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NUMERICAL DETERMINATION OF THE MOMENT LYAPUNOV EXPONENTS

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Abstract

Two numerical methods for the determination of the pth moment Lyapunov exponents of a two-dimensional system under bounded noise or real noise parametric excitation are presented. The first method is an analytical-numerical approach, in which the partial differential eigenvalue problems governing the moment Lyapunov exponents are established using the theory of stochastic dynamical systems. The eigenfunctions are expanded in double series to transform the partial differential eigenvalue problems to linear algebraic eigenvalue problems, which are then solved numerically. The second method is a Monte Carlo simulation approach. The numerical values obtained are compared with approximate analytical results with weak noise amplitudes.

1 Introduction

In general, the study of the dynamics of many engineering structures under random loadings leads to a d-dimensional stochastic differential equations of the form

$$\dot{X}^i = f^i(t, \mathbf{X}, \boldsymbol{\xi}), \qquad i = 1, 2, \ldots, d, \tag{1}$$

where $\mathbf{X} = (X^1, X^2, \ldots, X^d)^T$ is the state vector of the system and $\boldsymbol{\xi}$ is a vector of stochastic processes characterizing the randomness of the loadings. The sample or almost-sure stability of the trivial solution of system (1) is determined by the Lyapunov exponent, which characterizes the average exponential rate of growth of the solutions of system (1) for t large, defined as

$$\lambda_{\mathbf{X}(t)} = \lim_{t\to\infty} \frac{1}{t} \log \big\|\mathbf{X}(t)\big\|, \tag{2}$$

where $\|\mathbf{X}(t)\| = \big(\mathbf{X}^T\mathbf{X}\big)^{1/2}$ denotes the Euclidean vector norm. The trivial solution of system (1) is stable with probability one if the top Lyapunov exponent is negative, whereas it is unstable w.p.1 if the top Lyapunov exponent is positive. The theory of Lyapunov exponents was placed on a rigorous mathematical foundation in the Multiplicative Ergodic Theorem (Oseledec, 1968). The Lyapunov exponent has been recognized as an ideal avenue for studying the behaviour of a dynamical system, because it provides not only the information about stability or instability, but also how rapidly the response grows or diminishes with time.

On the other hand, the stability of the pth moment of the trivial solution of system (1), is determined by the moment Lyapunov exponent

$$\Lambda_{\mathbf{X}(t)}(p) = \lim_{t\to\infty} \frac{1}{t} \log E\big[\big\|\mathbf{X}(t)\big\|^p\big], \tag{3}$$

where $E[\cdot]$ denotes expected value. If the moment Lyapunov exponent is negative, then the pth moment approaches 0 as time $t \to \infty$. The pth moment Lyapunov exponent is a convex analytic function in p, which passes through the origin and its slope at the origin is equal to the top Lyapunov exponent. The non-trivial zero of the moment Lyapunov exponent is called the stability index.

In order to have a complete picture of the dynamic stability of system (1), it is important to study both the sample and moment stability and to determine both the top Lyapunov exponent and the pth moment Lyapunov exponent.

Although the moment Lyapunov exponents are important in the study of dynamic stability of stochastic systems, the actual evaluations of the moment Lyapunov exponents are very difficult. Various approximate analytical methods have been devised to actually carry out the computation for a number of engineering structural systems.

The Lyapunov exponents of a general n-dimensional stochastic system can be determined numerically using the algorithm developed in Wolf *et al.* (1985). However, there are no numerical algorithms for evaluating the moment Lyapunov exponents. Because of this reason, almost all published work has been on the analytical determination of the moment Lyapunov exponents under weak noise excitations. Xie (2001a) evaluated numerically the moment Lyapunov exponents of a near-nilpotent system under stochastic parametric excitation. The second-order ordinary differential eigenvalue problem governing the moment Lyapunov exponent is converted to a two-point boundary-value problem, which is solved numerically by the method of relaxation.

Numerical determination of the moment Lyapunov exponents is important for three reasons. Numerically accurate results of the moment Lyapunov exponents are essential in assessing the validity and the ranges of applicability of the approximate analytical results. In many engineering applications, the amplitudes of noise excitations are not small and the approximate analytical methods, such as the method of perturbation and stochastic averaging, cannot be applied. Numerical approaches have to be employed to evaluate the moment Lyapunov exponents. Furthermore, for systems under noise excitations that cannot be described in analytical forms, such as filtered white noise or bounded noise, or if only the time series of the response of the system is known, Monte Carlo simulation approaches have to be resorted to.

This paper presents the first study of numerical determination of the moment Lyapunov exponents of stochastic systems using two different methods. In Section 2, two-dimensional systems under bounded noise and real noise excitations are introduced. The first method, presented in Section 3, is an analytical-numerical approach, in which the second-order partial differential eigenvalue problems are established. Double series expansions of the eigenfunctions, in terms of orthogonal functions, are used to convert the partial differential eigenvalue problems to linear algebraic eigenvalue problems. The second method is a Monte Carlo simulation approach, which is presented in Section 4.

2 Two-Dimensional Systems under Bounded and Real Noise Excitations

Consider a two-dimensional system under random noise excitation

$$\frac{d^2 q(\tau)}{d\tau^2} + 2\beta \frac{dq(\tau)}{d\tau} + \left[\omega_0^2 - \varepsilon_0\, \eta(\tau)\right] q(\tau) = 0, \tag{4}$$

where τ is the time variable, $q(\tau)$ the generalized coordinate, β the damping constant, ω_0 the circular natural frequency of the system, ε_0 the amplitude of the random fluctuation. The random noise $\eta(\tau)$ can be a bounded noise or a real noise process.

A bounded noise process is given by

$$\eta(\tau) = \cos\left[\nu_0 \tau + \sigma_0 W(\tau) + \theta\right], \tag{5}$$

in which θ is a uniformly distributed random number in $(0,\,2\pi)$, and $W(\tau)$ is the standard Wiener process in time τ. The inclusion of the phase angle θ in equation (5) makes the bounded noise $\eta(\tau)$ a stationary process.

A real noise process modelled by an Ornstein-Uhlenbeck process is given by

$$d\eta(\tau) = -\alpha_0\, \eta(\tau)\, d\tau + \sigma_0\, dW(\tau), \tag{6}$$

The Ornstein-Uhlenbeck process is a simple, Gaussian, explicitly representable stationary process that is often used to model a realizable noise process.

Equation (4) can be simplified by removing the damping term using the transformation $q(\tau) = x(\tau)\, e^{-\beta\tau}$ and time scaling $t = \omega\tau$, where $\omega^2 = \omega_0^2 - \beta^2$, to yield

$$\frac{d^2 x(t)}{dt^2} + \left[1 - \varepsilon\, \xi(t)\right] x(t) = 0, \tag{7}$$

where $\varepsilon = \varepsilon_0/\omega^2$.

For the bounded noise excitation (5),

$$\xi(t) = \cos\zeta(t), \qquad d\zeta(t) = \nu\, dt + \sigma\, dW(t), \tag{8}$$

where $\nu = \nu_0/\omega$, $\sigma = \sigma_0/\sqrt{\omega}$, and $W(t)$ is a standard Wiener process in time t.

