k, \tag{55}$$

with C independent of h, and

$$\|\sigma\|_{h,U} = \|\sigma\|_U + \sup_{\tau_h \in U_h} \frac{h\left(\operatorname{div} \sigma, \operatorname{div} \tau_h\right)_h}{\|\operatorname{div} \tau_h\|_h} \quad \forall\, \tau_h \in U_h, \operatorname{div} \tau_h \neq 0, \sigma \in U. \tag{56}$$

The proof of Theorem 3.1 relies on next lemma which is similar to the verification of hypothesis (H1) for the Galerkin method but for a modified discrete operator $A_h(\cdot)$.

Lemma ($K_h(f)$-ellipticity of A_h). Let

$$K_h(f) = \left\{\tau_h \in U_{0h}^l, b(\tau_h, \mathbf{v}_h) = f(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h^k\right\}. \tag{57}$$

Then, for $\delta > 0$ and $k \geq 2$, there exists a constant $\gamma_h > 0$, such that

$$\left(A_h(\sigma_h) + A_h(\chi_h), \sigma_h - \chi_h\right) \geq \gamma_h \|\sigma_h - \chi_h\|_U^2 \quad \forall\, \sigma_h, \chi_h \in K_h(f). \tag{58}$$

The δ term, corresponding to a least-square residual of the equilibrium equation, added to the Galerkin approximation is crucial to the stability of $\mathbf{PG}_h^0$. Admitting that the exact solution $(\sigma^0, \mathbf{u}^0)$ of **Problem** $\mathbf{M}^0$ has enough regularity and using classical results of finite element interpolation theory (Ciarlet, 1978), from (55) we obtain the following *a priori* error estimate:

$$\|\sigma^0 - \sigma_h^0\|_{h,U} + \|\mathbf{u}^0 - \mathbf{u}_h^0\|_V \leq C(\sigma^0, \mathbf{u}^0) h^k, \quad (59)$$

with $C(\sigma^0, \mathbf{u}^0)$ independent of h. The same rates of convergence were derived in Loula, Hughes, Franca and Miranda (1987) and in Franca, Hughes, Loula, and Miranda (1988) for the mixed Petrov–Galerkin formulation applied to the Timoshenko beam and linear incompressible elasticity problems. Nonuniqueness of the pressure field for the Petrov–Galerkin formulation occurs when nontrival solutions of

$$\left(\bar{\mathbf{p}}_h^0, \operatorname{div} \mathbf{v}_h\right) = 0 \quad \forall\, \mathbf{v}_h \in V_h^k \quad (60)$$

exist, with $\bar{\mathbf{p}}_h^0$ being a piecewise constant pressure field. This situation only happens when $k = 1$, where a checkerboard-type mode is present, but this checkerboard mode can be filtered out by post-processing the pressure (Hughes, Liu, and Brooks, 1979). For the transient creep problem we have the following Petrov–Galerkin finite element approximation.

Problem $\boldsymbol{PG_h}$: For each $t \in [t_0, t_1]$, find $(\sigma_h, \mathbf{u}_h) \in U_h^l \times V_h^k$, such that

$$c(\dot{\sigma}_h, \tau_h) + \left(A_h(\sigma_h), \tau_h\right) = b(\tau_h, \mathbf{u}_h) - g_h(\tau_h) \quad \forall\, \tau_h \in U_h^l, \quad (61)$$

$$b(\sigma_h, \mathbf{v}_h) = f(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h^k, \quad (62)$$

with

$$\left(\sigma_h(t_0), \tau_h\right) = \left(\sigma^e, \tau_h\right) \quad \forall\, \tau_h \in U_h^l. \quad (63)$$

The next theorem on the asymptotic behavior of the approximate solution of the transient creep problem is proved in Guerreiro (1988).

Theorem 3.2. If $(\sigma_h(t), \mathbf{u}_h(t))$ and $(\sigma_h^0, \mathbf{u}_h^0)$ are solutions of **Problem** $\mathbf{PG}_h$ and **Problem** $\mathbf{PG}_h^0$, respectively, then for $k \geq 2$ and $\delta > 0$

$$\lim_{t \to \infty} \sigma_h(t) = \sigma_h^0 + c_h \mathbf{I}, \quad (64)$$

$$\lim_{t \to \infty} \mathbf{u}_h(t) = \mathbf{u}_h^0, \quad (65)$$

with $c_h \in R$, representing a constant hydrostatic pressure.

