

# Convex Hull Pricing in Electricity Markets: Formulation, Analysis, and Implementation Challenges

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## Abstract

Widespread interest in Convex Hull Pricing has unfortunately not been accompanied by an equally broad understanding of the method. This paper attempts to narrow the gap between enthusiasm and comprehension. Most importantly, Convex Hull Pricing is developed in an understandable manner – starting with a discussion of basic electricity market processes and ending with a new mathematical formulation of Convex Hull Pricing. From this mathematical formulation, a variety of important properties are derived and discussed. To illustrate that the [sometimes counterintuitive] properties of Convex Hull Pricing are not merely theoretical, several simple examples are presented. It is hoped that this paper will spur additional research on the pricing scheme so that an informed judgment can be made regarding its costs and benefits.

## I. Introduction

Most participants in wholesale electricity markets desire the “right price” for electricity and ancillary services, but no one can clearly define the “right price” or how it should be derived mathematically. Consequently, each Independent System Operator (ISO) uses different assumptions to simplify its underlying problem and formulate a tractable pricing algorithm with reproducible results. Although it is hoped that the resulting prices are acceptable to participants, there is arguably room for improvement.

Mathematically, the ISO pricing methods differ in sometimes subtle but important ways and, as a consequence, generate prices that incorporate different pricing elements and assumptions. Although ISOs use different naming conventions when referring to pricing methods, three well-established energy pricing principles are

- Ex ante pricing: price based on the marginal cost of serving load at the dispatch solution
- Ex post pricing (as originally implemented by PJM): price based on the marginal cost of serving load at the actual resource output levels
- Relaxed commitment pricing: price based on the marginal cost of serving load at a special dispatch solution obtained by relaxing certain bid-in resource parameters

Because all ISO pricing methods are based on the concept of optimal shadow prices, none are categorically wrong. The “best” pricing method, however, is open to debate.

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A pricing scheme that has recently risen in prominence is *Extended LMP* (ELMP). Mathematically, the entirety of this pricing method is better described as *Convex Hull Pricing* (ConvHP) and possesses several interesting properties. Unfortunately, the special properties of this pricing method are commonly misunderstood or incorrectly summarized by its proponents. These misconceptions are problematic for both ISOs and market participants: ISOs may implement Convex Hull Pricing without fully understanding its consequences while market participants may find it difficult to understand market outcomes.

The overarching purpose of this paper is to present Convex Hull Pricing and its properties in an understandable form through both mathematics and examples. This presentation is timely because basic knowledge about ConvHP is lacking even though it has been discussed in a variety of forums. Furthermore, the initial formulation and analysis of ConvHP (Gribik, Hogan, & Pope, 2007) was specialized for an electricity market and thus did not convey the generality of the method. It is hoped that the general formulation of this paper is more amenable to rigorous economic analysis.

In Section II, three ISO processes are described, the basic mathematical foundation of pricing is presented, and the underlying Convex Hull Pricing principle is motivated. Section III continues this development by summarizing the ConvHP explanation provided in (Gribik, Hogan, & Pope, 2007) and mathematically formulating the complete method. A proof of the side-payment minimization property of Convex Hull Pricing is also provided. In Section IV, other ConvHP properties are described and example problems are solved; the examples demonstrate the counterintuitive nature of some Convex Hull Pricing outcomes. Section V discusses the challenges of Convex Hull Pricing. Section VI concludes the paper.

## II. The basic ISO processes

For an ISO to run a wholesale electricity market, it must perform at least three basic tasks: commitment, dispatch, and pricing. These decisions are made in order (commitment before dispatch before pricing) and the results are communicated to market participants.<sup>2</sup> Common pricing schemes are closely related to dispatch-type problems because these problems naturally allow for a “marginal cost pricing” concept. Unfortunately, electricity markets do not fall neatly into that framework because physical constraints restrict the divisibility of goods (e.g., economic minimum, block-loading). To identify prices in this less-than-perfect setting, Convex Hull Pricing applies the marginal cost pricing concept to a modified commitment-type problem. In this section, the commitment, dispatch, and pricing optimization problems will be described so that the ConvHP problem can be appreciated.

### 1. Commitment, dispatch, and pricing

**Assumption 1.** The Commitment, Dispatch, and Pricing problems have nonempty, compact feasible regions.

The *Commitment problem* minimizes the total cost of meeting forecasted load<sup>3</sup> over a multi-hour time horizon by making **unit commitment decisions** and anticipated dispatch decisions. The objective function considers start-up costs, no-load costs, and incremental costs of all resources. Constraints enforced within the Commitment problem include bid-in resource capabilities, transmission or security constraints, forecasted energy balances, and ancillary service constraints.

Mathematically, the Commitment problem is usually formulated as a mixed integer linear program (MILP) but a more general mixed integer program (MIP) is studied in this paper. Denote resources by index  $i$  and time intervals by index  $t$ . After moving resource cost functions to the constraint set by introducing new variables, the commitment problem can be formulated as

$$\begin{aligned}
 \min_{c, x, u} \quad & \sum_t \sum_i c_{it} \\
 \text{s.t.} \quad & \sum_i A_{it} x_{it} \geq \mathbf{b}_t \quad \forall t \\
 & (c_i, x_i, u_i) \in X_i \quad \forall i.
 \end{aligned} \tag{1}$$

In this formulation,  $c_{it}$  is the cost function variable,  $u_{it}$  is the commitment variable  $\rightarrow$  *desired commitment state*, and  $x_{it}$  represents all other variables (e.g., cleared energy quantity, cleared ancillary service quantities). Vectors of these variables over the time horizon are denoted by  $c_i$ ,  $u_i$ , and  $x_i$ ,

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<sup>2</sup> In several ISOs, the dispatch and pricing steps are integrated in the sense that the optimal dispatch instructions and uniform prices are products of the same social surplus maximization problem. This integrated approach can easily be framed as the separate steps described here.

<sup>3</sup> Loads are treated as fixed in this paper. More general models of commitment, dispatch, and pricing maximize social surplus by clearing price-sensitive (as opposed to fixed) load.

respectively. The constraint  $\sum_i A_{it} x_{it} \geq b_t$  captures system-level constraints that are shared by the resources for each time interval:

- energy balance, transmission, security, ancillary service requirements, etc.

The sets  $X_i$  define resource-specific feasible regions over the entire time horizon:

- cost function specification<sup>4</sup>, cost function upper bound, production limits, ramp constraints, commitment-related properties such as minimum up/down times, etc.

It is assumed that each  $X_i$  is a compact, nonconvex set. This assumption is natural for electricity markets and only requires the addition of sufficiently large cost function upper bounds (included in above  $X_i$  feasible region description).

For the upcoming development of Convex Hull Pricing, it is useful to view the resource-specific feasible regions of (1) in a different but equivalent form. Namely, completely enumerate the possible commitment sequences  $u_i$  for each resource and denote each resulting compact [but not necessarily convex] set by  $X_i^j$  where  $j \in J_i$  with  $J_i$  being the enumerated set of commitment sequences. For example, a two-interval problem would give each resource the possible commitment sequence enumeration

$$\{(\text{On}, \text{On}), (\text{On}, \text{Off}), (\text{Off}, \text{On}), (\text{Off}, \text{Off})\}.$$

For certain  $j \in J_i$ , it is possible for  $X_i^j = \emptyset$  because constraints such as minimum up/down times make the specified commitment sequence infeasible.

Using the enumerated  $X_i^j$  regions, the Commitment problem (1) can be represented without explicit  $u_i$  vectors as the disjunctive program<sup>5</sup>

$$\begin{aligned} \min_{c, x} \quad & \sum_t \sum_i c_{it} \\ \text{s.t.} \quad & \sum_i A_{it} x_{it} \geq b_t \quad \forall t \\ & (c_i, x_i) \in \bigvee_{j \in J_i} X_i^j \quad \forall i. \end{aligned} \tag{2}$$

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<sup>4</sup> Resource cost functions are commonly assumed to be piecewise linear. This type of cost function can be expressed in the constraint set  $X_i$  as a series of lower bounds. For example, the piecewise linear cost function

$$\left\{ \begin{array}{l} x, \quad \text{if } x \leq 5 \\ 2x-5, \quad \text{if } x \geq 5 \end{array} \right\} \text{ can be modeled by } \min_{c, x} \quad c \quad \text{s.t.} \quad \begin{array}{l} c \geq x \\ c \geq 2x-5. \end{array}$$

<sup>5</sup> The term ‘‘disjunctive’’ comes from the fact that a set of constraints is an *inclusive or*.

The constraints of Commitment problem (2) require that the  $(c_i, x_i)$  pair of each resource  $i$  be in at least one  $X_i^j$ , thereby implying that there is at least one  $u_i$  commitment sequence for which  $(c_i, x_i)$  is feasible.

Following the Commitment problem, the **Dispatch problem**<sup>6</sup> is solved to minimize the production cost of satisfying forecasted load and ancillary service requirements by making **dispatch decisions for all committed units**. Transmission line constraints, security constraints, and resource capabilities are respected just as in the Commitment problem. Mathematically, the Dispatch problem can be formulated in a manner very similar to (1):

$$\begin{aligned}
 \min_{c,x} \quad & \sum_t \sum_i c_{it} \\
 \text{s.t.} \quad & \sum_i A_{it} x_{it} \geq b_t \quad \forall t \\
 & (c_i, x_i, u_i^*) \in X_i \quad \forall i, t.
 \end{aligned} \tag{3}$$

The only difference between (1) and (3) is that the latter Dispatch problem fixes the commitment sequence of each resource to its optimal value from the Commitment problem, hence the \* superscript. If the Commitment problem is a MILP, (3) is a reasonably sized linear programming problem. The *desired output* is  $x_i$ .

The final ISO process is the determination of market clearing prices for energy and ancillary services (*ISO price signal*). Each ISO pricing method determines prices based on the optimal **shadow prices** of a linear **Pricing problem**, where a shadow price is [loosely speaking] the objective function change induced by a marginal increase or decrease of the associated constraint's bound.

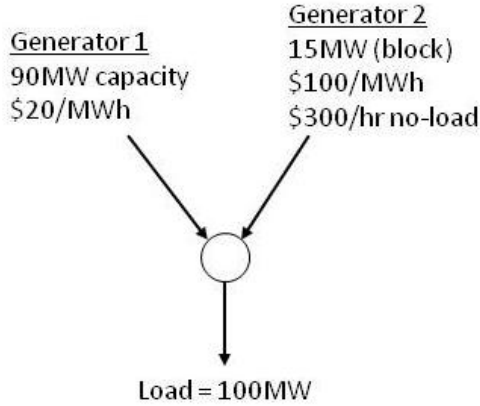
**Remark 1.** Because shadow prices reflect objective function changes, costs that are fixed or not considered in the objective function of a Pricing problem **cannot** be reflected in market prices. The incorporation of *fixed costs* in shadow prices is only possible if they are somehow modeled as *variable costs*. There are obviously consequences to such a modeling change because the Pricing problem will no longer reflect actual system production costs.

The following simple example illustrates how these three ISO processes interact for a simple problem.