For the real noise modelled by an Ornstein-Uhlenbeck process (6),

$$\xi(t) = \zeta(t), \qquad d\zeta(t) = -\alpha\zeta(t)\, dt + \sigma\, dW(t), \tag{9}$$

where $\alpha = \alpha_0/\omega$, $\sigma = \sigma_0/\sqrt{\omega}$.

The moment Lyapunov exponent of systems (4), and (7) are related by

$$\Lambda_{q(\tau)}(p) = -p\beta + \omega \Lambda_{x(t)}(p). \tag{10}$$

3 Analytical-Numerical Approach

3.1 Eigenvalue Problems for the Moment Lyapunov Exponents

Two-Dimensional System under Bounded Noise Excitation

The eigenvalue problem governing the pth moment Lyapunov exponent of system (7) can be derived using an approach originally applied by Wedig (1988) for a two-dimensional linear Itô stochastic system.

Under bounded noise excitation (8), equations (7) may be considered as a three dimensional system

$$\mathrm{d}\begin{Bmatrix} x_1 \\ x_2 \\ \zeta \end{Bmatrix} = \begin{Bmatrix} x_2 \\ (-1+\varepsilon\cos\zeta)\,x_1 \\ \nu \end{Bmatrix}\mathrm{d}t + \begin{Bmatrix} 0 \\ 0 \\ \sigma \end{Bmatrix}\mathrm{d}W.$$

Apply the Khasminskii transformation (Khasminskii, 1967)

$$s_1 = \frac{x_1}{a} = \cos\varphi, \quad s_2 = \frac{x_2}{a} = \sin\varphi, \quad a = \sqrt{x_1^2 + x_2^2}, \qquad (11)$$

and define a pth norm $P = a^p$. The Itô equations for P and φ can be obtained by Itô's lemma

$$\mathrm{d}P = \varepsilon\, pP\cos\zeta\,\cos\varphi\sin\varphi\mathrm{d}t,\ \ \mathrm{d}\varphi = (-1+\varepsilon\cos\zeta\,\cos^2\varphi)\mathrm{d}t.$$

Applying a linear stochastic transformation

$$S = T(\zeta,\varphi)P, \quad P = T^{-1}(\zeta,\varphi)S, \qquad (12)$$

where $-\infty < \zeta < \infty, 0 \leqslant \varphi < \pi$, the Itô equation for the new pth norm process S is given by, from Itô's lemma,

$$\mathrm{d}S = \left[\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} + \nu T_\zeta - (1-\varepsilon\cos\zeta\cos^2\varphi)\,T_\varphi + \varepsilon\, p\cos\zeta\,\cos\varphi\sin\varphi\, T\right]P\,\mathrm{d}t + \sigma T_\zeta P\,\mathrm{d}W. \qquad (13)$$

For bounded and non-singular transformation $T(\zeta,\varphi)$, both processes P and S are expected to have the same stability behaviour. Therefore, $T(\zeta,\varphi)$ is chosen so that the drift term of the Itô differential equation (13) is independent of the noise process $\zeta(t)$ and the phase process φ so that

$$\mathrm{d}S = \Lambda S\,\mathrm{d}t + \sigma T_\zeta T^{-1} S\,\mathrm{d}W. \qquad (14)$$

Comparing equations (13) and (14), it is seen that such a transformation $T(\zeta,\varphi)$ is given by the following equation

$$\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} + \nu T_\zeta - (1-\varepsilon\cos\zeta\cos^2\varphi)\,T_\varphi + \varepsilon\, p\cos\zeta\,\cos\varphi\sin\varphi\, T = \Lambda T, \quad -\infty<\zeta<\infty, \quad 0\leqslant\varphi<\pi, \qquad (15)$$

in which $T(\zeta,\varphi)$ is a periodic function in φ of period π and is bounded when $\zeta \to \pm\infty$. Equation (15) defines an eigenvalue problem for a second-order differential operator with Λ being the eigenvalue and $T(\zeta,\varphi)$ the associated eigenfunction. From equation (14), the eigenvalue Λ is seen to be the Lyapunov exponent of the pth moment of system (7), i.e. $\Lambda = \Lambda_{x(t)}(p)$.

Two-Dimensional System under Real Noise Excitation

The eigenvalue problem governing the moment Lyapunov exponent for system (7) under real noise excitation (9) can be derived using the same procedure. Equations (7) and (9) may be considered as a three dimensional system

$$\mathrm{d}\begin{Bmatrix} x_1 \\ x_2 \\ \zeta \end{Bmatrix} = \begin{Bmatrix} x_2 \\ (-1+\varepsilon\zeta)x_1 \\ -\alpha\zeta \end{Bmatrix}\mathrm{d}t + \begin{Bmatrix} 0 \\ 0 \\ \sigma \end{Bmatrix}\mathrm{d}W.$$

Applying the Khasminskii transformation (11), the Itô equations for $P = a^p$ and φ are

$$\mathrm{d}P = \varepsilon\, pP\,\zeta\,\cos\varphi\sin\varphi\,\mathrm{d}t, \quad \mathrm{d}\varphi = (-1+\varepsilon\zeta\cos^2\varphi)\,\mathrm{d}t.$$

Applying a linear stochastic transformation given by equation (12), the Itô equation for the new pth norm process S is given by

$$\mathrm{d}S = \left[\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} - \alpha\zeta\, T_\zeta - \left(1-\varepsilon\,\zeta\cos^2\varphi\right)T_\varphi + \varepsilon\, p\zeta\,\cos\varphi\sin\varphi\, T\right]P\,\mathrm{d}t + \sigma T_\zeta P\,\mathrm{d}W. \qquad (16)$$

The transformation $T(\zeta,\varphi)$ is chosen so that the drift term of the Itô differential equation (16) is independent of the real noise process $\zeta(t)$ and the phase process φ so that equation (16) is of then form (14). Comparing equations (16) and (14), the transformation $T(\zeta,\varphi)$ satisfies

$$\tfrac{1}{2}\sigma^2 T_{\zeta\zeta} - \alpha\zeta\, T_\zeta - \left(1-\varepsilon\zeta\cos^2\varphi\right)T_\varphi + \varepsilon\, p\zeta\,\cos\varphi\sin\varphi\, T = \Lambda T, \quad -\infty<\zeta<\infty, \quad 0\leqslant\varphi<\pi, \qquad (17)$$

which defines an eigenvalue problem of a second-order differential operator with $\Lambda = \Lambda_{x(t)}(p)$, the pth moment Lyapunov exponent, being the eigenvalue and $T(\zeta,\varphi)$ the associated eigenfunction.

3.2 Transformation of the Eigenvalue Problems for the Moment Lyapunov Exponents

In this subsection, series expansions of the eigenfunctions are employed to convert the second-order partial differential eigenvalue problems (15) and (17), governing the pth moment Lyapunov exponents, to linear algebraic eigenvalue problems, which can then be easily solved numerically.

