

Pruning and Higher-Order Perturbation Solutions

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Abstract

This paper evaluates the pruning procedure proposed by Kim, Kim, Schaumburg, and Sims (2008) that ensures that higher-order perturbation solutions are not explosive. This procedure induces substantial distortions *even* when the simulation is not on an explosive trajectory. In fact, the procedure turns the policy *functions* into policy *correspondences*. An accuracy procedure is proposed to evaluate the severity of the exposed disadvantages of pruning for the problem at hand. A simple alternative to pruning is proposed.

Key Words: accuracy, nonlinear numerical solutions

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1 Introduction

Perturbation solution techniques have become a popular choice to solve dynamic stochastic general equilibrium (DSGE) models. Unfortunately, higher-order approximations are not guaranteed to be stable. den Haan and de Wind (2009), henceforth denoted by dHdW, give examples in which instability occurs for values of the state variables close to the steady state. The pruning procedure developed in Kim, Kim, Schaumburg, and Sims (2008) ensures stability. Pruning is not only used in academic papers,¹ but also taught to students.²

Pruning is problematic because it can severely distort the numerical solution *even* when the simulation is not on an explosive trajectory. In fact, with pruning the numerical approximations to the policy functions are *correspondences* not *functions*. Nevertheless, pruning may improve the accuracy of approximations even when non-pruned simulations are not explosive. We propose an accuracy test that can be used to evaluate the severity of the distortion introduced by pruning.

2 Model

The model used is the following:

$$\begin{aligned} \max_{\{c_t, a_t\}_{t=1}^{\infty}} \quad & \mathbb{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \left(\frac{c_t^{1-\nu} - 1}{1-\nu} - P(a_t) \right) \\ \text{s.t.} \quad & \\ & c_t + \frac{a_t}{1+r} = a_{t-1} + \theta_t, \\ & \theta_t = \bar{\theta} + \varepsilon_t \text{ and } \varepsilon_t \sim N(0, \sigma^2). \end{aligned} \tag{1}$$

Here, c_t is the agent's consumption level, a_t the amount of assets chosen in period t , and θ_t an exogenous random income component. The Euler equation is given by

$$\frac{c_t^{-\nu}}{1+r} + \frac{\partial P(a_t)}{\partial a_t} = \beta \mathbb{E}_t [c_{t+1}^{-\nu}]. \tag{2}$$

¹See, e.g., Andreasen (2008), Fahr and Smets (2008), Doh (2009), and Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2009).

²See, e.g., Christiano (2009) and Schmitt-Grohe (2009).

The penalty function, $P(a_t)$ is given by

$$P(a_t) = \frac{\eta_1}{\eta_0} \exp(-\eta_0 a_t) + \eta_2 a_t. \quad (3)$$

The value of η_0 controls the curvature of the penalty function and the nonlinearity of the problem. dHdW show that for $\eta_0 = 20$ the nonlinearities are strong enough to run into the instability problem that requires an adjustment like pruning for the 2nd and 3rd-order solution, but not for the 4th and 5th-order solution.³

3 Pruning procedure

The N^{th} -order perturbation solution can be written as

$$\begin{aligned} p_N(a_{t-1} - \bar{a}_N, \theta_t - \bar{\theta}) \\ = \\ \gamma_{N,a}(a_{t-1} - \bar{a}_N) + \gamma_{N,\theta}(\theta_t - \bar{\theta}) + \tilde{p}_N(a_{t-1} - \bar{a}_N, \theta_t - \bar{\theta}), \end{aligned} \quad (4)$$

where \bar{a}_N is the *stochastic* steady state and $\tilde{p}_N(\bar{a}_{t-1} - \bar{a}_N, \theta_t - \bar{\theta})$ is the non-linear part of the perturbation solution. The pruning procedure consists of the following steps.

1. Simulate a_t^* using

$$a_t^* - \bar{a}_N = \gamma_{N,a}(a_{t-1}^* - \bar{a}_N) + \gamma_{N,\theta}(\theta_t - \bar{\theta}). \quad (5)$$

2. Simulate $a_{p,t}$ using

$$\begin{aligned} a_{p,t} - \bar{a}_N \\ = \gamma_{N,a}(a_{p,t-1} - \bar{a}_N) + \gamma_{N,\theta}(\theta_t - \bar{\theta}) + \tilde{p}_N(a_{t-1}^* - \bar{a}_N, \theta_t - \bar{\theta}), \end{aligned} \quad (6)$$

where the values of θ_t used are identical to those used in step 1. $\tilde{p}_N(a_{t-1}^* - \bar{a}_N, \theta_t - \bar{\theta})$ is stationary because both a_t^* and θ_t are stationary. Consequently, $a_{p,t}$ is stationary.