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<sup>6</sup> For simplicity, it is assumed that the Dispatch problem is solved with the same time horizon as the Commitment problem using the same system and forecast information. In practice, multi-interval Dispatch problems usually have shorter time horizons than Commitment problems (e.g., the California ISO uses a 1.5 hour horizon for its Dispatch problem but its Commitment problem looks ahead 4–4.5 hours).

**Example 1.** Consider a single-node, single-interval problem. Generator 1 must remain online and Generator 2 is an offline but available fast-start (FS) unit that can be block-loaded.



The Commitment problem (1) determines whether Generator 2 should be committed by solving

$$\begin{aligned} \min_{c,x,u} \quad & c_1 + c_2 \\ \text{s.t.} \quad & x_1 + x_2 = 100 \\ & c_1 = 20x_1 \\ & 0 \leq x_1 \leq 90 \\ & c_2 = 100x_2 + 300u_2 \\ & x_2 = 15u_2 \\ & u_2 \in \{0,1\}. \end{aligned}$$

The only feasible solution commits Generator 2, making the Dispatch problem (3)

$$\begin{aligned} \min_{c,x} \quad & c_1 + c_2 \\ \text{s.t.} \quad & x_1 + x_2 = 100 \quad (\lambda) \\ & c_1 = 20x_1 \\ & 0 \leq x_1 \leq 90 \\ & c_2 = 100x_2 + 300 \\ & x_2 = 15. \end{aligned}$$

The optimal dispatch solution entails Generator 1 producing 85MW and Generator 2 producing 15MW. If the Pricing problem is the same as the Dispatch problem, the LMP is \$20/MWh (the marginal cost of serving load at the dispatch solution). Mathematically, this LMP is the optimal shadow price  $\lambda^*$  of the energy balance constraint. It does not reflect the production cost of Generator 2 because that production cost is independent of a marginal load change.

## 2. Motivation behind Convex Hull Pricing

To understand Convex Hull Pricing, it is advantageous to have some intuition about the relationship between dispatch-following incentives and optimal shadow prices. This relationship, as will be explored, is based on the concept of a competitive partial equilibrium – an equilibrium exists if (a) every participant maximizes its utility given the payments it receives, and (b) the market clears. Because resources can be paid both within the market (e.g., energy and ancillary service payments) and outside of the market (e.g., make-whole payments), payments can be classified as *market-based payments* and *side-payments*,

respectively. In order to appropriately incentivize resources to follow their cleared quantities, a combination of market-based payments and side-payments must eliminate incentives to deviate.

**Definition 1.** Uniform market clearing prices are called *market-based incentive compatible* for a resource if the resource does not have a strong incentive to deviate from its cleared quantities.

The distinguishing aspect of the *market-based* property of Definition 1 is that the resource does not have an incentive to deviate given market prices (and only market prices). This is the traditional economics definition for property (a) of a competitive partial equilibrium. When side-payments are required, a traditional equilibrium does not exist because incentives are no longer purely “market-based.” Therefore, Definition 1 is generalized to Definition 2 here.

**Definition 2.** The combination of uniform market clearing prices and side-payments is called *incentive compatible* for a resource if the resource does not have a strong incentive to deviate from its cleared quantities.

For any resource, it should be obvious that *market-based* incentive compatibility implies incentive compatibility. The converse is not necessarily true.

The basic incentive-shadow price relationship can be most clearly observed by examining the necessary and sufficient **Karush-Kuhn-Tucker optimality conditions** (KKT conditions) of a linear optimization problem (a similar property holds for convex optimization problems under a suitable regularity condition).

**Theorem 1.** (Nocedal & Wright, 2006)

Consider the linear optimization problem

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \quad (\lambda). \end{aligned}$$

The **necessary** and **sufficient** KKT optimality conditions are

$$\begin{aligned} c - A^T \lambda &= 0 \\ Ax &\geq b \\ \lambda &\geq 0 \\ \lambda^T (Ax - b) &= 0. \end{aligned}$$

“Necessity” means that these conditions must be satisfied by every optimal solution, and “sufficiency” means that any vector satisfying the KKT conditions is an optimal solution.

Without loss of generality, a linear Pricing problem can be expressed as

$$\begin{aligned}
 \min_{c,x} \quad & \sum_i c_i \\
 \text{s.t.} \quad & Ax \geq b \quad (\lambda) \\
 & B_i c_i + C_i x_i \geq d_i \quad (\mu_i) \quad \forall i
 \end{aligned}$$

In this problem,  $c$  is the cost variable,  $x$  is the quantity variable,  $\lambda$  is the shadow price vector for the system constraints, and  $\mu_i$  is the shadow price vector for the private constraint set of participant  $i$ . The optimal solution of this problem consists of the **pricing run quantities**  $x^*$  and **optimal shadow prices**  $\lambda^*$  and  $\mu^*$ . Because the Pricing problem identifies market prices, only the optimal shadow prices  $\lambda^*$  and  $\mu^*$  are used by the ISO. The pricing run quantities can be thought of as artificial values associated with the optimal shadow prices. From a basic analysis of the KKT conditions (see Appendix A), it can be concluded that optimal shadow prices are *market-based* incentive compatible with each resource's pricing run quantities. Unfortunately, pricing run quantities do **NOT** necessarily agree with cleared quantities (see Remark 1). For example, economic minimum (EcoMin) values of committed FS resources are relaxed to 0MW in the Midcontinent System Operator's (MISO's) Pricing problem. As a consequence, the pricing run quantity for a FS resource can be less than its bid-in EcoMin. Because the Dispatch problem enforces bid-in values, the cleared quantities for FS resources may be different than their pricing run quantities. These differences indicate that *market-based* incentive compatibility does not always hold, thus the need for side-payments to achieve incentive compatibility for each resource.

**Example 2.** In Example 1, consider relaxing the EcoMin of Generator 2 to 0MW and amortizing its \$300/hr no-load cost over its 15MW output block. Mathematically, the Pricing problem is

$$\begin{aligned}
 \min_{c,x} \quad & c_1 + c_2 \\
 \text{s.t.} \quad & x_1 + x_2 = 100 \\
 & c_1 = 20x_1 \\
 & 0 \leq x_1 \leq 90 \\
 & c_2 = 120x_2 \\
 & 0 \leq x_2 \leq 15.
 \end{aligned}$$

The optimal solution of this Pricing problem is 90MW from Generator 1 and 10MW from Generator 2. The LMP is \$120/MWh. Recall that the actual dispatch solution is 85MW for Generator 1 and 15MW for Generator 2. Therefore, *market-based* incentive compatibility does not hold for Generator 1 (it would strictly prefer to produce 90MW given the \$120/MWh LMP). *Market-based* incentive compatibility holds for Generator 2.

If commitment decisions are not treated as fixed, a side-payment may be necessary even without disagreement between the pricing run quantities and the cleared quantities.



**Example 3.** Consider Example 1 again. The LMP is \$20/MWh if the Pricing problem is identical to the Dispatch problem. Given this LMP, Generator 1 earns its optimal as-bid profit of \$0 when producing at its actual dispatch level of 85MW. Generator 2, on the other hand, has an incremental cost of \$100/MWh and a no-load cost of \$300/hr. If Generator 2 was not constrained by its commitment status, it would strictly prefer to be offline and therefore requires a side-payment even though its cleared quantity and pricing run quantity are identical. These conclusions are the reverse of those reached in Example 2.

**Remark 2.** In general, incentive compatibility requires side-payments when the original market is *nonconvex* because a *convex* Pricing problem cannot perfectly reflect model nonconvexities. Current two-settlement electricity markets naturally include nonconvexities in the form of EcoMin values and three-part bids, so side-payments are in some sense unavoidable under a uniform pricing framework.

Side-payments are commonly separated into two categories based on their intended purpose. First, a resource can be unhappy because its cleared quantity profit is negative given the market clearing prices. This is the situation faced by Generator 2 in Example 3. Second, a resource can be unhappy because its cleared quantity profit is not the maximum possible given prices. Knowing these two incentive issues, side-payments may be separated as follows.<sup>7</sup>

1. **Make-whole payments** (MWP): ensure that each resource receives at least its cleared bid-in cost → earn at least \$0 as-bid profit
2. **Lost opportunity costs** (LOCs): ensure that each resource receives its maximum possible profit given prices and its bid-in constraints<sup>8</sup>

A MWP is, in essence, a special type of LOC where the maximum as-bid profit is taken to be \$0. In ISO New England, Net Commitment Period Compensation payments fulfill the role of MWPs (and LOC-type payments in special situations) and administrative penalties are imposed in lieu of universal LOC payments. Under the Convex Hull Pricing framework, only LOCs are considered with the understanding that MWPs may be taken into account by the LOC calculation:

Maximum possible as-bid profit – As-bid cleared quantity profit.

A resource that follows its quantity instruction and receives the above LOC payment will earn its maximum possible as-bid profit.

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<sup>7</sup> In certain situations, it may be impossible to distinguish between the two reasons for unhappiness specified here. For instance, it is possible for a resource to have a negative cleared quantity profit and a maximum possible profit of \$0.

<sup>8</sup> LOCs can be defined differently, but this definition from (Gribik, Hogan, & Pope, 2007) is the most complete from the authors' perspective. It fully reflects the idea that LOCs ensure that a unit achieves its maximum possible as-bid profit, regardless of what this profit level is.

**Remark 3.** As written, the LOC value is independent of the resource’s “performance” relative to its cleared quantities. This independence will not naturally result in incentive compatibility so an additional LOC restriction must be introduced. The simplest modification would be the elimination of LOC payments for resources that do not “perform” close enough to their cleared quantities.

For participants, MWPs and LOCs are discriminative and not transparent. They are undesirable because loads must estimate and recover these side-payments in their hedging strategies. Generators may also be unhappy with these payments because they cannot be effectively hedged in forward markets. The general consensus in electricity markets is that MWPs and LOCs should be reduced. The endpoint of this reasoning defines Convex Hull Pricing:

Identify energy and ancillary service prices that minimize total side-payments.

### III. Development of Convex Hull Pricing

The original presentation of (Gribik, Hogan, & Pope, 2007) graphically illustrates the concept of **convex hull prices** for a simple problem. This exposition is useful for building some intuition about the pricing method and is therefore summarized below for a short example. After the example, a general mathematical formulation of Convex Hull Pricing is provided. This general formulation is easier to understand than the two-stage formulation provided in (Gribik, Hogan, & Pope, 2007) and is shown to give the same solution for the example problem. The section concludes with a proof that the presented ConvHP formulation is indeed “Convex Hull Pricing” – it minimizes total side-payments.

#### 1. Graphical method for simple problems

For simple problems, convex hull prices can be graphically calculated in three steps.