Two-Dimensional System under Bounded Noise Excitation

For the two-dimensional system (7) under bounded noise excitation (8), the pth moment Lyapunov exponent satisfies the eigenvalue problem (15). Since the coefficients of the equation (15) are periodic functions in ζ of period 2π and in φ of period π, the eigenfunction $T(\zeta,\varphi)$ can be expanded in double Fourier series in the complex form, which is much more compact than the real form, as follows (Tolstov, 1962)

$$T(\zeta,\varphi) = \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} C_{l,k}\,\mathrm{e}^{\mathrm{i}(l\zeta+2k\varphi)}, \qquad (18)$$

where the coefficients $C_{l,k}$ are complex numbers. Substituting equation (18) into equation (15), multiplying the resulting equation by $e^{-i(m\zeta+2n\varphi)}$, integrating with respect to ζ from 0 to 2π and with respect to φ from 0 to π yields a linear algebraic eigenvalue problem of infinite dimension. In numerical analysis, only a finite number of terms can be taken in the double Fourier series (18). Hence, let m takes the values $-M, -M+1, \ldots, M-1, M$, and n takes the values $-N, -N+1, \ldots, N-1, N$; that is, there are $2M+1$ terms in ζ and $2N+1$ terms in φ in the double Fourier series (18).

For the easy of formulation, the two-dimensional array of the coefficients $C_{m,n}$ is transformed to the one-dimensional array $y_j = C_{m,n}$, where $j = (2N+1)(M+m)+N+n+1$, for $m = -M : M$, $n = -N : N$. The linear algebraic eigenvalue problem can be written as

$$\mathbf{A}\,\mathbf{y} = \Lambda\,\mathbf{y}, \tag{19}$$

in which the dimension of matrix $\mathbf{A}$ is $(2M+1)(2N+1)\times(2M+1)(2N+1)$. For the jth row, $j = (2N+1)(M+m)+N+n+1$, $m = -M : M$, $n = -N : N$, the non-zero elements of $\mathbf{A}$ are

$$A_{j,j} = -\tfrac{1}{2}\sigma^2 m^2 + i m\nu - i2n, \qquad A_{j,J^-} = A_{j,J^+} = i\tfrac{1}{2}\varepsilon n,$$
$$A_{j,J^- -1} = A_{j,J^+ -1} = i\tfrac{1}{8}\varepsilon[2(n-1)-p],$$
$$A_{j,J^- +1} = A_{j,J^+ +1} = i\tfrac{1}{8}\varepsilon[2(n+1)+p],$$

where $J^{\pm} = (2N+1)(M+m\pm 1)+N+n+1$.

Two-Dimensional System under Real Noise Excitation

The moment Lyapunov exponent of the two-dimensional system (7) under real noise excitation (9) is governed by the eigenvalue problem (17). The coefficients of equation (17) are periodic functions in φ of period 2π. The eigenfunction $T(\zeta,\varphi)$ are expanded in terms of sinusoidal functions and Hermite polynomials as

$$T(\zeta,\varphi) = \sum_{l=0}^{\infty}\sum_{k=-\infty}^{\infty} C_{l,k}\, h_l(\zeta)\, e^{i2k\varphi}, \tag{20}$$

where $h_l(\zeta)$ is the normalized Hermite polynomial

$$h_l(\zeta) = \frac{1}{(2^l\, l!\,\sqrt{\pi})^{1/2}} \exp\Big(-\frac{\zeta^2}{2}\Big) H_l(\zeta), \qquad l = 0, 1, \ldots, \tag{21}$$

in which $H_l(\zeta)$ is the Hermite polynomial (Lebedev, 1972). The normalized Hermite polynomials $h_l(\zeta)$, $l = 0,\ 1,\ \ldots$, form an orthonormal system on the interval $(-\infty, +\infty)$, i.e.

$$\int_{-\infty}^{+\infty} h_l(\zeta)\, h_m(\zeta)\, d\zeta = \delta_{l,m}. \tag{22}$$

Some relevant properties of the normalized Hermite polynomials are given in Appendix A.

Substituting equation (20) into (17), multiplying the resulting equation by $h_m(\zeta)\, e^{-i2n\varphi}$, for $m = 0 : +\infty$ and $n = -\infty : +\infty$, integrating with respect to ζ from $-\infty$ to $+\infty$ and with respect to φ from 0 to π, and utilizing the orthogonality condition (22) and identities (5) leads to a system of infinity homogeneous linear algebraic equations for the unknown coefficients $C_{m,n}$, $m = 0 : +\infty$, $n = -\infty : +\infty$. For numerical analysis, the series expansion (20) must be truncated, i.e. m takes the values $0, 1, \ldots, M$, and n takes the values $-N, -N+1, \ldots, N-1, N$. Map the two-dimensional array of coefficients $C_{m,n}$ to the one-dimensional array $y_j = C_{m,n}$, $j = (2N+1)m+N+n+1$. The system of infinity homogeneous linear algebraic equations can then be written in the form of a linear algebraic eigenvalue problem (19), in which the dimension of matrix $\mathbf{A}$ is $(M+1)(2N+1)\times(M+1)(2N+1)$. For row $j = (2N+1)m+N+n+1$, the non-zero elements of matrix $\mathbf{A}$ are

$$A_{j,j} = -\frac{\sigma^2}{2}\Big(m+\frac{1}{2}\Big) + \frac{\alpha}{2} + i2n,$$
$$A_{j,J^- -1} = i\varepsilon\frac{2(n-1)-p}{4}\sqrt{\frac{m}{2}}, \quad A_{j,J^-} = i\varepsilon n\sqrt{\frac{m}{2}},$$
$$A_{j,J^- +1} = i\varepsilon\frac{2(n+1)+p}{4}\sqrt{\frac{m}{2}},\ A_{j,J^+ -1} = i\varepsilon\frac{2(n-1)-p}{4}\sqrt{\frac{m+1}{2}},$$
$$A_{j,J^+} = i\varepsilon n\sqrt{\frac{m+1}{2}}, \quad A_{j,J^+ +1} = i\varepsilon\frac{2(n+1)+p}{4}\sqrt{\frac{m+1}{2}},$$

$$A_{j,J_2^-} = \Big(\frac{\sigma^2}{2}+\alpha\Big)\frac{\sqrt{m(m-1)}}{2},\ A_{j,J_2^+} = \Big(\frac{\sigma^2}{2}-\alpha\Big)\frac{\sqrt{(m+2)(m+1)}}{2},$$

where $J^{\pm} = (2N+1)(m\pm 1)+N+n+1$ and $J_2^{\pm} = (2N+1)(m\pm 2)+N+n+1$.

3.3 Numerical Results and Discussions

Double series expansions of the eigenfunctions using orthogonal functions have been applied to transform the partial differential eigenvalue problems (15) and (17) into linear algebraic eigenvalue problems of the form (19). The resulting large square matrix $\mathbf{A}$ is highly sparse. To solve system (19) numerically, one must take full advantage of the sparsity of matrix $\mathbf{A}$ in developing or selecting numerical algorithms. *Matlab 6* has an excellent sparse matrix handling facility and functions for determining the eigenvalues of a large sparse matrix. In this paper, the function eigs in *Matlab 6* is used to evaluate a few eigenvalues of system (19). Numerical results are presented in the following.

Two-Dimensional System under Bounded Noise Excitation

The bounded noise excitation (8) reduces to harmonic excitation, i.e. $\xi(t) = \cos\nu t$, when $\sigma = 0$. It is well known that the resulting Mathieu's equation (7) under harmonic excitation is in parametric resonance when $\nu = 2, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. The effect of the noise on the parametric resonance or the stability of system (7) is of particular interest.