³Furthermore, $\beta = 0.9$, $r = 0.03$, $\nu = 3$, $\bar{\theta} = 1.5$, and $\sigma = 0.15$. η_1 and η_2 are such that both the mean and the standard deviation of a_t in the model with the penalty function are identical to the values found in the model with an inequality constraint.

4 Distortion induced by pruning

The pruning procedure introduces an additional state variable, a_{t-1}^* , which means that the choice of a_t is no longer unique given values of a_{t-1} and θ_t , although it should be according to the model. The severity of the distortion is easy to document for our model, since we assume that θ_t is i.i.d., which means that there is only one state variable, $x_t = a_{t-1} + \theta_t$. Figure 1 plots for the 3rd and the 5th-order perturbation solutions the choices made according to the pruning solution and the choice according to the underlying perturbation solution.⁴ Pruning substantially distorts the numerical solution *even* for values of the state variable when the simulated series is not on an explosive trajectory. Also, pruning does not correct the clearly inaccurate oscillation that is present in the 5th-order approximation.

5 Accuracy

Table 1 documents model properties calculated with and without pruning and also with a very accurate projection method.⁵ The 1st-order solution is clearly not accurate. Statistics based on the 2nd and 3rd-order solutions cannot be calculated without pruning, because the simulated series (quickly) explode.

Although pruning distorts perturbation solutions even when non-pruned solutions are not explosive, the table documents that pruning can increase accuracy; for the 4th-order solution the statistics based on pruned data are closer to the truth for all cases considered, sometimes substantially so. For the 5th-order solution this is true for some statistics, but not for others.

In practice one does not have an accurate benchmark and one would calculate Euler equation errors to assess accuracy. This is a one-period test, however, which would not detect small systematic errors accumulating to big errors over time. This is especially problematic for the pruning procedure, since it introduces path dependence into the solution that is not part of the true solution to the model. Therefore, we propose to use a

⁴The 2nd and 3rd-order solutions turn out to be explosive, because the slopes become too steep for high values of the state variable.

⁵See dHdW for details.

dynamic version of the standard Euler-equation accuracy test. Let

$$z_t = h(z_{t-1}, \theta_t) \quad (7)$$

be a numerical approximation. This formulation allows for standard solutions in which case $z_t = a_t$ and for pruned solutions in which case $z_t = [a_{p,t}, a_t^*]'$ and

$$\begin{bmatrix} a_t^* - \bar{a}_N \\ a_{p,t} - \bar{a}_N \end{bmatrix} = \begin{bmatrix} p_1(a_{t-1}^* - \bar{a}_N, \theta_t - \bar{\theta}) \\ \gamma_{N,k}(a_{p,t-1} - \bar{a}_N) + \gamma_{N,\theta}(\theta_t - \bar{\theta}) + \tilde{p}_N(a_{t-1}^* - \bar{a}_N, \theta_t - \bar{\theta}) \end{bmatrix}. \quad (8)$$

The idea behind the dynamic Euler-equation accuracy test is to compare a long series z_t constructed using $z_t = h(z_{t-1}, \theta_t)$ with an alternative series that starts at the same initial value and uses the same realizations for θ_t , but uses $\tilde{z}_t = \tilde{h}(\tilde{z}_{t-1}, \theta_t)$, where $\tilde{h}(z_{t-1}, \theta_t)$ is calculated as follows:

- $\tilde{z}_t = h(\tilde{z}_{t-1}, \theta_t)$, where \tilde{z}_t is only a temporary variable,
- \tilde{z}_t and $h(\tilde{z}_t, \theta_{t+1})$ are used to calculate the conditional expectation in Equation (2),⁶ and
- \tilde{z}_t is calculated using the Euler equation and the budget constraint.

That is, \tilde{z}_t is calculated each period as it is directly implied by the budget constraint and the Euler equation; the approximation, $h(\cdot)$, is only used indirectly, namely to calculate the conditional expectation. If the solution is accurate, then z_t and \tilde{z}_t should be close to each other.

Figure 2 plots the time paths of z_t and \tilde{z}_t together with the time path according to a very accurate projection procedure. The Figure focuses on part of the simulation where the series display large fluctuations. The top panel reports the results for the standard 1st-order perturbation solution and the bottom panel reports the results for the pruned 5th-order perturbation solution. For the 1st-order perturbation solution, the series of z_t and \tilde{z}_t are systematically far apart, that is, the 1st-order solution is clearly not accurate.

⁶We use Gaussian quadrature, i.e., approximate $E[g(\varepsilon_{t+1})]$ with $\sum_{i=1}^I \omega_i g(\sigma\sqrt{2}\zeta_i)/\sqrt{\pi}$, where ω_i (ζ_i) is the i^{th} Gaussian Hermite weight (node) and we set $I = 30$.