Step 1. Construct the **optimal total cost curve** as a function of load for the Commitment problem

Step 2. Find the **convex hull of the optimal total cost curve**

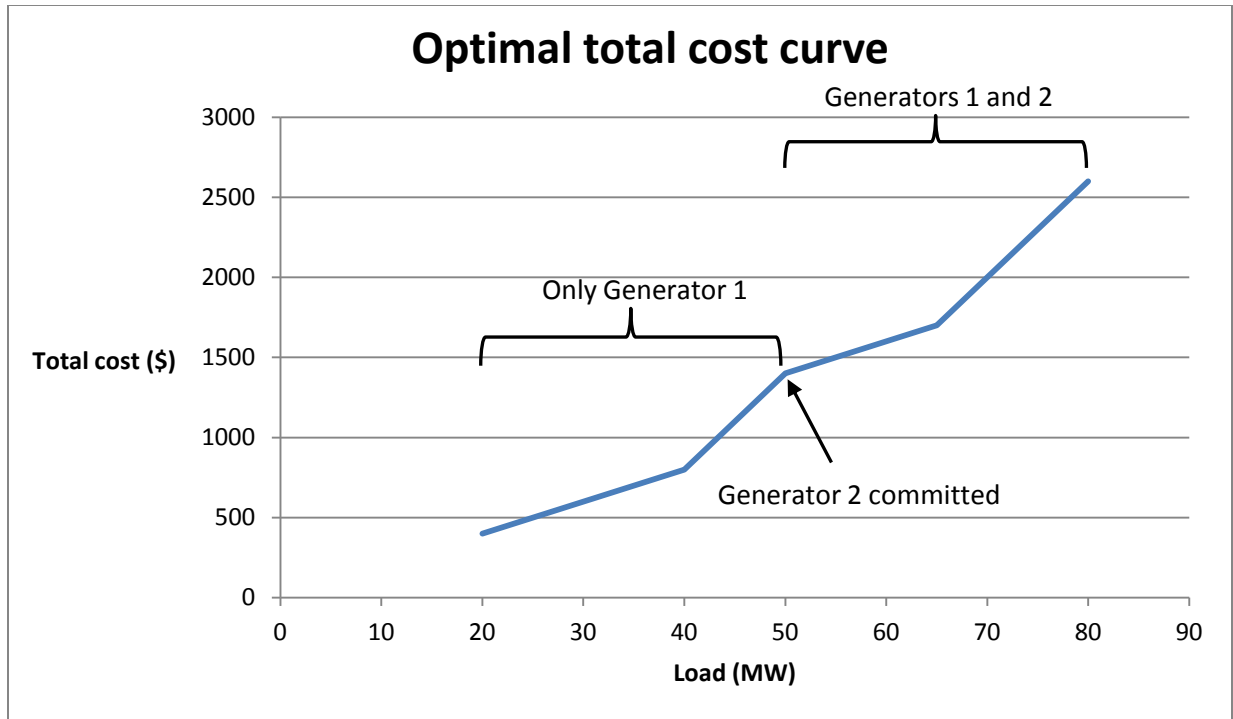
Step 3. The convex hull price is the slope of the convex hull of the optimal total cost curve at the actual load

This graphical method only works when the Pricing problem (a) has one time interval, (b) is single-node, (c) does not consider marginal losses, and (d) does not include ancillary services. Extensions of the method to problems that violate these conditions are exceedingly difficult because the optimal total cost curve becomes nontrivial and the corresponding graphical representation becomes high dimensional. Since a majority of electricity market problems include transmission constraints and ancillary services, a rigorous and mathematical (as opposed to graphical) formulation of Convex Hull Pricing is needed.

Consider a single-interval, single-node Pricing problem with no ancillary services and two generators. Let Generator 1 be online and incapable of shutting down. Let Generator 2 be an offline but available FS unit. The generator bids/offers are provided in the following table.

	Generator 1	Generator 2
Economic minimum (MW)	20	25
Economic maximum (MW)	55	25
Bid block 1 (MW range, \$/MWh)	0 – 40, 20	-
Bid block 2 (MW range, \$/MWh)	40 – 55, 60	-
No-load cost (\$/hr)	-	900

Step 1 leads to the following **optimal total cost curve**.

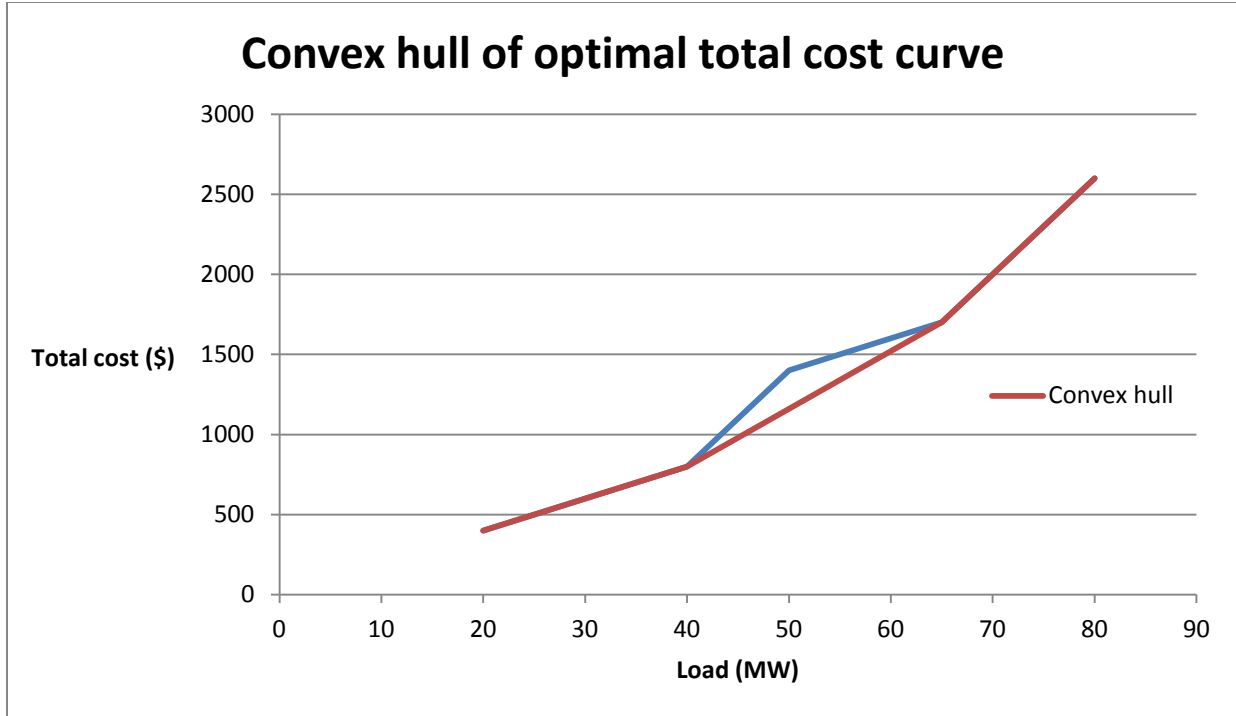


For loads of 20 – 50MW, the least cost solution is to produce from Generator 1 only. Above 50MW, the least cost solution is to commit Generator 2 and produce from both generators. The decrease in marginal total cost at the 50MW load level is caused by the 25MW block-loading of Generator 2 decreasing Generator 1’s dispatch to 25MW.

Step 2 of Convex Hull Pricing entails constructing the **convex hull of the optimal total cost curve**:

‘the greatest convex function that is bounded above by the optimal total cost curve.’

A more traditional definition of a convex hull will be provided in the upcoming mathematical ConvHP formulation. Step 2 gives the following figure.



According to Step 3, the convex hull price is the slope of this convex hull at the actual load. In the following table, it can be seen that the relative relationship between the convex hull price and the slope of the total cost curve itself (i.e., *dispatch price*) changes with load.

Load level (MW)	Dispatch price (\$/MWh)	Convex hull price (\$/MWh)
30	20	20
45	60	36
55	20	36
70	60	60

It is proven in (Gribik, Hogan, & Pope, 2007) that, for the specified problem form, Steps 1 – 3 lead to uniform energy prices that minimize side-payments:

$$\sum_i (\text{Maximum possible as-bid profit} - \text{As-bid cleared quantity profit})_i.$$

Unfortunately, the provided proof only explicitly applies to the specified problem form with a claim of validity for problems with inequality constraints (e.g., transmission constraints, reserve constraints). This validity with inequality constraints must be proven because, although true, it requires an expanded definition of side-payments. These theoretical deficiencies will be rectified next via a rigorous mathematical formulation of Convex Hull Pricing.

## 2. Mathematical Convex Hull Pricing formulation and side-payment minimization proof<sup>9</sup>

To be a viable pricing method, the graphical development of Convex Hull Pricing must be generalized mathematically for realistic problems that include (among other elements) multiple time intervals, a meshed network with transmission constraints, losses, reserves, and security constraints. A Lagrangian dual formulation of the ConvHP problem was presented in (Gribik, Hogan, & Pope, 2007), but the resulting problem form is not easy to solve and is specialized for an electricity market setting. A simple and effective primal formulation is provided here; the formulation also gives rise to a straightforward side-payment minimization proof. The interested reader is referred to (Falk, 1969) for a detailed theoretical analysis of the relationship between the upcoming Convex Hull Pricing formulation and the formulation in (Gribik, Hogan, & Pope, 2007).

To begin, the traditional definition for a **convex hull of a set** is needed.

**Definition 3.** The convex hull of set  $S$ ,  $\text{conv}(S)$ , is the intersection of all convex sets containing  $S$ .

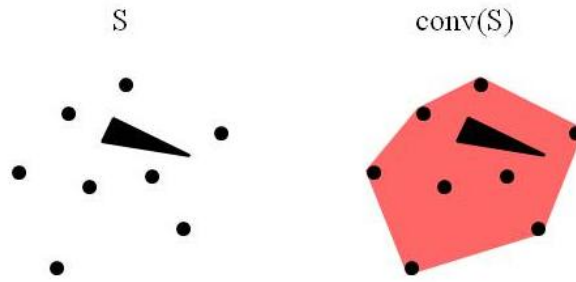


Figure 1. Convex hull of a set

Applying Definition 3, it is claimed that the Convex Hull Pricing problem corresponding to the Commitment problem (2) is

$$\begin{aligned}
 \min_{c, x} \quad & \sum_t \sum_i c_{it} \\
 \text{s.t.} \quad & \sum_i A_{it} x_{it} \geq b_t \quad (\lambda_t) \quad \forall t \\
 & (c_i, x_i) \in \text{conv}\left(\bigvee_{j \in J_i} X_i^j\right) \quad \forall i.
 \end{aligned} \tag{4}$$

Convex hull prices are the optimal shadow prices  $\lambda^*$  of (4). It is important to note that “convexification” is performed on a resource-specific level, not a system-wide level. If each  $X_i^j$  is a compact polyhedron (a common assumption in electricity markets), (4) is a linear program and therefore has a number of desirable properties. Most importantly, every solution is globally optimal. From a computational standpoint, a number of solution methods are available when (4) is a linear program (e.g., simplex

<sup>9</sup> This section can be skipped without a loss of understanding.

method, interior point method, etc.). The computational methods available for the minimax formulation of (Gribik, Hogan, & Pope, 2007) are significantly more limited.

For the linear example problems in this paper, it is useful to have an explicit definition of the unit-specific convex hulls in (4).<sup>10</sup> The following result from (Balas, 1998) provides this formulation.