Using a method of regular perturbation, Xie (2003) obtained a weak noise expansion of the moment Lyapunov exponent of system (7) under bounded noise excitation (8). The second-order perturbation of the moment Lyapunov exponent is given by, for small ε,

$$\Lambda_{x(t)}(p) = \varepsilon^2 \frac{p(p+2)\,S(2)}{16} + O(\varepsilon^4), \tag{23}$$

where $S(2)$ is the spectral density function $S(\omega)$ with $\omega = 2$ of the bounded noise $\xi(t)$ given by

$$S(2) = \frac{\sigma^2\left(4+\nu^2+\frac{1}{4}\sigma^4\right)}{2\left[(2+\nu)^2+\frac{1}{4}\sigma^4\right]\left[(2-\nu)^2+\frac{1}{4}\sigma^4\right]}.$$

In the numerical solution of the linear algebraic eigenvalue problem (19), the numbers of terms of the series expansion are taken as $N = M = 50$, resulting in the dimension of matrix **A** being 10201×10201. The number of non-zero elements is 69800 and the density is

$$\frac{\text{Number of non-zero element}}{\text{Total number of elements}} = \frac{69800}{10201^2} = 6.708 \times 10^{-4},$$

which indicates that matrix **A** is highly sparse. The approximate analytical result (23) can be used as a seed in the *Matlab* function eigs to determine the largest real eigenvalue of system (19).

Numerical results of the moment Lyapunov exponents are shown in Figures 1–3 for $\nu = 2.0$, 1.0, and 3.0, respectively, and various values of ε. It is seen that, for all values of ν, the approximate analytical result agrees extremely well with the numerical results for ε up to 0.2. For $\varepsilon = 0.1$, the relative error between the approximate analytical result and the numerical result is within 1% for all values of p. When $\nu = 2.0$ and 1.0, the two-dimensional system is in the primary and secondary parametric resonance, respectively. Discrepancy between the two results increases rapidly when the value of ε is increased as shown in Figures 1 and 2. When $\nu = 3.0$, as shown in Figure 3, system (7) is not in parametric resonance and the two results agree extremely well for ε as large as 2.0.

Two-Dimensional System under Real Noise Excitation

When the noise amplitude parameter ε is small, a second-order perturbation of the moment Lyapunov exponent of system (7) under the real noise excitation (9) is given by (Xie, 2001b)

$$\Lambda_{x(t)}(p) = \varepsilon^2 \frac{p(p+2)\,\sigma^2}{16\,(\alpha^2+4)} + O(\varepsilon^4). \tag{24}$$

One extra parameter can be eliminated in equations (7) and (9) as

$$\ddot{x}(t) + [1 - \hat{\varepsilon}\,\hat{\xi}(t)]\,x(t) = 0,$$

$$\mathrm{d}\hat{\xi}(t) = -\alpha\,\hat{\xi}(t)\,\mathrm{d}t + \mathrm{d}W,$$

where $\hat{\xi}(t) = \xi(t)/\sigma$, $\hat{\varepsilon} = \varepsilon\sigma$. Hence, without loss of generality, the parameter σ can be taken as 1.

In the numerical analysis, the numbers of the double series expansions are taken as $M = N = 50$. The dimension of matrix **A** in the linear algebraic eigenvalue problem (19) is 5151×5151 and the number of non-zero elements is 44849. The density of non-zero elements is $44849/5151^2 = 1.690 \times 10^{-3}$. Hence, matrix **A** is also highly sparse and, as in the case under bounded noise excitation, the eigs function in *Matlab* is used to determine the largest real eigenvalue as the moment Lyapunov exponent, with the approximate analytical result (24) used as the seed.

Typical results of the moment Lyapunov exponents obtained are shown in Figure 4 for $\alpha = 2.0, \sigma = 1.0$, and various values of the noise amplitude parameter ε. The approximate analytical result and the numerical result agree very well as shown in Figure 4. Good agreement is observed even for ε close to 1.0.

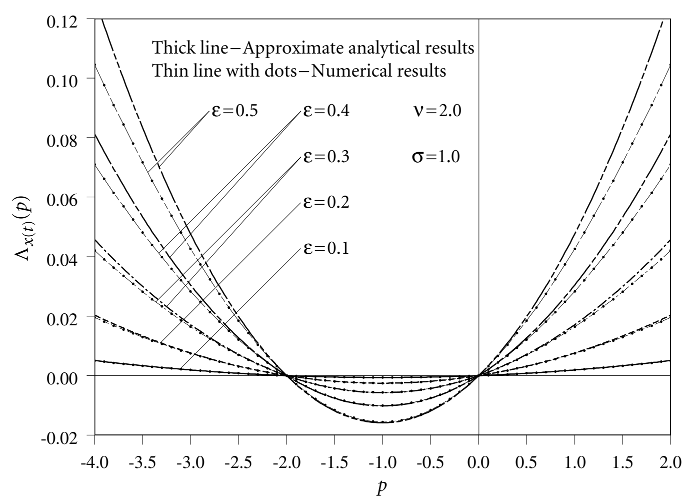

Figure 1. Moment Lyapunov Exponent (Bounded Noise)

4 Monte Carlo Simulation

4.1 Determination of the pth Moment

The state vector of system (1) is augmented to rewrite system (1) as a D-dimensional system of autonomous Itô stochastic differential equations

$$\mathrm{d}Y^i = a^i(\mathbf{Y})\,\mathrm{d}t + \sum_{k=1}^{K} b^{i,k}(\mathbf{Y})\,\mathrm{d}W^k, \qquad i = 1, 2, \ldots, D, \tag{25}$$

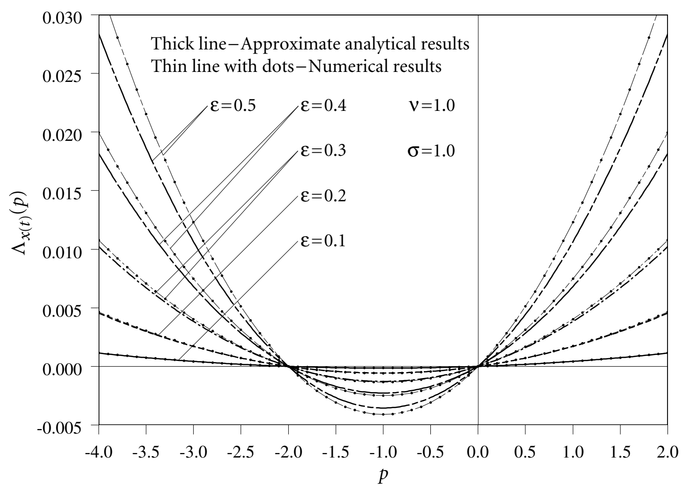

Figure 2. Moment Lyapunov Exponent (Bounded Noise)