Theorem 3.2 shows that the transient Petrov–Galerkin approximation, like the transient solution of the continuous problem, tends to the corresponding steady-state solution up to the natural hydrostatic pressure. Therefore, in this case we can accurately compute the solution of the steady state problem as the limit of the corresponding transient solution.

4. ALGORITHMS AND NUMERICAL RESULTS

To solve the nonlinear system of equations generated by **Problem** $\mathbf{PG}_h^0$, we derived an Uzawa-type algorithm (Loula and Guerreiro, 1989b), consisting of finding the sequence of functions $(\sigma_h^m, \mathbf{u}_h^m) \in U_h^l \times V_h^k$ satisfying, for all m,

Problem $\boldsymbol{PG_h^m}$: Find $(\mathbf{S}_h^{m+1}, \mathbf{p}_h^{m+1}, \mathbf{u}_h^{m+1}) \in U_{Th}^l \times Q_h^l \times V_h^k$, such that

$$\left(A_h(\mathbf{S}_h^{m+1}), \mathbf{T}_h\right) = b(\mathbf{T}_h, \mathbf{u}_h^m) - g_h(\mathbf{T}_h) - \frac{\delta h^2}{\vartheta}\left(\nabla \mathbf{p}_h^m, \operatorname{div} \mathbf{T}_h\right)_h \quad \forall\, \mathbf{T}_h \in U_{Th}^l, \quad (66)$$

$$-\frac{\delta h^2}{\vartheta}\left(\nabla \mathbf{p}_h^{m+1}, \nabla \mathbf{q}_h\right)_h + \left(\mathbf{q}_h, \operatorname{div} \mathbf{u}_h^{m+1}\right) = \frac{\delta h^2}{\vartheta}\left(\nabla \mathbf{q}_h, \mathbf{f}\right)_h + \frac{\delta h^2}{\vartheta}\left(\operatorname{div} \mathbf{S}_h^{m+1}, \nabla \mathbf{q}_h\right)_h \quad \forall\, \mathbf{q}_h \in Q_{0h}^l, \quad (67)$$

$$d\left(\mathbf{u}_h^{m+1} - \mathbf{u}_h^m, \mathbf{v}_h\right) + \rho_m\left(\mathbf{p}_h^{m+1}, \operatorname{div} \mathbf{v}_h\right) = \rho_m\left(f(\mathbf{v}_h) - b\left(\mathbf{S}_h^{m+1}, \mathbf{v}_h\right)\right) \quad \forall\, \mathbf{v}_h \in V_h^k, \quad (68)$$

where $d(\cdot,\cdot)$ is a bilinear form symmetric, continuous and V-elliptic, that is, there exist constants $M < \infty$ and $\xi > 0$ such that

$$d(\mathbf{u}, \mathbf{v}) \leq M\|\mathbf{u}\|_V \|\mathbf{v}\|_V \quad \forall\, \mathbf{u}, \mathbf{v} \in V, \quad (69)$$

$$d(\mathbf{v}, \mathbf{v}) \geq \xi \|\mathbf{v}\|_V^2, \quad \mathbf{v} \in V. \quad (70)$$

It is interesting to note that, at each iteration, the nonlinear equation (66) is solved on the element level while (67) and (68) are coupled but linear. To make possible the elimination of the pressure degrees of freedom elementwise, we added the regularization term $\epsilon(\mathbf{p}_h^{m+1}, \mathbf{q})$ to (67), as usual. In Loula and Guerreiro (1989b), the following result on convergence of this algorithm is proved:

Theorem 4.1. If $(\mathbf{S}_h^0, \mathbf{p}_h^0, \mathbf{u}_h^0)$ is the solution of **Problem** $\mathbf{PG}_h^0$, and γ_h and ξ are the ellipticity constants appearing in (58) and (70), respectively, then for $0 < \rho_m < 2\xi\gamma_h$

$$\lim_{m \to \infty} \mathbf{S}_h^m = \mathbf{S}_h^0, \quad (71)$$

$$\lim_{m \to \infty} \mathbf{p}_h^m = \mathbf{p}_h^0, \quad (72)$$

$$\lim_{m \to \infty} \mathbf{u}_h^m = \mathbf{u}_h^0. \quad (73)$$

To solve **Problem** $\mathbf{PG}_h$, corresponding to the transient creep analysis, we adopted a first-order finite-difference approximation for the time derivative of the stress field, resulting in the following sequence of discrete problems:

Problem PG_h^Δ: Find $(\sigma_h^{n+1}, \mathbf{u}_h^{n+1}) \in U_h^l \times V_h^k$, satisfying

$$c\left(\frac{\sigma_h^{n+1} - \sigma_h^n}{\Delta t}, \tau_h\right) + \left(A_h(\sigma_h^{n+1}), \tau_h\right)$$

$$= b\left(\tau_h, \mathbf{u}_h^{n+1}\right) - g_h(\tau_h) \ \forall \ \tau_h \in U_h^l, \quad (74)$$

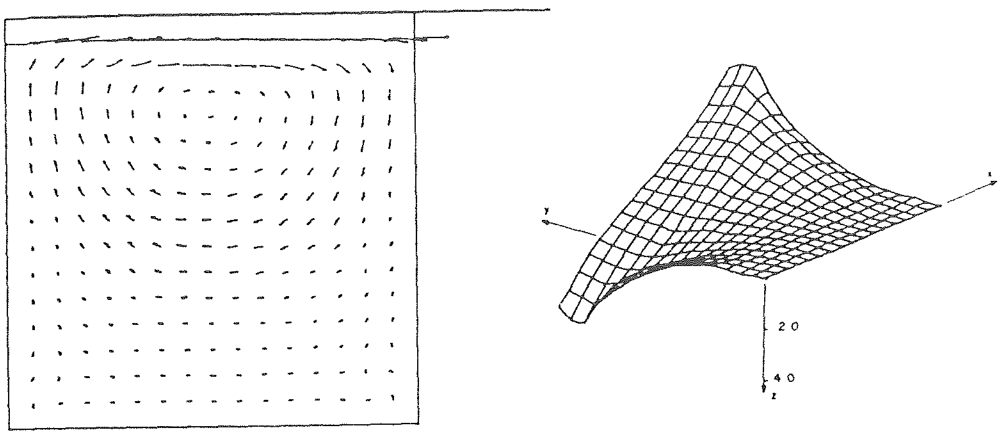
$$b\left(\sigma_h^{n+1}, \mathbf{v}_h\right) = f(\mathbf{v}_h) \quad \forall \ \mathbf{v}_h \in V_h^k, \quad (75)$$

for any $n = 1, 2, \ldots, N$, $\Delta t = (t_1 - t_0)/N$, with N being the total number of time steps, and

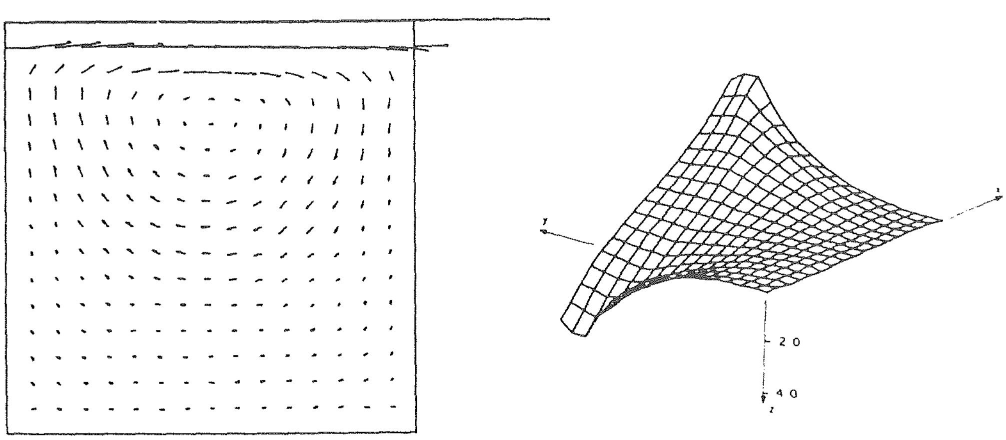
$$\left(\sigma_h^1, \tau\right) = \left(\sigma^e, \tau\right) \quad \forall \ \tau_h \in U_h^l. \quad (76)$$

At each step n, **Problem** $\mathbf{PG}_h^\Delta$ can also be solved using the algorithm derived for the steady state problem.