**Theorem 2.** (Balas, 1998)

Let  $\bigvee_{k \in K} \begin{pmatrix} \mathbf{F}y \geq \mathbf{f} \\ \mathbf{G}^k y \geq \mathbf{g}^k \\ y \geq 0 \end{pmatrix}$  be a finite set of compact polyhedra,  $Y = \left\{ y \in \mathbb{R}^n \mid \bigvee_{k \in K} \begin{pmatrix} \mathbf{F}y \geq \mathbf{f} \\ \mathbf{G}^k y \geq \mathbf{g}^k \\ y \geq 0 \end{pmatrix} \right\}$ ,

$Y^k = \{y \in \mathbb{R}^n \mid \mathbf{F}y \geq \mathbf{f}, \mathbf{G}^k y \geq \mathbf{g}^k, y \geq 0\}$ , and  $K^* = \{k \in K \mid Y^k \neq \emptyset\}$ . If  $Y \neq \emptyset$ , then

$$\text{conv}(Y) = \left\{ \begin{array}{l} y \in \mathbb{R}^n \mid y = \sum_{k \in K^*} \xi^k \\ \mathbf{F}\xi^k - \mathbf{f}\xi_0^k \geq 0, \quad k \in K^* \\ \mathbf{G}^k \xi^k - \mathbf{g}^k \xi_0^k \geq 0, \quad k \in K^* \\ \sum_{k \in K^*} \xi_0^k = 1 \\ \xi^k, \xi_0^k \geq 0, \quad k \in K^* \end{array} \right\}.$$

Using Theorem 2, the lower bound of the feasible region of (4) is identical to the convex hull obtained from the graphical method for the example at the beginning of this section: the example's Convex Hull Pricing problem for load  $L$  is

$$\begin{aligned} \min_{c, x, \xi_0} \quad & c_1 + c_2 \\ \text{s.t.} \quad & x_1 + x_2 = L \quad (\lambda) \\ & 20x_1 \leq c_1 \\ & 60x_1 - 1600 \leq c_1 \leq 1700 \\ & 20 \leq x_1 \leq 55 \\ & c_2 = 900\xi_0 \\ & x_2 = 25\xi_0 \\ & 0 \leq \xi_0 \leq 1. \end{aligned}$$

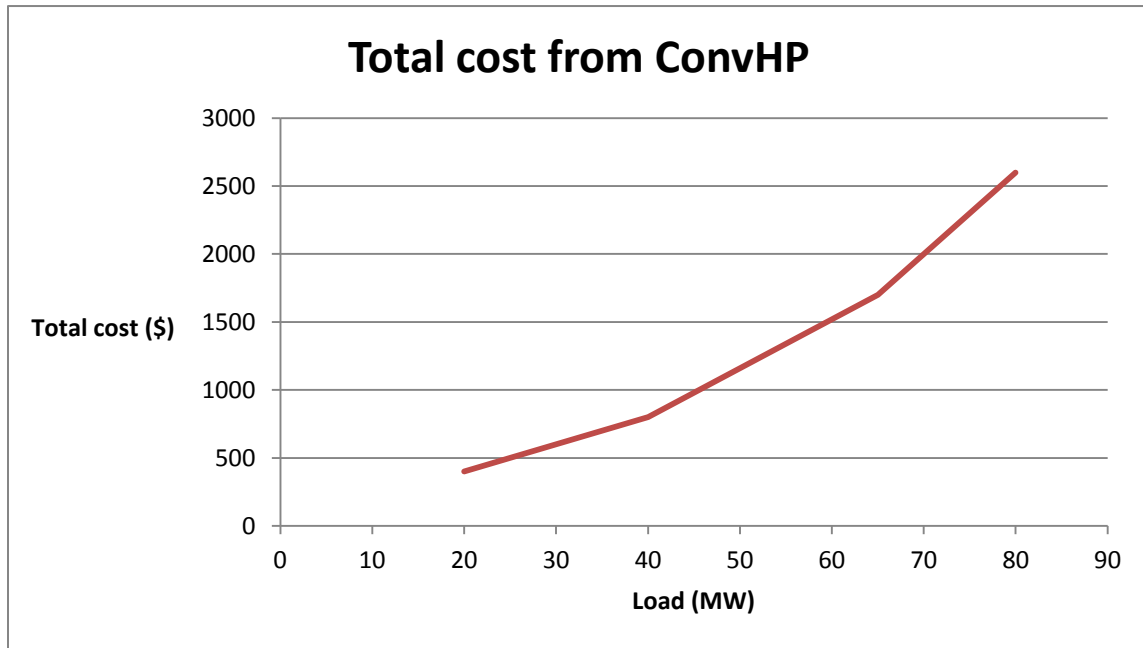
In this problem, the “marginal cost” of Generator 2 is modeled as  $\frac{\$900/\text{h}}{25\text{MW}} = \$36/\text{MWh}$  (i.e., increasing  $\xi_0$  by 0.04 provides 1MWh for \$36). This “marginal cost” is not real in the sense that

<sup>10</sup> Although the explicit form of the convex hull is readily available for polyhedral  $X_i^j$ , the fact that (4) is the Convex Hull Pricing problem for (2) only requires Assumption 1 and compact private constraint sets.

Generator 2 is block-loaded and cannot provide marginal MWs; instead, it represents a relaxation of the Convex Hull Pricing framework. The marginal costs will be accepted in the order:

\$20/MWh (Generator 1; Block 1), \$36/MWh (Generator 2), \$60/MWh (Generator 1; Block 2).

The following graph shows the optimal Convex Hull Pricing objective value as a function of load  $L$ . Because this function is the same as the “convex hull of the optimal total cost curve,” (4) will produce the same convex hull prices as the graphical method.



To prove that (4) is indeed the correct generalization of the graphical Convex Hull Pricing method for realistic problems, it must be proven that its solution minimizes total side-payments. Consider the Lagrangian dual problem of (4) with the shared constraints relaxed:

$$\max_{\lambda \geq 0} \left\{ \begin{array}{l} \min_{c, x} \sum_t \sum_i c_{it} - \sum_t \lambda_t \left( \sum_i A_{it} x_{it} - b_t \right) \\ \text{s.t.} \quad (c_i, x_i) \in \text{conv} \left( \bigvee_{j \in J_i} X_i^j \right) \quad \forall i \end{array} \right\}. \quad (5)$$

**Assumption 2.** Slater’s condition holds for the convex optimization problem (4).

Given Assumption 2, strong duality holds between (4) and (5).<sup>11</sup> By Assumption 1, the existence of a primal (and therefore a dual) optimal solution is guaranteed. Rearranging terms,

<sup>11</sup> Strong duality obviously holds if (4) is a linear optimization problem.





- The third minimized term corresponds to the sum of payment differences for each market product. Because the quantity instructions must enforce  $\sum_i A_{it} x_{it}^{\text{DDP}} \geq b_t$ , this final term will be nonnegative since  $\lambda \geq 0$ .

When combined, the first two minimized terms give

$$\sum_i (\text{Maximum possible as-bid profit} - \text{As-bid cleared quantity profit})_i,$$

the total LOC payment required by the resources. As noted above, this total LOC payment will be nonnegative.

The third term ensures that, given prices, the payments made by load cover the total payments required by generators. Consider a general shared constraint  $a^T x \geq b$ . For concreteness, this constraint can be thought of as a reserve or transmission constraint. The third minimized term of (8) is only needed when (a) the constraint is not binding as dispatched, and (b) the Pricing problem associates a positive optimal shadow price with the constraint. Therefore, the third term will hereafter be called an **excess product payment**<sup>12</sup> – “excess” meaning that the payment is only required when the ISO dispatch results in the constraint being slack, and “product” meaning that the minimized term can be expressed as the summation of payments for different constraint products (e.g., reserve, transmission).<sup>13</sup> The concept of excess product payments will be discussed in much detail in the next two sections.

The following theorem summarizes the Convex Hull Pricing result.

**Theorem 3.**

Given Assumptions 1 and 2, (4) is the Convex Hull Pricing problem for (2).

<sup>12</sup> This terminology is not ideal and ISO New England is exploring more natural names for this type of side-payment.

<sup>13</sup> A transmission-only version of this payment was called **FTR uplift** in (Gribik, Hogan, & Pope, 2007).

#### IV. Convex Hull Pricing: Properties and examples

Convex Hull Pricing has several interesting theoretical properties, the most commonly cited of which is side-payment minimization (proven at the end of Section III). This section explicitly details the most important Convex Hull Pricing properties. Short examples are also solved to demonstrate that the stated properties hold for simple problems.

**Property #1**

Convex Hull Pricing identifies uniform energy and ancillary service prices that **minimize total side-payments**.

Property #1 is the most commonly cited property of Convex Hull Pricing. Without additional clarification on what counts as a side-payment, this statement can be misleading. From (8), a rigorous statement about side-payment minimization under ConvHP can be made.

**Property #2**

The total side-payment minimized by Convex Hull Pricing consists of two parts:  
LOC payments and excess product payments.

Each LOC payment is calculated at a resource-specific level for the entire time horizon:

$$\text{Maximum possible as-bid profit} - \text{As-bid cleared quantity profit.}$$

The first term in this LOC calculation is determined by the most profitable feasible commitment and dispatch quantities given market clearing prices. The second term is based on the cleared quantities and market clearing prices. It should be obvious that this LOC payment is nonnegative.

**Remark 4.** In order to calculate LOC payments for real-time electricity markets, it must be assumed that ISO cleared quantities are known for the entire time horizon. For multi-interval time horizons, this is unrealistic because system condition forecasts are never perfect and the ISO may change its anticipated cleared quantities as time proceeds.

Unlike LOC payments which are made in certain situations under ISO New England's current pricing scheme, it is impossible to incur the latter **excess product payments** with today's pricing methodology (see discussion after (8) in Section III). The necessity of excess product payments is intimately related to another key observation about Convex Hull Pricing.

**Property #2a**

Nonbinding shared constraints based on cleared quantities (e.g., reserves, transmission) may receive positive prices from Convex Hull Pricing.

Because of the novelty of Property #2a, some additional discussion is in order. As an example, this property can manifest itself by the ISO clearing excess reserves *and* generating a positive reserve price from Convex Hull Pricing. An initial reaction to this positive reserve price is that it violates the marginal cost pricing concept - a marginal change in the reserve requirement will not change the optimal objective function cost of the Dispatch problem. However, it must be remembered that the Convex Hull Pricing problem makes different assumptions than the Dispatch problem. The positive reserve price is indeed based on the marginal cost pricing concept for the Pricing problem; given that different assumptions are made in the Dispatch and Pricing problems, there is no reason to expect that the solutions of these problems will agree.

**Property #2b**

In Convex Hull Pricing, an excess product payment ensures that the total load payment for a market product covers the total generator payment for that product.