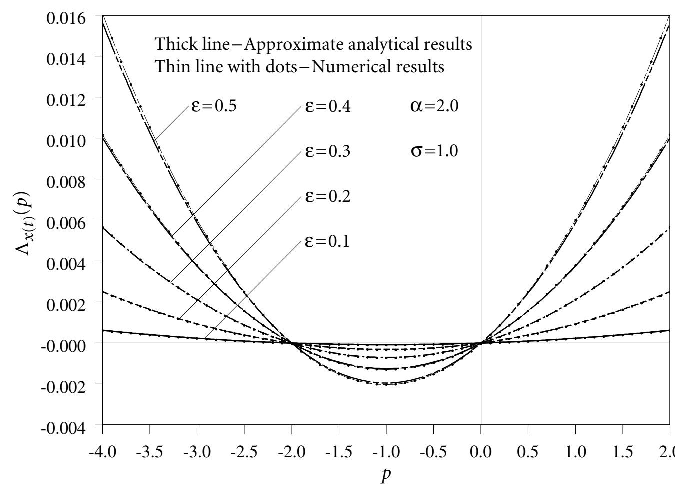

Figure 4. Moment Lyapunov Exponent (Real Noise)

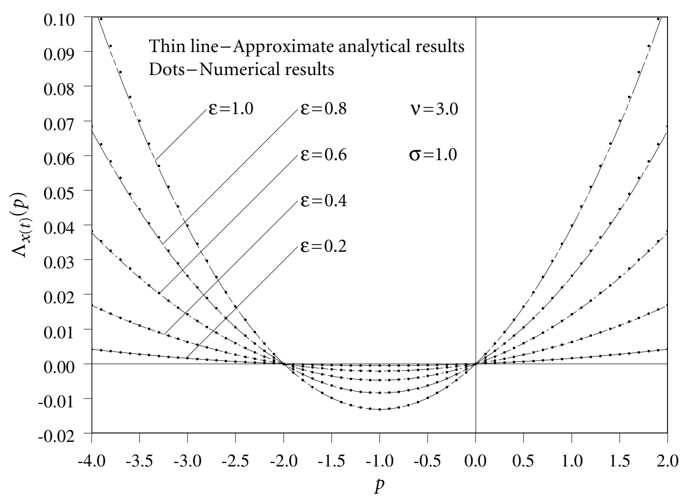

Figure 3. Moment Lyapunov Exponent (Bounded Noise)

where $\mathbf{Y} = (Y^1, Y^2, \ldots, Y^D)^T$, in which $Y^i = X^i$, for $i = 1, 2, \ldots, d$. In the remainder of this paper, vector $\mathbf{X}$ and the vector containing the first d elements of vector $\mathbf{Y}$ are interchangeable for the easy of presentation.

Since the moment Lyapunov exponent is related to the exponential rate of growth or decay of the pth moment, only the numerical approximation of the pth moment of the solution of system (1) or (25) is of interest in the Monte Carlo simulation. As a result, pathwise approximations of the solutions of the stochastic differential equations (1) or (25) are not necessary. A much weaker form of convergence in probability distribution is required. For the numerical solutions of the stochastic differential equations (25), weak Taylor approximations may be applied. To evaluate the pth moment $E\big[\|\mathbf{X}\|^p\big]$, S samples of the solutions of equations (25) are generated.

If the functions $a^i(\mathbf{Y})$ and $b^i(\mathbf{Y})$, $i = 1, 2, \ldots, D$, are six times continuously differentiable, the simplified order 2.0 weak Taylor scheme of the sth realization of equations (25) at the mth iteration with $t_m - t_{m-1} = \Delta$, where Δ is the time step of integration, is given by (Kloeden and Platen, 1992), for $i = 1, 2, \ldots, D$,

$$\begin{aligned} Y^i_{s,m} &= Y^i_{s,m-1} + a^i_{s,m-1} \cdot \Delta + \frac{1}{2} L^0 a^i_{s,m-1} \cdot \Delta^2 \\ &\quad + \sum_{k=1}^{K} \Big[b^{i,k}_{s,m-1} + \frac{1}{2} \Big(L^0 b^{i,k}_{s,m-1} + L^k a^i_{s,m-1} \Big) \Big] \cdot \Delta W^k_{s,m-1} \\ &\quad + \frac{1}{2} \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} L^{k_1} b^{i,k_2}_{s,m-1} \Big(\Delta W^{k_1}_{s,m-1} \cdot \Delta W^{k_2}_{s,m-1} + V^{k_1,k_2}_{s,m-1} \Big), \end{aligned} \tag{26}$$

where the subscript s stands for the sth sample, $s = 1, 2, \ldots, S$, and the operators L^0, L^k are defined as

$$L^0 = \sum_{j=1}^{D} a^j \frac{\partial}{\partial Y^j} + \frac{1}{2} \sum_{j,l=1}^{D} \sum_{k=1}^{K} b^{j,k} b^{l,k} \frac{\partial^2}{\partial Y^j \partial Y^l},$$
$$L^k = \sum_{j=1}^{D} b^{j,k} \frac{\partial}{\partial Y^j}, \quad k = 1, 2, \ldots, K,$$

in which the functions a^j and $b^{j,k}$ are evaluated at time t_{m-1} of the sth sample. $\Delta W^k_{s,m-1}$ can be taken as a normally distributed random number with mean 0 and standard deviation $\sqrt{\Delta}$

$$\Delta W^k_{s,m-1} = R^k_{s,m-1} \sqrt{\Delta}, \tag{27}$$

where $R^k_{s,m-1}$ is a standard normal random number $N(0,1)$. $V^{k_1,k_2}_{s,m-1}$ are independent random numbers with the following two-point distribution, for $k_1 = 1, 2, \ldots, K$,

$$\begin{aligned} &P\big(V^{k_1,k_2}_{s,m-1} = \pm\Delta\big) = \tfrac{1}{2}, && \text{for } k_2 = 1, 2, \ldots, k_1 - 1, \\ &V^{k_1,k_2}_{s,m-1} = -\Delta, && \text{for } k_2 = k_1, \\ &V^{k_1,k_2}_{s,m-1} = -V^{k_2,k_1}_{s,m-1}, && \text{for } k_2 = k_1 + 1, k_1 + 2, \ldots, K. \end{aligned} \tag{28}$$

For the special case when $K=1$, equations (26) are reduced to, for $s=1,2,\ldots,S$, and $i=1,2,\ldots,D$,

$$\begin{aligned}Y^i_{s,m} = Y^i_{s,m-1} + a^i_{s,m-1}\cdot\Delta + b^i_{s,m-1}\cdot\Delta W_{s,m-1} + \tfrac{1}{2}L^0 a^i_{s,m-1}\cdot\Delta^2 \\ + \tfrac{1}{2}L^1 b^i_{s,m-1}\Big[\big(\Delta W_{s,m-1}\big)^2 - \Delta\Big] + L^1 a^i_{s,m-1}\cdot\Delta Z_{s,m-1} \\ + L^0 b^i_{s,m-1}\Big[\Delta W_{s,m-1}\cdot\Delta - \Delta Z_{s,m-1}\Big], \quad (29)\end{aligned}$$

where $\Delta W_{s,m-1}$ and $\Delta Z_{s,m-1}$ are a pair of correlated normally distributed random numbers generated as

$$\Delta W_{s,m-1} = R^1_{s,m-1}\sqrt{\Delta}, \quad \Delta Z_{s,m-1} = \frac{1}{2}\Delta^{3/2}\Big(R^1_{s,m-1} + \frac{R^2_{s,m-1}}{\sqrt{3}}\Big),$$

where $R^1_{s,m-1}$ and $R^2_{s,m-1}$ are two independent standard normally distributed random numbers.