For the pruned 5th-order solution the time paths of z_t and \tilde{z}_t are typically much closer to each other, but there are episodes where the solution clearly is inaccurate.

Table 1 reports the maximum and average absolute difference between z_t and \tilde{z}_t and between z_t and a very accurate projection solution. For the 4th-order solution, pruning reduces both the maximum and the average error. For the 5th-order solution, pruning reduces the maximum error but not the average error. Consistent with the comparison based on model properties, the 4th-order is clearly more accurate than the 5th-order solution. Given that the 2nd and 3rd-order solutions explode, pruning obviously improves the results in these cases.

6 Alternative to pruning

Figure 2 documents that \tilde{z}_t based on 1st-order perturbation is actually quite accurate. In fact, the maximum and average error are equal to 1.35% and 0.21% which are both lower than any of the numbers reported in Table 1. Thus, if the current-period decision is based on an explicit calculation of the conditional expectation, then the accuracy is vastly improved even if future behavior is still based on the linear approximation. Interestingly, if we compare the different \tilde{z}_t series than the 1st-order solution is still the best in terms of the maximum error and in terms of the average error is only substantially improved upon by the 4th-order solution.⁷

The disadvantage of 1st-order perturbation is that close to the steady state it is a worse approximation than the ones based on higher-order. The disadvantages of higher-order perturbations are that they induce oscillations. Moreover, in case of non-pruned solutions they could generate instability and in case of pruned solutions distortions. If the current decision is based on an explicitly calculated conditional expectation and the next period's behavior on a perturbation solution, then the disadvantages of higher-order perturbation outweigh the disadvantage of 1st-order perturbation in our example.

Calculating \tilde{z}_t only requires a procedure to calculate the conditional expectation and—

⁷The average difference between \tilde{z}_t based on the 4th-order pruned (not-pruned) solution and the "truth" is equal to 0.14% (0.096%).

in contrast to a full projection solution—does not require constructing a grid, choosing the appropriate approximating functional form or interpolation procedure, and a procedure to calculate the fixed point. Given that it is so easy to calculate and guaranteed to be stable, it would be worthwhile to explore further whether \tilde{z}_t based on 1st-order perturbation is a valid alternative to higher-order perturbation.

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Figure 1: Policy "functions" with and without pruning