To understand Property #2b, it is necessary to examine the payments required for each market product. Consider a reserve constraint generically expressed as  $a^T x \geq b$ . From the Dispatch problem, the ISO finds clearing quantities that satisfy  $a^T x^{\text{DDP}} \geq b$ . From the Convex Hull Pricing problem, a nonnegative reserve price  $\lambda^*$  may be produced. The total payment required by generators is  $\lambda^* a^T x^{\text{DDP}}$  while the total payment made by load is  $\lambda^* b$  if load is only obligated to pay up to the demanded reserve quantity. Because

$$\lambda^* a^T x^{\text{DDP}} \geq \lambda^* b,$$

the payment required by generators may exceed the payment made by load (a result of the Dispatch and Pricing problems using different assumptions). The excess product payment serves to eliminate this payment discrepancy:

When a shared constraint (a) is not binding in the Dispatch solution, and (b) has a positive optimal shadow price in the ConvHP solution, the excess product payment for the constraint is

$$\lambda^* a^T x^{\text{DDP}} - \lambda^* b > 0.$$

This nontraditional side-payment is commonly ignored when Property #1 of ConvHP is described, but it is important for ISO revenue neutrality. It also represents an additional side-payment that must be estimated by loads and factored into their hedging strategies.

May 1, 2015

The reader may mistakenly assume here that excess product payments could be completely avoided by modeling every shared requirement as an equality constraint (instead of an inequality constraint). While this suggestion may sound reasonable, it is false.

1. Not all shared inequality constraints can be validly formulated as equalities. Inequality constraints may be necessary because of nonconvex bids (e.g., block-bidding of regulation), physical properties (e.g., transmission), and product substitution behavior (e.g., 10-minute reserves can be counted towards meeting the 30-minute reserve requirement). Indeed, the excess product payment associated with transmission constraint inequalities was called **FTR uplift** in (Gribik, Hogan, & Pope, 2007).
2. If a shared requirement can validly be modeled as an equality constraint, the ISO will only clear capabilities up to the requirement. Unfortunately, the excess product payment associated with that shared constraint does not simply disappear! Because convex hull prices do not depend on the shared constraint  $= / \geq$  relationship (as long as both formulations are valid), resources may have undesignated product capability that faces a positive price. If the resource's cost of providing that undesignated product is less than convex hull price, the unrealized profit would need to be paid as an LOC payment to provide an adequate quantity incentive. The excess product payment does not disappear – it gets shifted into the LOC payments!

In conclusion, changing the model formulation will not affect the total side-payment amount even though the removal of certain excess product payment terms may be possible via constraint reformulations. This property will be shown after the next example.

**Example 4.** Consider a single-interval, single-node problem with energy and reserve requirements of 75MW and 20MW, respectively. Consider the following generators.

	Generator 1	Generator 2
Economic minimum (MW)	30	20
Economic maximum (MW)	80	20
Bid (\$/MWh)	30	-
No-load cost (\$/hr)	-	2000
Maximum online reserves (MW)	25	0

Assume that Generator 1 must remain online and Generator 2 is an offline but available FS unit. The Commitment and Convex Hull Pricing problems are

Commitment

$$\begin{aligned}
 &\min_{c, x, r} \quad c_1 + c_2 \\
 \text{s.t.} \quad &x_1 + x_2 = 75 \\
 &r_1 \geq 20 \\
 &c_1 = 30x_1 \\
 &30 \leq x_1 \leq 80 - r_1 \\
 &0 \leq r_1 \leq 25 \\
 &\left( \begin{matrix} c_2 = 0 \\ x_2 = 0 \end{matrix} \right) \vee \left( \begin{matrix} c_2 = 2000 \\ x_2 = 20 \end{matrix} \right),
 \end{aligned}$$

Convex Hull Pricing

$$\begin{aligned}
 &\min_{c, x, r, \xi_0} \quad c_1 + c_2 \\
 \text{s.t.} \quad &x_1 + x_2 = 75 \quad (\lambda) \\
 &r_1 \geq 20 \quad (\mu) \\
 &c_1 = 30x_1 \\
 &30 \leq x_1 \leq 80 - r_1 \\
 &0 \leq r_1 \leq 25 \\
 &c_2 = 2000\xi_0 \\
 &x_2 = 20\xi_0 \\
 &0 \leq \xi_0 \leq 1.
 \end{aligned}$$

Generator 2 is committed in the Commitment solution. If the system forecast is perfect and the ISO designates the maximum possible reserve quantities, the ISO dispatch instructions are  $(x_1^*, r_1^*, x_2^*) = (55, 25, 20)$  (alternate optimal solutions exist if maximum reserve designations are not assumed). The optimal production quantities and prices for the ConvHP problem are  $(x_1^*, r_1^*, x_2^*, \lambda^*, \mu^*) = (60, 20, 15, 100, 70)$ .

Given the \$100/MWh LMP and \$70/MWh reserve market clearing price (RMCP), neither unit requires an LOC payment (Generator 1 is indifferent between energy and reserves; Generator 2 is setting the energy price). However, there are an infinite number of (LMP, RMCP) pairs with total LOC payments of \$0: namely, any  $LMP \geq 100$  and  $RMCP = LMP - 30$ . The convex hull prices minimize total side-payments:

The optimal cleared quantities provide 25MW of reserves but only 20MW of reserves are required. If load pays for its requirement, it would pay  $20\text{MWh} \times \$70/\text{MWh} = \$1400$ . Generators, on the other hand, need to receive \$1750 ( $= 25 \times 70$ ) based on the RMCP and their designated quantities. This \$350 difference is the excess product payment required to cover the reserve revenue shortfall. The minimum total side-payment is \$350 with the convex hull prices.

In Example 4, it can easily be seen that the \$350 excess product payment will become an LOC payment if only 20MW of reserves was cleared by the ISO. First, note that Generator 1 would receive a 20MW reserve designation and would have 5MW of undesignated capacity that could be used as reserves. The convex hull prices are unaffected by changing the reserve constraint from an inequality to an equality, so the RMCP remains \$70/MWh. Generator 1 faces a LOC of \$350 in this situation because it sees the \$70/MWh reserve price and would prefer to have all of its 25MW of reserve capability cleared.

Another interesting property of Convex Hull Pricing is the ability of offline resources to set prices.<sup>14</sup> Potomac Economics, MISO's Independent Market Monitor, suggested that MISO delay its implementation of a new pricing method after observing this property during parallel operations (Patton, 2014). The primary concerns raised by Potomac Economics were:

- Offline resources utilized in the pricing solution appeared to be either not truly feasible or not truly economic;
- System marginal prices were significantly affected by offline resource utilization in the pricing solution;
- Offline resource utilization in the pricing solution may be caused by inter-market coordination constraints that do not necessarily need to be satisfied.

To address these concerns, Potomac Economics suggested that the pricing algorithm be changed to prevent offline resource price-setting, amortize offline unit commitment costs over 5 minutes (i.e., one time interval), impose shift factor cutoff values for offline resources considered in the pricing run, remove certain inter-market coordination constraints, and eliminate offline pump storage units from pricing consideration.

<p style="text-align: center;"><b><u>Property #3</u></b></p> <p style="text-align: center;">Offline units can set market prices in Convex Hull Pricing.</p>
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The following example illustrates Property #3.

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<sup>14</sup> "Offline" here refers to resources that are ISO-scheduled to be offline.

**Example 5.** Consider a single-interval, single-node problem with an energy requirement of 255MW. The generators have the following properties.

	Generator 1	Generator 2	Generator 3
EcoMin (MW)	50	100	50
EcoMax (MW)	200	150	100
Bid block 1 (MW range, \$/MWh)	0-200, 50	0-100, 60	0-50, 200
Bid block 2 (MW range, \$/MWh)	-	100-150, 65	50-100, 250
No-load cost (\$/hr)	-	30000	5000

Assume that Generator 1 must remain online while Generators 2 and 3 are offline and available FS units. The Commitment and Convex Hull Pricing problems are

Commitment

$$\begin{aligned} \min_{c, x} \quad & c_1 + c_2 + c_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 255 \\ & c_1 = 50x_1 \\ & 50 \leq x_1 \leq 200 \end{aligned}$$

$$\left( \begin{array}{l} c_2 = 0 \\ x_2 = 0 \end{array} \right) \vee \left( \begin{array}{l} 60x_2 + 30000 \leq c_2 \\ 65x_2 + 29500 \leq c_2 \leq 39250 \\ 100 \leq x_2 \leq 150 \end{array} \right)$$

$$\left( \begin{array}{l} c_3 = 0 \\ x_3 = 0 \end{array} \right) \vee \left( \begin{array}{l} 200x_3 + 5000 \leq c_3 \\ 250x_3 + 2500 \leq c_3 \leq 27500 \\ 50 \leq x_3 \leq 100 \end{array} \right),$$

Convex Hull Pricing

$$\begin{aligned} \min_{c, x, \xi} \quad & c_1 + c_2 + c_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 255 \quad (\lambda) \\ & c_1 = 50x_1 \\ & 50 \leq x_1 \leq 200 \\ & 60x_2 + 30000\xi_{0,2} \leq c_2 \\ & 65x_2 + 29500\xi_{0,2} \leq c_2 \leq 39250\xi_{0,2} \\ & 100\xi_{0,2} \leq x_2 \leq 150\xi_{0,2} \\ & 200x_3 + 5000\xi_{0,3} \leq c_3 \\ & 250x_3 + 2500\xi_{0,3} \leq c_3 \leq 27500\xi_{0,3} \\ & 50\xi_{0,3} \leq x_3 \leq 100\xi_{0,3} \\ & 0 \leq \xi_{0,2}, \xi_{0,3} \leq 1. \end{aligned}$$

The least cost solution of the Commitment problem commits Generator 3. This commitment is less expensive than committing Generator 2 because the latter unit has a higher commitment cost and EcoMin. If the system forecast is perfect, the Dispatch problem solution will be identical to the optimal Commitment problem quantities  $(x_1^*, x_2^*, x_3^*) = (200, 0, 55)$ . The optimal production quantities and prices from the ConvHP problem are  $(x_1^*, x_2^*, x_3^*, \lambda^*) = (200, 55, 0, 261.67)$ .

In this problem, the average cost of the offline Generator 2 sets the LMP:

$$\frac{\text{Gen. 2 cost at EcoMax}}{\text{Gen. 2 EcoMax}} = \frac{30000 + 60 \times 100 + 65 \times 50}{150} = \$261.67 / \text{MWh}.$$

Another important property of Convex Hull Pricing relates to its time horizon. First, Convex Hull Pricing is based on the Commitment problem. Therefore, it must be solved over the same time horizon as the Commitment problem: for every ISO, a multi-hour time horizon would be required. Given that such



look-ahead solutions are based on forecasts that change over time, fixing future prices based on these forecasted conditions may be difficult to justify.

**Property #4**

Convex Hull Pricing must have the same time horizon as the Commitment problem. Therefore, it is inherently a multi-interval pricing scheme for electricity markets.

One important consequence of Property #4 is that total side-payments are only minimized over the time horizon under consideration. Therefore, a multi-interval ConvHP implementation only minimizes multi-interval side-payments; no claims about side-payment minimization over shorter time horizons can be made.

**Corollary 1 of Property #4**

Convex Hull Pricing only minimizes total side-payments over its specified time horizon.