Having obtained s samples of the solutions of the stochastic differential equations, the pth moment can be determined as follows

$$E\big[\|\mathbf{X}(t_m)\|^p\big] = \frac{1}{S}\sum_{s=1}^{S}\|\mathbf{X}_{s,m}\|^p, \quad \|\mathbf{X}_{s,m}\| = \big(\mathbf{X}^T_{s,m}\mathbf{X}_{s,m}\big)^{1/2}, \quad (30)$$

where $X^i_{s,m} = Y^i_{s,m}$, for $i=1,2,\ldots,d$.

4.2 Determination of the pth Moment Lyapunov Exponents

Having obtained the pth moment $E\big[\|\mathbf{X}\|^p\big]$ at any time instance t, the moment Lyapunov exponent $\Lambda_{\mathbf{X}}(p)$ can be determined using equation (3). However, since the pth moment grows or decays exponentially in time, periodic normalization of the pth moment must be applied in order to avoid numerical overflow or underflow and to correctly determine the moment Lyapunov exponent.

Take the initial condition of $\mathbf{X}_s(0)$ such that $\|\mathbf{X}_s(0)\| = 1$, $s=1,2,\ldots,S$. Note that $Y^i_s = X^i_s$, for $i=1,2,\ldots,d$. Normalization of the first d elements of the state vector $\mathbf{Y}_s$ is applied after every time period ($M\Delta$).

At the time instance $n(M\Delta)$, $n=1,\,2,\,\ldots,\,N$, the following ratio is determined for all values of p of interest

$$\rho_n(p) = \frac{E\big[\|\mathbf{X}\big(n(M\Delta)\big)\|^p\big]}{E\big[\|\mathbf{X}\big((n-1)(M\Delta)\big)\|^p\big]}, \quad \text{for } n=1,\,2,\,\ldots,\,N. \quad (31)$$

The state vector $\mathbf{Y}_s$ is then normalized, in the sense that $\|\mathbf{X}_s\| = 1$, using

$$Y^i_s\big(n(M\Delta)\big) = \frac{Y^i_s\big(n(M\Delta)\big)}{\|\mathbf{X}_s\big(n(M\Delta)\big)\|}, \qquad i=1,\,2,\,\ldots,\,d. \quad (32)$$

After the normalization, numerical solution of the stochastic differential equations is continued.

From equation (31), it can be easily shown that

$$\rho_1(p)\cdot\rho_2(p)\cdots\rho_N(p) = E\big[\|\mathbf{X}\big(N(M\Delta)\big)\|^p\big]. \quad (33)$$

Using equations (3) and (33), the pth moment Lyapunov exponent is given by, for all values of p of interest,

$$\begin{aligned}\Lambda_{\mathbf{X}}(p) &= \frac{1}{N(M\Delta)}\log E\big[\|\mathbf{X}\big(N(M\Delta)\big)\|^p\big] \\ &= \frac{1}{N(M\Delta)}\log\big[\rho_1(p)\cdot\rho_2(p)\cdots\rho_N(p)\big] \\ &= \frac{1}{N(M\Delta)}\sum_{n=1}^{N}\log\rho_n(p), \quad \text{for large } N.\end{aligned} \quad (34)$$

4.3 Monte Carlo Simulation of the pth Moment Lyapunov Exponents

The results presented in Sections 4.1 and 4.2 are summarized in the following procedure for the Monte Carlo simulation of the pth moment Lyapunov exponent.

1. **Setting the Initial Conditions**

   For sample s, $s=1,\,2,\,\ldots,\,S$, set the initial conditions of the first d elements of the state vector $\mathbf{Y}_s$ as $Y^i_s(0) = 1/\sqrt{d}$, $i=1$, 2, $\ldots$, d. $Y^i_s(0)$, $i=d+1,\,d+2,\,\ldots,\,D$, can be set to any values; for simplicity of implementation, they may also be set to $1/\sqrt{d}$.

2. **Conducting the Monte Carlo Simulation**

   For time iterations $n=1,2,\ldots,N$, conduct the Monte Carlo simulation. For each increment in n, the increase in time is $M\Delta$.

   (a) For $m=1,2,\ldots,M$, and sample $s=1,2,\ldots,S$, perform the numerical integration of the stochastic differential equations. For each increment in m, the increment in time is Δ.

      i. Generate $3K$ standard normally distributed random numbers to evaluate $\Delta W^k_{s,m-1}$, $\Delta W^{k_1}_{s,m-1}$, $\Delta W^{k_2}_{s,m-1}$, $k,k_1,k_2=1,\,2,\,\ldots,\,K$, using equation (27).

      ii. Generate $\frac{1}{2}K(K-1)$ uniformly distributed random numbers in $(0,1)$ to evaluate V^{k_1,k_2}_s, $k_1=1$, $2,\ldots,K$, $k_2=k_1+1, k_1+2,\ldots,K$, using equation (28).

      iii. Evaluate $\mathbf{Y}_s\big([(n-1)M+m]\Delta\big)$ in time step Δ using the iterative equation (26).

   For the special case when $K=1$, the following simplified steps are taken:

      i. Generate two standard normally distributed random numbers to evaluate $\Delta W_{s,m-1}$ and $\Delta Z_{s,m-1}$.

ii. Evaluate $\mathbf{Y}_s([(n-1)M+m]\Delta)$ in time step Δ using the iterative equation (29).

(b) For all values of p of interest and sample $s = 1, 2, \ldots, S$, determine the pth norms $\|\mathbf{X}_s(n(M\Delta))\|^p$ using $\|\mathbf{X}\| = (\mathbf{X}^T\mathbf{X})^{1/2}$, where $X_s^i = Y_s^i, i = 1, 2, \ldots, d$.

(c) Determine the pth moments $E[\|\mathbf{X}_s(n(M\Delta))\|^p]$ using equation (30) for all values of p of interest.

(d) Evaluate the ratio $\rho_n(p) = E[\|\mathbf{X}_s(n(M\Delta))\|^p]$ using equation (31) for all values of p of interest.

(e) Normalize the state vector $\mathbf{Y}_s(n(M\Delta))$ using equation (32).

3. **Determining the pth Moment Lyapunov Exponent**

Determine the pth moment Lyapunov exponent $\Lambda_{\mathbf{X}}(p)$ using equation (34) for all values of p of interest.

4.4 Numerical Results

Two-Dimensional System under Bounded Noise Excitation

The two-dimensional system (4) under bounded noise excitation (5) can be converted to a three-dimensional autonomous stochastic system as:

$$\begin{cases} \mathrm{d}Y^1 = Y^2\,\mathrm{d}\tau, \\ \mathrm{d}Y^2 = \left[-2\beta\,Y^2 - \left(\omega_0^2 - \varepsilon_0 \cos Y^3\right) Y^1\right]\mathrm{d}\tau, \\ \mathrm{d}Y^3 = \nu_0\,\mathrm{d}\tau + \sigma_0\,\mathrm{d}W, \end{cases} \tag{35}$$

where $Y^1 = q(\tau)$, $Y^2 = \mathrm{d}q(\tau)/\mathrm{d}\tau$, $Y^3 = \xi(\tau)$. Y^1 and Y^2 are related to the state variables of the original system (4) and are used to calculated the pth norm $\|\mathbf{Y}\|^p = \left[(Y^1)^2 + (Y^2)^2\right]^{p/2}$.