To illustrate the performance of Galerkin and Petrov–Galerkin methods, we present numerical results obtained with these formulations for the classical wall-driven cavity problem, governed by the Odqvist–Norton law with $p = 4$ and $\mu = 1$ and Hook's law for homogeneous and isotropic materials with $E = 1$ and $\nu = 0.3$, corresponding to steady state solutions and limit solutions of the transient problem for sufficiently large time. In this analysis we consider a unit square domain with $u_1(x, 1) = 1$, $u_2(x, 1) = 0$, and $\mathbf{u} = 0$ on the other boundaries, and adopted a uniform mesh with 17×17 nodes with the nine-node element (biquadratic discontinuous stress and biquadratic continuous velocity interpolations). In Fig. 1 we present the plots of the velocity vectors and pressure elevations obtained with the Petrov–Galerkin formulation, $\delta = 1$. The pressure approximations were post-processed using a least-square smoothing, for plotting. Note the perfect agreement between the steady solution and the limit solution for both velocity and pressure fields. In Fig. 2 we present the same type of

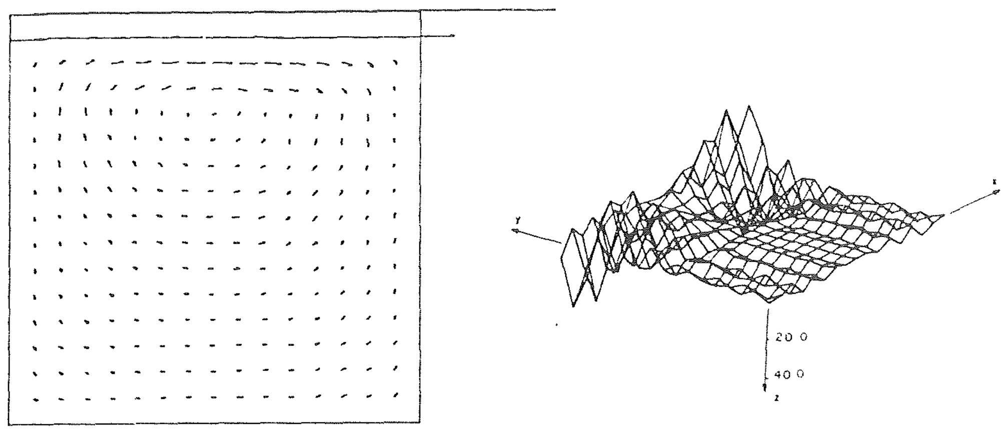


(a) Steady state solution



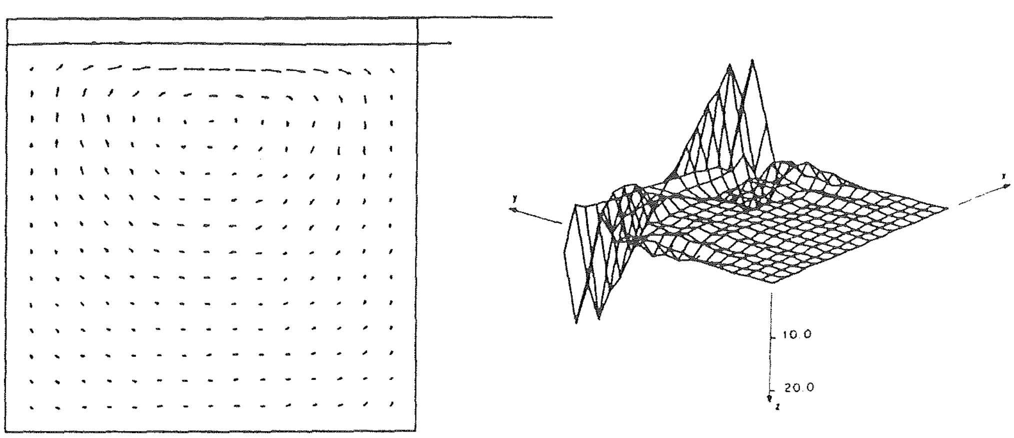
(b) Limit solution

FIG. 1. Petrov–Galerkin formulation. Velocity vector and pressure elevations for the driven cavity problem.



(a) **Steady state solution**

FIG. 2. Galerkin formulation. Velocity vector and pressure elevations for the driven cavity problem.



(b) **Limit solution**

plots for the Galerkin formulation, which is unstable. Observe that, although in this case the velocity field doesn't lock completely, as it does for the equal order bilinear element, the pressure obtained with the Galerkin method is totally unstable for both transient and steady state approximations. And most important, the limit solution differs from the steady state one.

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