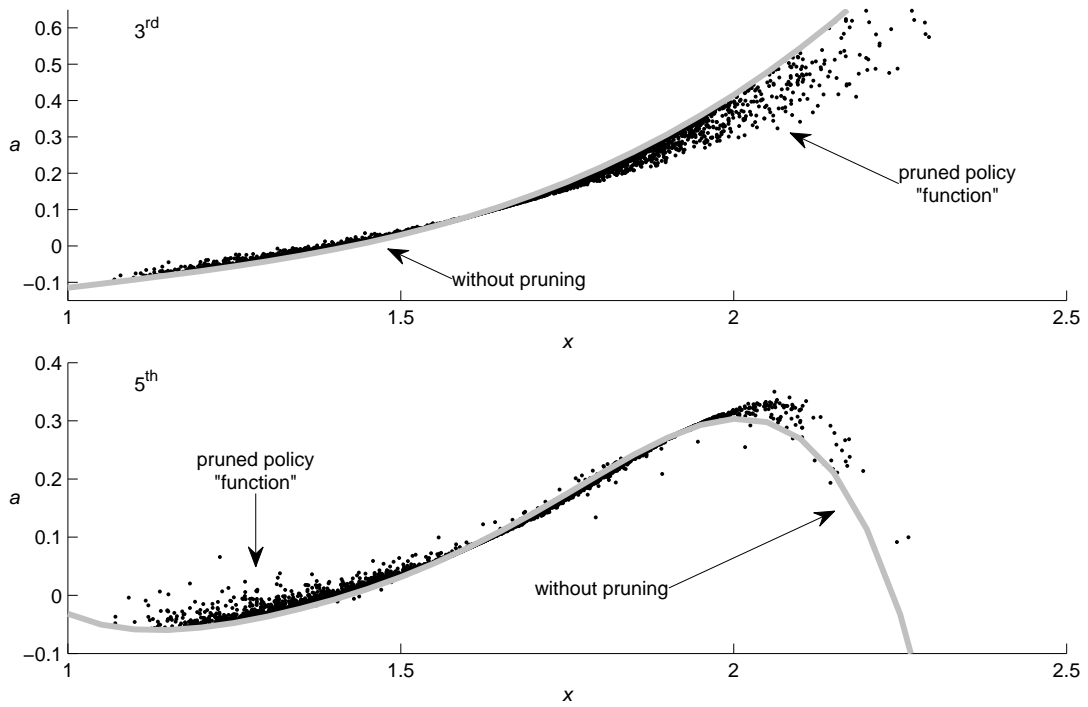


Figure 2: Accuracy test

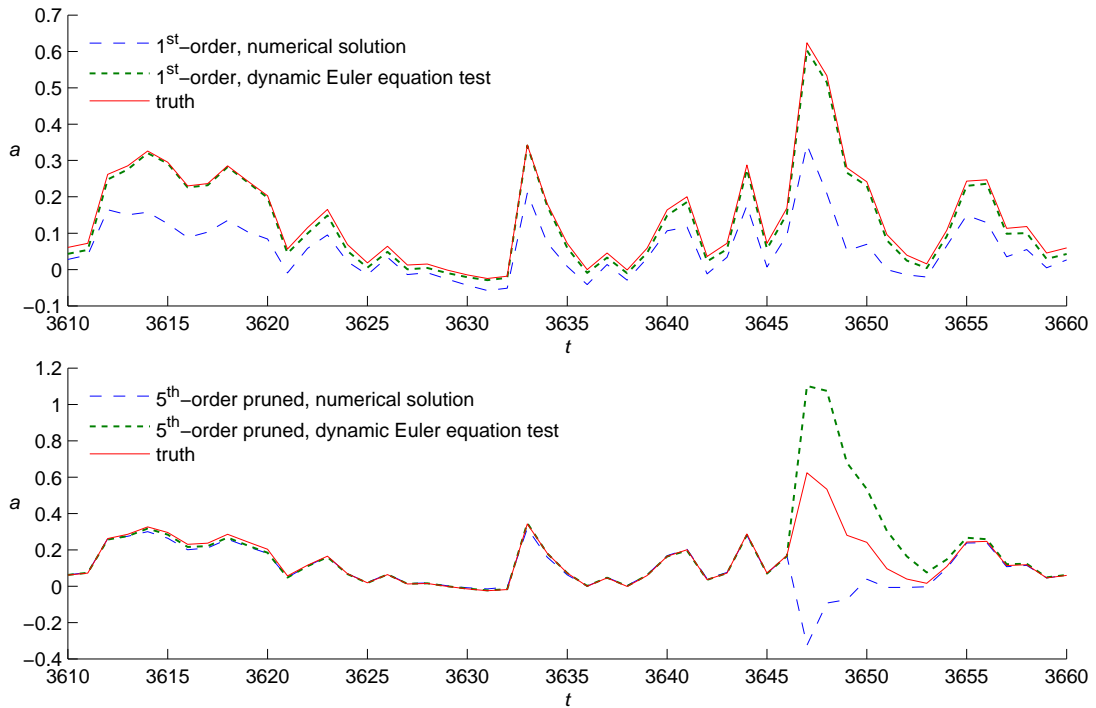


Table 1: Model properties

	"Truth"	Perturbation				
		1 st	2 nd	3 rd	4 th	5 th
No pruning						
μ_a	0.0849	0.0295	–	–	0.0965	0.0826
σ_a	0.0973	0.0698	–	–	0.1112	0.0869
a_{\min}	–0.0941	–0.1844	–	–	–0.1288	–0.4951
a_{\max}	0.6352	0.3202	–	–	0.5796	0.3038
$\rho(a_{t+1}, y_t)$	0.8127	0.9067	–	–	0.7676	0.8353
$\rho(c_t, y_t)$	0.8358	0.9067	–	–	0.8031	0.8476
Euler-equation						
consumption error						
max	–	11.11%	–	–	12.53%	160.7%
average	–	1.16%	–	–	0.30%	0.47%
true error						
max	–	11.33%	–	–	6.98%	63.52%
average	–	1.11%	–	–	0.37%	0.37%
With pruning						
μ_a	0.0849	–	0.0796	0.0890	0.0901	0.0848
σ_a	0.0973	–	0.0821	0.1036	0.0991	0.0838
a_{\min}	–0.0941	–	–0.0426	–0.0973	–0.1266	–0.1465
a_{\max}	0.6352	–	0.6465	0.8863	0.6205	0.3500
$\rho(a_{t+1}, y_t)$	0.8127	–	0.8478	0.8002	0.8182	0.8489
$\rho(c_t, y_t)$	0.8358	–	0.8588	0.7906	0.8143	0.8559
Euler-equation						
consumption error						
max	–	–	10.96%	30.48%	6.62%	131.3%
average	–	–	0.61%	0.50%	0.32%	0.51%
true error						
max	–	–	6.19%	16.09%	4.87%	52.79%
average	–	–	0.63%	0.32%	0.24%	0.39%

Notes: This table reports summary statistics based on a sample of 9,000 observations. "truth" refers to a very accurate projections solution. μ_a is the mean of a , σ_a is the standard deviation, and $\rho(x_t, y_t)$ is the correlation between x_t and y_t .