The order 2.0 weak Taylor scheme is given by, from equation (29),

$$\begin{aligned} Y_{s,m}^1 &= Y_{s,m-1}^1 + Y_{s,m-1}^2 \cdot \Delta + \tfrac{1}{2} R_{s,m-1} \cdot \Delta^2, \\ Y_{s,m}^2 &= Y_{s,m-1}^2 + R_{s,m-1} \cdot \Delta - \tfrac{1}{2}\Big[\left(\omega_0^2 - \varepsilon_0 \cos Y_{s,m-1}^3\right) Y_{s,m-1}^2 \\ &\qquad + 2\beta R_{s,m-1} - \varepsilon_0\left(\nu_0 \sin Y_{s,m-1}^3 + \tfrac{1}{2}\sigma_0^2 \cos Y_{s,m-1}^3\right) \\ &\qquad \cdot Y_{s,m-1}^1\Big] \cdot \Delta^2 - \varepsilon_0 \sigma_0 Y_{s,m-1}^1 \sin Y_{s,m-1}^3 \cdot \Delta Z, \\ Y_{s,m}^3 &= Y_{s,m-1}^3 + \nu_0 \cdot \Delta + \sigma_0 \cdot \Delta W, \\ R_{s,m-1} &= -\left(\omega_0^2 - \varepsilon_0 \cos Y_{s,m-1}^3\right) Y_{s,m-1}^1 - 2\beta\, Y_{s,m-1}^2. \end{aligned}$$

Numerical results of the pth moment Lyapunov exponents $\Lambda_{q(\tau)}(\tau)$ from Monte Carlo simulation are plotted in Figures 5–6 for $\nu = 2.0$, 1.0, respectively, $\beta = 0.01$, $\omega_0 = 1.0$, $\nu_0 = 1.0$, $\sigma_0 = 1.0$ and various values of ε_0. It is observed that the numerical results compare well with a fourth-order approximation of the pth moment Lyapunov exponent determined using a method of regular perturbation in (Xie, 2003) for small $\varepsilon_0 > 0$.

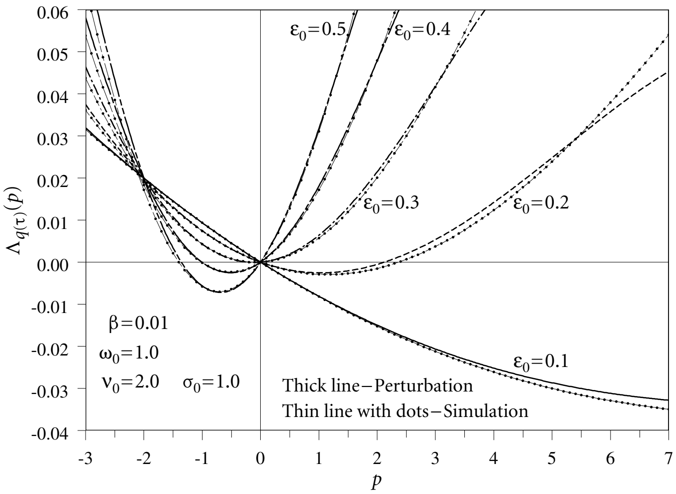

Figure 5. Moment Lyapunov Exponent (Bounded Noise)

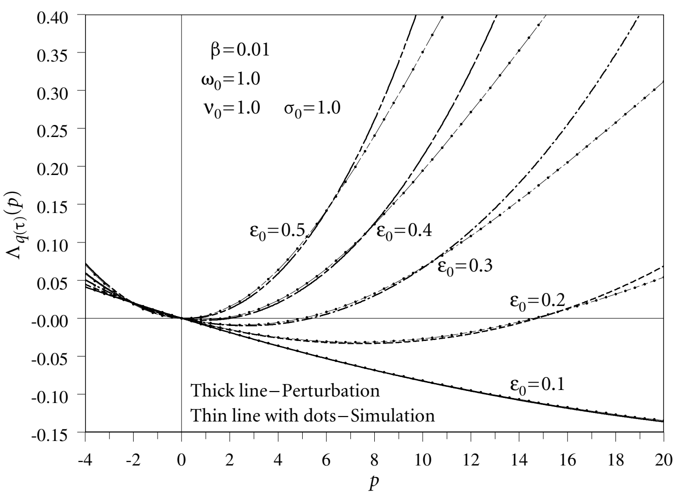

Figure 6. Moment Lyapunov Exponent (Bounded Noise)

Two-Dimensional System under Real Noise Excitation

The two-dimensional system (4) under real noise excitation (6) can be written as a three-dimensional autonomous stochastic system as:

$$\begin{cases} \mathrm{d}Y^1 = Y^2\,\mathrm{d}\tau, \\ \mathrm{d}Y^2 = \left[-2\beta\,Y^2 - (1 - \varepsilon_0\,Y^3)\,Y^1\right]\mathrm{d}\tau, \\ \mathrm{d}Y^3 = -\alpha_0\,Y^3\,\mathrm{d}\tau + \sigma_0\,\mathrm{d}W, \end{cases} \tag{36}$$

where $Y^1 = q(\tau)$, $Y^2 = \mathrm{d}q(\tau)/\mathrm{d}\tau$, $Y^3 = \xi(\tau)$, and Y^1 and Y^2 are used to calculated the pth norm of the state vector of the system $\|\mathbf{Y}\|^p = \left[(Y^1)^2 + (Y^2)^2\right]^{p/2}$.

Using equation (29), the order 2.0 weak Taylor scheme is given by

$$\begin{aligned}
Y^1_{s,m} &= Y^1_{s,m-1} + Y^2_{s,m-1}\cdot\Delta + \tfrac{1}{2}R_{s,m-1}\cdot\Delta^2,\\
Y^2_{s,m} &= Y^2_{s,m-1} + R_{s,m-1}\cdot\Delta + \varepsilon_0\sigma_0 Y^1_{s,m-1}\cdot\Delta Z\\
&\quad + \tfrac{1}{2}\big[-\omega_0^2 Y^2_{s,m-1} - 2\beta R_{s,m-1}\\
&\quad + \varepsilon_0 Y^3_{s,m-1}\big(Y^2_{s,m-1} - \alpha Y^1_{s,m-1}\big)\big]\cdot\Delta^2,\\
Y^3_{s,m} &= Y^3_{s,m-1} - \alpha_0 Y^3_{s,m-1}\cdot\Delta + \sigma_0\cdot\Delta W\\
&\quad + \tfrac{1}{2}\alpha_0^2 Y^3_{s,m-1}\cdot\Delta^2 - \alpha_0\sigma_0\cdot\Delta Z,\\
R_{s,m-1} &= -2\beta Y^2_{s,m-1} - \big(\omega_0^2 - \varepsilon_0 Y^3_{s,m-1}\big) Y^1_{s,m-1}.
\end{aligned}$$

The pth moment Lyapunov exponents of system (36) obtained using Monte Carlo simulation are presented in Figures 7–8 for $\alpha_0 = 2.0, 1.0$, respectively, $\beta = 0.01$, $\omega_0 = 1.0$, $\sigma_0 = 1.0$ and various values of ε_0. The simulation results compare well with a sixth-order approximation of the pth moment Lyapunov exponent obtained in (Xie, 2001b) using a method of regular perturbation for small values of $\varepsilon_0 > 0$.