An additional consequence of Property #4 deals with incentive compatibility. Namely, the LOC being minimized by ConvHP is explicitly the maximum possible profit over the time horizon less the as-bid cleared quantity profit over the time horizon. Therefore, incentive compatibility does not necessarily hold for each time interval independently. This property may become important in situations where participants do not trust the ISO system forecast and have a different expectation about future prices.

**Corollary 2 of Property #4**

Convex Hull Pricing satisfies incentive compatibility for the specified time horizon. It does not satisfy incentive compatibility in each individual time interval.

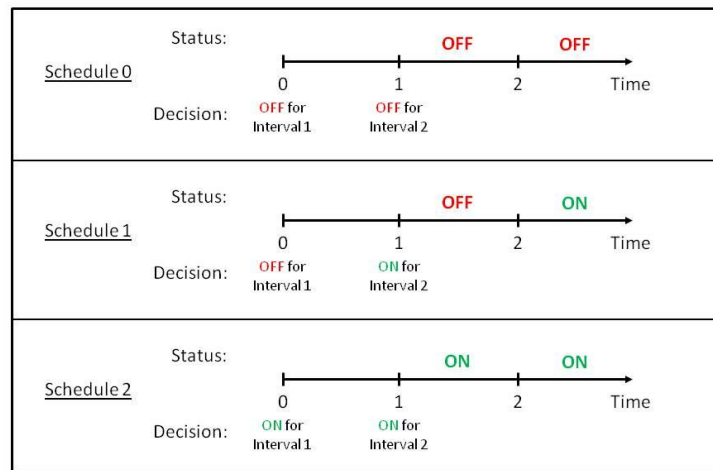
Property 4 and its corollaries should be obvious from the relationship between the Commitment problem (2) and the Convex Hull Pricing problem (4). Most of the examples in this paper are single-interval purely for the sake of simplicity; Convex Hull Pricing cannot theoretically be implemented in electricity markets via a single-interval framework because Commitment problems are multi-interval.

The following example demonstrates Property #4 and its corollaries.

**Example 6.** Consider a two-interval, single-node problem with energy requirements of 80MW (Interval 1) and 110MW (Interval 2). The generators have the following properties.

	Generator 1	Generator 2
EcoMin (MW)	0	25
EcoMax (MW)	75	55
Bid block 1 (MW range, \$/MWh)	0-75, 50	0-25, 100
Bid block 2 (MW range, \$/MWh)	-	25-55, 200
Notification & start-up time (min)	-	15
Minimum run time (min)	-	30

Assume that each time interval is 15 minutes long and that each commitment decision is made 15 minutes in advance. Because of Generator 2’s properties, its feasible commitment schedules are presented below.



The obvious optimal solution for the two-period Commitment problem is to commit Generator 2 for both time intervals → Schedule 2. Given a perfect forecast, the optimal dispatch solution is  $(x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*) = (55, 75, 25, 35)$  where  $x_{it}$  is the production level of Generator  $i$  for Interval  $t$ .

The convex hull prices are \$50/MWh (Interval 1) and \$241.67/MWh (Interval 2) – see Appendix B for the problem formulation. Given these prices and generator operational parameters, maximum profits over different time horizons (Interval 1 only, Interval 2 only, or both Intervals) are provided in the following table.

MAXIMUM PROFIT (\$)	Interval 1 only	Interval 2 only	Both intervals
Generator 1	0	14,375.25	14,375.25
Generator 2, Schedule 0	0	0	0
Generator 2, Schedule 1	0	4,791.85	4,791.85
Generator 2, Schedule 2	-1,250	4,791.85	3,541.85

Generator 2 has a dispatch-following profit of \$2,708.45. Because this is less than its maximum possible “Both intervals” profit, an LOC of \$2,083.40 is required. This LOC incentivizes two-interval production as a whole but not Interval 1 production alone.

May 1, 2015

Finally, Convex Hull Pricing is a mathematical pricing scheme, not a traditional economic principle. This novelty introduces concerns about its overall effects on markets because, unlike marginal cost pricing that has been rigorously studied for convex markets and has been historically applied to nonconvex markets such as electricity, little economic intuition exists for Convex Hull Pricing. Its effects on other market elements are also unknown and long-term investment incentives have not been adequately explored.

**Property #5**

Convex Hull Pricing is not a well-known economic concept.

V. Issues with Convex Hull pricing

There are several foreseeable issues with Convex Hull Pricing. ISO New England finds these issues problematic and is averse to the introduction of the Convex Hull Pricing at this time.

**Issue 1. Convex Hull Pricing does not minimize “uplift” as commonly defined**

The main attraction of Convex Hull Pricing is usually its “uplift minimization” property. Poorly informed advocates of the method claim that this means that make-whole payments will decrease under Convex Hull Pricing. This statement is demonstrably false as shown in the following example.

**Example 7.** Consider a single-interval, single-node problem. Generator 1 must remain online and Generator 2 is an offline but available FS unit that can be block-loaded.

<b>Generator 1</b> 10 – 50MW \$50/MWh	<b>Generator 2</b> 50MW (block) \$10/MWh
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The Commitment problem is

$$\begin{aligned}
 \min_{c,x} \quad & c_1 + c_2 \\
 \text{s.t.} \quad & x_1 + x_2 = 35 \\
 & c_1 = 50x_1 \\
 & 10 \leq x_1 \leq 50 \\
 & \left( \begin{matrix} c_2 = 0 \\ x_2 = 0 \end{matrix} \right) \vee \left( \begin{matrix} c_2 = 500 \\ x_2 = 50 \end{matrix} \right).
 \end{aligned}$$

The trivial solution of this problem is to produce only from Generator 1 (Generator 2 remains offline).

Given the optimal commitment decision, the optimal dispatch solution is 35MW from Generator 1 and 0MW from Generator 2. The current ISO New England pricing method produces an LMP of \$50/MWh. The Convex Hull Pricing problem is

$$\begin{aligned}
 \min_{c,x} \quad & c_1 + c_2 \\
 \text{s.t.} \quad & x_1 + x_2 = 35 \quad (\lambda) \\
 & c_1 = 50x_1 \\
 & 10 \leq x_1 \leq 50 \\
 & c_2 = 500\xi_0 \\
 & x_2 = 50\xi_0 \\
 & 0 \leq \xi_0 \leq 1.
 \end{aligned}$$

The ConvHP LMP is \$10/MWh. Under the current \$50/MWh price, neither unit needs a make-whole payment even though Generator 2 faces an LOC of \$2000 (could have made \$40/MWh on its 50MW capability). However, the convex hull price gives Generator 1 a make-whole payment of \$1400 (loss of \$40/MWh for its 35MW dispatch). Generator 2 does not require a make-whole payment. Thus, the make-whole payment increased with Convex Hull Pricing even though the total side-payment was reduced.

As was observed in Example 7, Convex Hull Pricing can increase the total make-whole payment. This can occur because, as described by Properties #1 – 2, Convex Hull Pricing minimizes a *very specific* side-payment summation. Since make-whole payments are not present in this side-payment calculation, they will obviously not be minimized. Furthermore, the minimum uplift justification for Convex Hull Pricing is in considerable trouble if the ISO decides to “make resources whole.”

**Issue 2. Convex Hull Pricing may produce positive reserve prices when the system has surplus reserve capability**

One issue that is often glossed over when discussing Convex Hull Pricing is its ability to set positive prices for shared constraints for which a marginal requirement change can be satisfied for free. This property was demonstrated in Example 4 without a detailed analysis.

**Example 4 (again).** Consider a single-interval, single-node problem with energy and reserve requirements of 75MW and 20MW, respectively. Consider the following generators.

	Generator 1	Generator 2
EcoMin (MW)	30	20
EcoMax (MW)	80	20
Bid (\$/MWh)	30	-
No-load cost (\$/hr)	-	2000
Maximum online reserves (MW)	25	0

Assume that Generator 1 must remain online and Generator 2 is an offline but available FS unit.

The optimal solution (assuming maximum reserve designations) is for Generator 1 to supply 55MW of energy and 25MW of reserves while Generator 2 provides 20MW energy. From Convex Hull Pricing, the LMP is \$100/MWh and the RMCP is \$70/MWh.

Because the reserve requirement is only 20MW, the ISO cleared quantities result in a 5MW reserve excess. However, the RMCP produced by Convex Hull Pricing is \$70/MWh. Thus, Convex Hull Pricing has produced a positive reserve price when there are excess dispatched reserves.

If the ISO had instead only designated 20MW of reserves on Generator 1, the reserve price according to ConvHP would still be \$70/MWh. This result still corresponds to a positive reserve price when a marginal unit of reserves can be obtained for free.

The pricing outcome from Example 4 does not agree with the [as-cleared] understanding of marginal cost pricing concept. If a marginal cost test is performed on the reserve product, a 1MW increase in the reserve requirement will not change the optimal dispatch solution and thus results in a \$0 objective function increase. Despite this test, Convex Hull Pricing produces a nonzero reserve price purely based on total side-payment minimization. In addition to being counterintuitive, this price creates a side-payment corresponding to either an excess product payment (if Generator 1 is assigned 25MW of reserves) or an LOC payment (if Generator 1 is assigned 20MW of reserves).

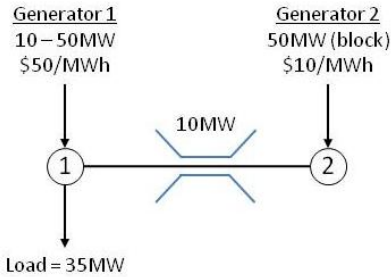
**Remark 5.** Convex Hull Pricing does **NOT** obey **as-cleared** marginal cost pricing.

The resource-specific convex hulls of (4) can provide some insight into this pricing outcome. In essence, every unit commitment sequence that is feasible for a resource specifies a well-defined evolution of the resource's feasible region over time. Abstractly, each of these evolutions can be thought of as a point in set  $S$  of Figure 1. Convex hull of a set. In (4), the feasible region for each resource is the convex hull of these points ( $\text{conv}(S)$  in Figure 1. Convex hull of a set), which allows for feasible region evolutions defined by a convex combination of the actual feasible evolutions. Therefore, the Pricing problem can simplistically be thought of as allowing a FS unit to be dispatched based on a "partial commitment." It follows the optimal Pricing problem solution can "partially commit" the same FS unit to satisfy the reserve requirement at minimum cost. As a result, the reserve price may be positive even though the system has surplus reserve capability.

**Issue 3. Convex Hull Pricing may produce positive congestion prices for transmission lines that are not congested as dispatched**

The phenomenon referenced in Issue 2 extends to transmission constraints. Namely, a transmission line that is not congested according to the dispatch solution may be assigned a positive congestion price in Convex Hull Pricing. Additional side-payments must be collected by the ISO when this situation occurs to cover potential revenue shortfalls for financial transmission rights (FTR) contracts. This revenue shortfall was noted in (Gribik, Hogan, & Pope, 2007).

**Example 8.** Consider a two-node, single-interval problem. Generator 1 must remain online and Generator 2 is an offline but available FS unit that can be block-loaded.