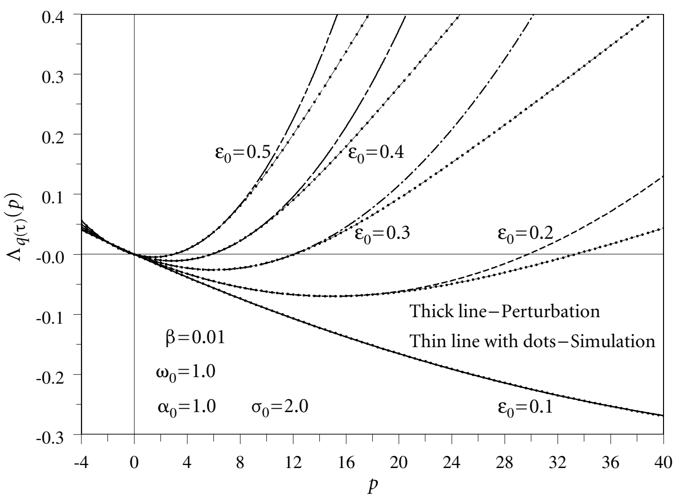

Figure 7. Moment Lyapunov Exponent (Real Noise)

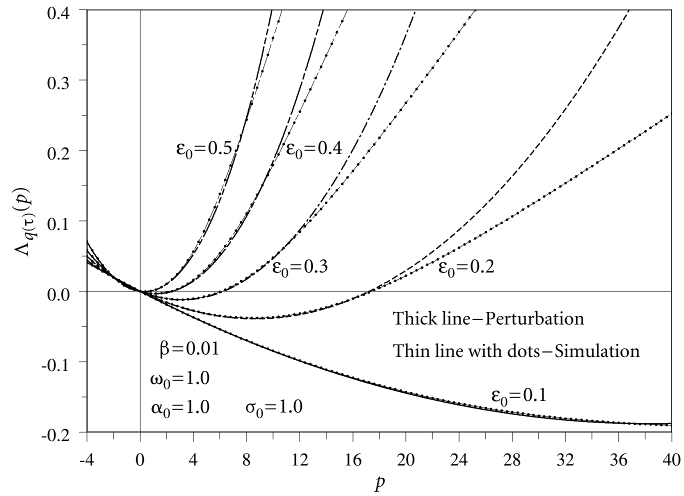

Figure 8. Moment Lyapunov Exponent (Real Noise)

5 Conclusions

In this paper, an analytical-numerical approached is employed to obtain numerical values of the moment Lyapunov exponents of a two-dimensional system under either a bounded noise or a real noise parametric excitation. The theory of stochastic dynamical systems is applied to establish the partial differential eigenvalue problems governing the pth moment Lyapunov exponents. Double series expansions of the eigenfunctions in terms of orthogonal functions are taken to transform the partial differential eigenvalue problems to linear algebraic eigenvalue problems. The eigs function in *Matlab* for determining a few eigenvalues of a large sparse matrix is then used to solve the linear eigenvalue problem to obtain the pth moment Lyapunov exponents. The numerical results are compared with the approximate analytical results with weak noise obtained earlier (Xie, 2001b and 2003). It is found that for the amplitude of the exciting noise ε in the order of 0.1, the approximate analytical results agree with the numerical results extremely well. Discrepancies increase for larger values of ε.

A Monte Carlo simulation procedure is also developed to numerically determine the pth moment Lyapunov exponents. The procedure can be easily implemented. The Monte Carlo simulation is a pure numerical method and is more general than the analytical-numerical approach. The method can be easily applied for higher dimensional systems and any noise excitations, even for those with only time series available.

This is the first paper that presents numerical methods for determining the pth moment Lyapunov exponents of stochastic systems under non-white noise excitations. Its usefulness and importance is twofold. Firstly, it verifies the validity of the approximate analytical results and determines the range of applicability of the parameter ε. Secondly, for many engineering applications, the amplitude of the noise ε is not small and numerical approaches must be employed to determine the pth moment Lyapunov exponents.

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Appendix A. Formulas of the Normalized Hermite Polynomials

It is well-known that the Hermite Polynomials satisfy the identities

$$H_{m+1}(\zeta) = 2x H_m(\zeta) - 2m H_{m-1}(\zeta), \;\; H_m{}'(\zeta) = 2m H_{m-1}(\zeta),$$

from which it can be shown that

$$\begin{aligned} x H_m(\zeta) &= \tfrac{1}{2} H_{m+1}(\zeta) + m H_{m-1}(\zeta), \\ x^2 H_m(\zeta) &= \tfrac{1}{4} H_{m+2}(\zeta) + (m+\tfrac{1}{2}) H_m(\zeta) + m(m-1) H_{m-2}(\zeta), \\ x H_m{}'(\zeta) &= m H_m(\zeta) + 2m(m-1) H_{m-2}(\zeta), \\ H_m{}''(\zeta) &= 4m(m-1) H_{m-2}(\zeta). \end{aligned}$$

Using these equations and the definition of the generalized Hermite polynomials, one obtains

$$\begin{aligned} x h_m(\zeta) &= \sqrt{\frac{m+1}{2}}\, h_{m+1}(\zeta) + \sqrt{\frac{m}{2}}\, h_{m-1}(\zeta), \\ x h_m{}'(\zeta) &= -\frac{\sqrt{(m+2)(m+1)}}{2}\, h_{m+2}(\zeta) - \tfrac{1}{2} h_m(\zeta) \\ &\quad + \frac{\sqrt{m(m-1)}}{2}\, h_{m-2}(\zeta), \\ h_m{}''(\zeta) &= \frac{\sqrt{(m+2)(m+1)}}{2}\, h_{m+2}(\zeta) - (n+\tfrac{1}{2})\, h_m(\zeta) \\ &\quad + \frac{\sqrt{m(m-1)}}{2}\, h_{m-2}(\zeta). \end{aligned}$$

Employing the orthogonality condition (22), the following results can be derived

$$\begin{aligned} \int_{-\infty}^{+\infty} x h_l(\zeta)\, h_m(\zeta)\, \mathrm{d}\zeta &= \sqrt{\frac{m+1}{2}}\, \delta_{l,m+1} + \sqrt{\frac{m}{2}}\, \delta_{l,m-1}, \\ \int_{-\infty}^{+\infty} x h_l{}'(\zeta)\, h_m(\zeta)\, \mathrm{d}\zeta &= -\frac{\sqrt{(m+2)(m+1)}}{2}\, \delta_{l,m+2} \\ &\quad - \tfrac{1}{2}\, \delta_{l,m} + \frac{\sqrt{m(m-1)}}{2}\, \delta_{l,m-2}, \\ \int_{-\infty}^{+\infty} h_l{}''(\zeta)\, h_m(\zeta)\, \mathrm{d}\zeta &= \frac{\sqrt{(m+2)(m+1)}}{2}\, \delta_{l,m+2} \\ &\quad - (n+\tfrac{1}{2})\, \delta_{l,m} + \frac{\sqrt{m(m-1)}}{2}\, \delta_{l,m-2}. \end{aligned}$$