The Commitment problem is

$$\begin{aligned} \min_{c,x} \quad & c_1 + c_2 \\ \text{s.t.} \quad & x_1 + x_2 = 35 \\ & -10 \leq x_2 \leq 10 \\ & c_1 = 50x_1 \\ & 10 \leq x_1 \leq 50 \\ & \left( \begin{matrix} c_2 = 0 \\ x_2 = 0 \end{matrix} \right) \vee \left( \begin{matrix} c_2 = 500 \\ x_2 = 50 \end{matrix} \right). \end{aligned}$$

The second constraint models the transmission limit. Committing Generator 2 is infeasible.

Given the optimal commitment decision, the optimal dispatch solution is 35MW from Generator 1 and 0MW from Generator 2. The Convex Hull Pricing problem is

$$\begin{aligned} \min_{c,x} \quad & c_1 + c_2 \\ \text{s.t.} \quad & x_1 + x_2 = 35 \quad (\lambda) \\ & -10 \leq x_2 \leq 10 \\ & c_1 = 50x_1 \\ & 10 \leq x_1 \leq 50 \\ & c_2 = 500\xi_0 \\ & x_2 = 50\xi_0 \\ & 0 \leq \xi_0 \leq 1. \end{aligned}$$

The LMP is \$50/MWh at node 1 and \$10/MWh at node 2. The congestion price for the transmission line is \$40/MWh. Since there is no flow along the transmission line according to the dispatch solution, Convex Hull Pricing has produced a positive congestion price for an uncongested line.

This congestion pricing property is counterintuitive and requires additional wealth transfers to cover FTR revenue shortfalls. When market participants observe the Example 8 congestion pattern, they will likely bid into the FTR market to obtain an FTR from node 2 to node 1 for 10MW (the transmission capacity). The FTR holder will then require an FTR payment of \$400 (\$40/MWh congestion price for the 10MW contract). However, the settlement in the energy market is balanced, contributing \$0 to the FTR revenue fund. Thus, either a shortfall of \$400 must be collected from market participants or the FTR payments must be prorated to \$0. Either way, participants are likely to dispute the outcome.

#### **Issue 4. Convex Hull Prices cannot be easily understood/explained**

Whenever a market change is proposed, participants should be aware that their current bidding strategies may not be optimal under the new framework. The necessity of bid adjustments is not purely hypothetical for Convex Hull Pricing because of the nontraditional properties detailed in Section IV. Unfortunately, modifying intuition to reflect ConvHP behavior can be exceedingly difficult because it does not admit a simple non-mathematical explanation for realistic problems. Some general reasons for this difficulty are as follows.

- i. LMPs from ConvHP cannot be verified by a simple examination of the dispatch solution and bid-in costs.<sup>15</sup> Therefore, it would be difficult for system operators to verify that the pricing solution is logical. Along the same line of reasoning, this property would make it more difficult for regulators to identify when a participant is exercising market power. From a participant perspective, it would become much less obvious how “close-to-marginal” a resource was; this may reduce the incentive for competitive bidding.
- ii. Reserve and congestion prices from ConvHP can be counterintuitive (see Issues 2 and 3). Furthermore, it is not clear whether these pricing outcomes would be predictable. For load, excess product payments would represent a new class of side-payments that must be estimated and factored into hedging strategies.
- iii. Convex Hull Pricing implicitly incorporates start-up and no-load costs in market clearing prices. Unfortunately, the method of incorporation is unclear and unpredictable: it does not follow a well-defined pattern such as fixed cost amortization over EcoMin and minimum run time.

#### **Issue 5. The “minimum uplift” goal of Convex Hull Pricing is not widely accepted**

Marginal cost pricing has been the subject of in-depth economic analysis. Most importantly, this pricing scheme is the equilibrium outcome of a decentralized convex market with perfect competition. As such, ISOs (centralized markets) have a defensible reason to utilize a marginal cost pricing scheme: replication of the idealized decentralized market outcome subject to additional reliability considerations.

Convex Hull Pricing, on the other hand, does not have a widely accepted economic justification. Although reducing uplift is commonly viewed as desirable because of increased transparency, it is debatable whether this should be the primary goal of a pricing method. If uplift minimization is indeed the most important property of a pricing method, an ISO would be well-served by ConvHP. If other properties (such as truthful bidding, incentive compatibility, consumer payments, and side-payment cost allocations) are important, the suitability of ConvHP is more questionable.

The short-term and long-term investment incentives induced by Convex Hull Pricing are also not understood, and the effect of Convex Hull Pricing on related markets is unknown. For instance, Convex Hull Pricing for electricity and ancillary services will likely affect clearings in the Forward Capacity

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<sup>15</sup> This difficulty in LMP verification applies to any pricing scheme involving a relaxation of bid-in parameters. It is not unique to Convex Hull Pricing.



Market and Forward Reserve Market to an unknown degree in ISO New England. This lack of economic understanding extends to future market design changes as well.

**Issue 6. Convex Hull Pricing does not agree with “dispatch-based pricing”**

The concept of “dispatch-based pricing” has been proposed in (Hogan, 2014; Ring, 1995) and put forth in several other venues. Although not mathematically rigorous, this principle advocates for prices that are based on the actual ISO system dispatch rather than on some unrealized dispatch. In Convex Hull Pricing, resources that are offline are allowed to set the market clearing price (see Example 5). Additionally, it is possible that an expensive online FS unit may not set the market clearing price even if it was ISO-committed and dispatched. These properties seem to conflict with the premise of dispatch-based pricing unless the concept of “actual dispatch” is expanded to include the 0MW “dispatch” of offline units. An expanded definition of “actual dispatch” would obviously allow many more pricing schemes to be called “dispatch-based pricing.”

**Issue 7. Implementation of Convex Hull Pricing would be computationally challenging**

In addition to the Convex Hull Pricing issues discussed above, there are significant computational difficulties for the method regardless of formulation.

- Problem (4) of Section III.2: Solving this Convex Hull Pricing problem requires an explicit formulation of the convex hull for each resource’s feasible region. Given that the time horizon of Convex Hull Pricing is identical to the time horizon of the Commitment problem, the explicit convex hull formulation (even in the linear case of Theorem 2) is unlikely to be computationally tractable.
- Lagrangian dual approach from (Gribik, Hogan, & Pope, 2007): Identifying a globally optimal solution for a Lagrangian dual problem is difficult. Therefore, it is questionable whether an optimal solution can be reliably found. Furthermore, experiments performed by ISO New England suggest that the dual objective function value is not very sensitive to price changes around the optimal solution. Therefore, even if an optimal pricing run quantity can be found, a wide range of prices would likely satisfy the specified convergence tolerance. Picking a price from the available options would be subjective.

A more nuanced computational difficulty related to Property #4 deals with the interactions between consecutive Convex Hull Pricing time horizons. In real-time electricity markets, dispatch and pricing must be performed with rolling time horizons. Unfortunately, each ConvHP problem necessarily uses a fixed time horizon. Without taking end-of-horizon conditions into consideration, there is no guarantee that the sum of side-payments over several independent time horizons will be minimized. It can also be concluded that there is no guarantee of realizing the initially minimized side-payment because projected optimal dispatch decisions fall within different rolling time horizons as time proceeds: even if every time horizon forecast was perfect, optimal dispatch decisions may change as future system conditions are revealed and included in the time horizon. The following figure illustrates the latter concept.

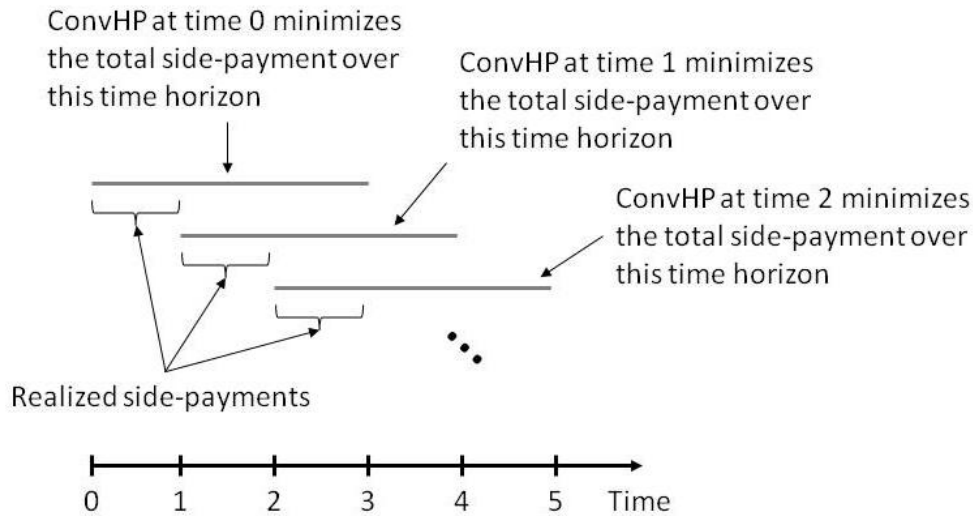


Figure 2. Realized side-payments with a rolling time horizon

**Remark 6.** The link between commitment states in consecutive time horizons is well-known. A difficulty of Convex Hull Pricing is that it introduces the same issue to the Pricing problem. Some suggestions have been made to address this difficulty, but it is unlikely that a perfect solution can be found.

**Issue 8. Convex Hull Pricing is an all-or-nothing pricing scheme**

Each current ISO pricing method is flexible in terms of mathematical formulation and resource modeling. If the ISO wants to change how market clearing prices are calculated, the Pricing problem can be modified without much difficulty because no rigorous theoretical property needs to be satisfied.<sup>16</sup> Convex Hull Pricing is much different in this respect. The fundamental side-payment minimization property of ConvHP will not be maintained if modifications to the method are made (see proof in Section III.2). Thus, no modification of the rigorous Convex Hull Pricing method will be “Convex Hull Pricing.”<sup>17</sup>

Convex Hull Pricing is all-or-nothing: either an ISO implements Convex Hull Pricing in its entirety and realizes all of its properties, or the ISO modifies Convex Hull Pricing and gets a completely different pricing scheme that (a) does not necessarily minimize total side-payments, and (b) does not necessarily have any of the other Convex Hull Pricing properties of Section IV. There is **NO** middle ground.

<sup>16</sup> Many ISOs claim to employ marginal cost pricing, but this concept is ambiguous for electricity markets. For example, several ISOs employ pricing methods that allow block-loaded units to set price. It is debatable whether this truly reflects “marginal cost pricing” because a block-loaded unit cannot possibly be “marginal” in the traditional sense.

<sup>17</sup> MISO has claimed that its ELMP method is an “approximation” of Convex Hull Pricing, but this claim relies on there being a rigorous definition of “approximate Convex Hull Pricing.” In the absence of such a rigorous definition, there is no basis for claiming that something is an “approximation” of Convex Hull Pricing.

## VI. Conclusion

This paper has formulated and discussed Convex Hull Pricing. After providing a brief overview of the wholesale electricity market, Section II motivated the Convex Hull Pricing method – Convex Hull Pricing minimizes the total side-payments needed to ensure that each resource has an adequate incentive to follow its quantity instruction. The graphical pricing method from (Gribik, Hogan, & Pope, 2007) was presented in Section III followed by the new mathematical formulation (4). Several important properties and implementation-related issues were presented and illustrated with examples in Sections IV and V. These properties include:

1. Convex Hull Pricing identifies uniform energy and ancillary service prices that minimize a very specific set of side-payments over a specified time horizon.
2. The total side-payment minimized by Convex Hull Pricing consists of two parts:
  - LOC payments and excess product payments.
    - a. LOC payments ensure that each online resource receives its maximum possible profit given prices and its bid-in constraints.
    - b. Excess product payments ensure that the total load payment for each market product covers the total generator payment for that product. This payment arises because Convex Hull Pricing may result in positive prices for products that are not short given the market clearing.
    - c. Make-whole payments are NOT considered by Convex Hull Pricing.
3. Offline resources can set market prices in Convex Hull Pricing.
4. Convex Hull Pricing must have the same time horizon as the Commitment problem. Therefore, Convex Hull Pricing is inherently a multi-interval pricing method for electricity markets.
5. Convex Hull Pricing is not an established economic concept and deviates from the traditional understanding of marginal cost pricing.

Overall, Convex Hull Pricing is an interesting mathematical pricing concept. However, ISO New England believes that it is not fully understood by participants or market designers who have been told that it identifies “perfect prices”; indeed, the appropriateness of its prices and side-payments is open to debate. Furthermore, Convex Hull Pricing would not have predictable outcomes if implemented. Market participants should also pause because their accumulated intuition on ISO pricing would become useless due to the vast differences between ConvHP and current pricing methods. From a computational perspective, Convex Hull Pricing for electricity markets would be problematic because of its inherent multi-interval property and the extremely large size of the Pricing problem. Lastly, Convex Hull Pricing requires a full implementation of (4); deviations from this [or an equivalent] formulation invalidate the herein proven properties of ConvHP.

For these reasons and many others, ISO New England believes that Convex Hull Pricing should be studied more rigorously to gain a better understanding of its short- and long-term consequences. It would be premature to suggest that Convex Hull Pricing is in any way preferred over common pricing methods at this time. Simpler pricing schemes may be more practical and transparent while achieving similar benefits.

## Appendix A. Analysis of the KKT conditions

Let the linear Pricing problem be

$$\begin{aligned}
 \min_{c, x} \quad & \sum_t \sum_i c_{it} \\
 \text{s.t.} \quad & \sum_i A_{it} x_{it} \geq \mathbf{b}_t \quad (\lambda_t) \quad \forall t \\
 & B_{it} c_{it} + C_{it} x_{it} \geq \mathbf{d}_{it} \quad (\mu_{it}) \quad \forall i, t.
 \end{aligned} \tag{9}$$

The necessary and sufficient KKT conditions of this problem are

$$\begin{aligned}
 1 - B_{it}^T \mu_{it} &= 0 \quad \forall i, t \\
 -A_{it}^T \lambda_t - C_{it}^T \mu_{it} &= 0 \quad \forall i, t \\
 \sum_i A_{it} x_{it} &\geq \mathbf{b}_t \quad \forall t \\
 B_{it} c_{it} + C_{it} x_{it} &\geq \mathbf{d}_{it} \quad \forall i, t \\
 \lambda_t &\geq 0 \quad \forall t \\
 \mu_{it} &\geq 0 \quad \forall i, t \\
 \lambda_t^T \left( \sum_i A_{it} x_{it} - \mathbf{b}_t \right) &= 0 \quad \forall t \\
 \mu_{it}^T (B_{it} c_{it} + C_{it} x_{it} - \mathbf{d}_{it}) &= 0 \quad \forall i, t.
 \end{aligned} \tag{10}$$

Let a feasible tuple for (10) be  $(c^*, x^*, \lambda^*, \mu^*)$ . By the sufficiency of the KKT conditions for linear problems, this feasible tuple is an optimal solution of (9).

Assume that the ISO sets market clearing prices according to the optimal shadow price vector  $\lambda^*$ . The optimization problem faced by profit-maximizing resource  $i$  depends on its production costs  $c_{it}$ , its market product outputs  $A_{it} x_{it}$ , its private constraints, and the exogenous market prices  $\lambda^*$ . Therefore, resource  $i$  solves

$$\begin{aligned}
 \max_{c_i, x_i} \quad & \sum_t \left[ (\lambda_t^*)^T A_{it} x_{it} - c_{it} \right] \\
 \text{s.t.} \quad & B_{it} c_{it} + C_{it} x_{it} \geq \mathbf{d}_{it} \quad (\hat{\mu}_{it}) \quad \forall t.
 \end{aligned} \tag{11}$$

The KKT conditions of this linear problem are

$$\begin{aligned}
 1 - \mathbf{B}_{it}^T \hat{\boldsymbol{\mu}}_{it} &= 0 & \forall t \\
 -\mathbf{A}_{it}^T \boldsymbol{\lambda}_t^* - \mathbf{C}_{it}^T \hat{\boldsymbol{\mu}}_{it} &= 0 & \forall t \\
 \mathbf{B}_{it} \mathbf{c}_{it} + \mathbf{C}_{it} x_{it} &\geq \mathbf{d}_{it} & \forall t \\
 \hat{\boldsymbol{\mu}}_{it} &\geq 0 & \forall t \\
 \hat{\boldsymbol{\mu}}_{it}^T (\mathbf{B}_{it} \mathbf{c}_{it} + \mathbf{C}_{it} x_{it} - \mathbf{d}_{it}) &= 0 & \forall t.
 \end{aligned}$$

Because

$$\begin{aligned}
 1 - \mathbf{B}_{it}^T \boldsymbol{\mu}_{it}^* &= 0 & \forall t \\
 -\mathbf{A}_{it}^T \boldsymbol{\lambda}_t^* - \mathbf{C}_{it}^T \boldsymbol{\mu}_{it}^* &= 0 & \forall t \\
 \mathbf{B}_{it} \mathbf{c}_{it}^* + \mathbf{C}_{it} x_{it}^* &\geq \mathbf{d}_{it} & \forall t \\
 \boldsymbol{\mu}_{it}^* &\geq 0 & \forall t \\
 (\boldsymbol{\mu}_{it}^*)^T (\mathbf{B}_{it} \mathbf{c}_{it}^* + \mathbf{C}_{it} x_{it}^* - \mathbf{d}_{it}) &= 0 & \forall t
 \end{aligned}$$

holds by the feasibility of  $(\mathbf{c}^*, x^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$  for (10), it follows that  $(\mathbf{c}_i^*, x_i^*, \boldsymbol{\lambda}_i^*, \boldsymbol{\mu}_i^*)$  is an optimal solution of (11). Therefore, resource  $i$  cannot realize more profit than it does from its pricing run quantities  $x_i^*$  and *market-based* incentive compatibility holds.

## Appendix B. Convex Hull Pricing problem for Example 6

Using the subscript  $it$  to denote the variable value for Generator  $i$  for Interval  $t$ , the two-interval Commitment problem for Example 6 can be expressed as

$$\begin{aligned}
 \min_{c, x} \quad & c_{11} + c_{12} + c_{21} + c_{22} \\
 \text{s.t.} \quad & x_{11} + x_{21} = 80 \\
 & x_{21} + x_{22} = 110 \\
 & c_{11} = 50x_{11} \\
 & c_{12} = 50x_{12} \\
 & 0 \leq x_{11} \leq 75 \\
 & 0 \leq x_{12} \leq 75 \\
 & \left( \begin{array}{l} c_{21} = 0 \\ x_{21} = 0 \\ c_{22} = 0 \\ x_{22} = 0 \end{array} \right) \vee \left( \begin{array}{l} c_{21} = 0 \\ x_{21} = 0 \\ c_{22} = 2500 \\ x_{22} = 25 \end{array} \right) \vee \left( \begin{array}{l} c_{21} = 2500 \\ x_{21} = 25 \\ 100x_{22} \leq c_{22} \\ 200x_{22} - 2500 \leq c_{22} \leq 8500 \\ 25 \leq x_{22} \leq 55 \end{array} \right).
 \end{aligned}$$

To formulate the corresponding Convex Hull Pricing problem, several new variables are introduced to allow the convex hull of Generator 2's feasible region to be written as a set of linear constraints (Theorem 2). Namely, the original variables get defined as

$$\begin{aligned}
 c_{21} &= \xi_{c_{21}}^0 + \xi_{c_{21}}^1 + \xi_{c_{21}}^2 \\
 x_{21} &= \xi_{x_{21}}^0 + \xi_{x_{21}}^1 + \xi_{x_{21}}^2 \\
 c_{22} &= \xi_{c_{22}}^0 + \xi_{c_{22}}^1 + \xi_{c_{22}}^2 \\
 x_{22} &= \xi_{x_{22}}^0 + \xi_{x_{22}}^1 + \xi_{x_{22}}^2
 \end{aligned}$$

where the superscript  $n$  indicates the commitment sequence. Additional constraints are included according to Theorem 2 to restrict each  $\xi$  value as appropriate. Because commitment sequences 1 and 2 don't allow for production in Interval 1, the simplification

$$\begin{aligned}
 c_{21} &= \xi_{c_{21}}^2 \\
 x_{21} &= \xi_{x_{21}}^2
 \end{aligned}$$

can be made. With this substitution, the Convex Hull Pricing problem is

May 1, 2015

$$\begin{aligned}
 \min_{c, x, \xi, \xi_0} \quad & c_{11} + c_{12} + c_{21} + c_{22} \\
 \text{s.t.} \quad & x_{11} + x_{21} = 80 \\
 & x_{21} + x_{22} = 110 \\
 & c_{11} = 50x_{11} \\
 & c_{12} = 50x_{12} \\
 & 0 \leq x_{11} \leq 75 \\
 & 0 \leq x_{12} \leq 75 \\
 & c_{21} = 2500\xi_0^2 \\
 & x_{21} = 25\xi_0^2 \\
 & c_{22} = \xi_{c_{22}}^1 + \xi_{c_{22}}^2 \\
 & x_{22} = \xi_{x_{22}}^1 + \xi_{x_{22}}^2 \\
 & \xi_{c_{22}}^1 = 2500\xi_0^1 \\
 & \xi_{x_{22}}^1 = 25\xi_0^1 \\
 & 100\xi_{x_{22}}^2 \leq \xi_{c_{22}}^2 \\
 & 200\xi_{x_{22}}^2 - 2500\xi_0^2 \leq \xi_{c_{22}}^2 \leq 8500\xi_0^2 \\
 & 25\xi_0^2 \leq \xi_{x_{22}}^2 \leq 55\xi_0^2 \\
 & 0 \leq \xi_0^1 + \xi_0^2 \leq 1 \\
 & \xi_0^1, \xi_0^2 \geq 0.
 \end{aligned}$$

Non-negativity of  $\xi_{c_{22}}^1, \xi_{x_{22}}^1, \xi_{c_{22}}^2, \xi_{x_{22}}^2$  does not have to be explicitly enforced because it is implied by the other constraints.

May 1, 2015

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