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Full- and reduced-order observer design for rectangular descriptor systems with unknown inputs☆

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Abstract

In this paper, methods are proposed to design Luenberger type full- and reduced-order observers for rectangular descriptor systems with unknown inputs. These methods are based on the effect of pre- and post-multiplicative operation of a linear transformation, derived here by means of simple matrix theory. Sufficient conditions for the existence of observers are given and proved. Numerical examples are given to illustrate the effectiveness of the proposed method.

Keywords: Descriptor systems, Unknown inputs, Detectability, Observer design.

1. Introduction

In the last few decades, many researchers have given a lot of attention on the analysis and design of descriptor systems as these are general enough to provide a solid understanding of the inner dynamics of any physical system [1–3]. Many physical systems can be modeled as the system of differential algebraic equations and can be written in the form of descriptor systems, but it is not always necessary that number of variables of interest and equations are same. Thus, we are here concerned with rectangular systems. Some real life applications are electrical circuits [4], chemical control processes [5], constrained mechanics [6], and biological systems [7] to name a few.

An observer is a mathematical realization which uses the input and output information of a given system and its output asymptotically approaches the true state values of the given system. Observer design problem for normal systems has received a great attention in the literature [8–14] and the techniques used for them have been extended successfully to descriptor linear [15–22] and nonlinear [23–26] systems. There are many practical situations where control systems arise with noise or disturbances, so the problem of observer design for systems with unknown inputs is of great importance. Descriptor systems are very sensitive to slight input changes, because differentiation terms of inputs exist in their solution [1]. Thus, observers design problem for descriptor systems with unknown inputs is more important than observer design for normal systems with unknown inputs. Many researchers have worked either on observer design of square descriptor system with unknown inputs or on rectangular descriptor system with unknown inputs only in dynamical part [27–32],

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but results on rectangular descriptor systems with unknown inputs in dynamical as well as in measurement equation are limited [33, 34]. Considering unknown inputs in measurement equation is vary important since, unlike the known inputs, unknown inputs can not be eliminated from the measurement equation without loss of generality. Concepts of generalized Sylvester equation and generalized inverse have been used for the design of observers for descriptor systems with unknown inputs [33]. Koenig [34] presents a method to design proportional multiple-integral observer for descriptor systems for estimating simultaneously the states, faults, and unknown inputs, but the order of observer is greater than the number of states in the given system. Ting et *al.* [35] have designed observer for normal system with unknown inputs by transforming it into descriptor system through a series of linear transformations in the state and output equations.

It has been shown that the condition of detectability is necessary for the existence of any Luenberger type observer for any descriptor system [2]. In this article, our assumptions on the system operators are imitated from the papers [33] and [34] as these conditions are very less restrictive for the design of observers for systems having unknown inputs. As compared to these articles, the proposed method is straightforward and simple to understand and implement. Our approach is based on the restricted system equivalent theory and does not require the concept of generalized Sylvester equation. In this note, one full column rank matrix R is designed in such a way that its pre-multiplication to some matrices gives the design approach for full-order observer and its post-multiplication reveals the reduced-order observer design approach. The order of the proposed reduced-order observer is less than the dynamical order of the given descriptor system.

Rest of the paper is organized as follows. In Section 2, descriptor system with unknown inputs in dynamical and measurement equations is transformed to descriptor system with unknown inputs only in dynamical part by using the known results. Section 3 presents full-order observer design approach. Based on the results of Sections 2 and 3, reduced-order observer is designed in Section 4. To illustrate the derived results, one numerical example is given in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

Consider the following linear time invariant descriptor system with unknown inputs

$$E^*\dot{x} = A^*x + B^*u + F^*v, \tag{1a}$$
$$y^* = C^*x + G^*v, \tag{1b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, $v \in \mathbb{R}^q$, and $y^* \in \mathbb{R}^p$ are the state vector, the control input vector, the unknown input vector, and the output vector, respectively. $E^* \in \mathbb{R}^{m\times n}$, $A^* \in \mathbb{R}^{m\times n}$, $B^* \in \mathbb{R}^{m\times k}$, $F^* \in \mathbb{R}^{m\times q}$, $C^* \in \mathbb{R}^{p\times n}$, and $G^* \in \mathbb{R}^{p\times q}$ are known constant matrices. We assume that rank $E^* = n_0 < max\{m, n\}$. Let us make the following assumptions on the system (1)

(H1) $\text{rank}\begin{bmatrix} E^* & A^* & F^* & 0 \\ 0 & E^* & 0 & F^* \\ 0 & C^* & G^* & 0 \\ 0 & 0 & 0 & G^* \end{bmatrix} = \text{rank}\begin{bmatrix} E^* & F^* \\ 0 & G^* \end{bmatrix} + n + q,$

(H2) $\text{rank}\begin{bmatrix} A^* - \lambda E^* & F^* \\ C^* & G^* \end{bmatrix} = n + q \ \forall \ \lambda \ \in \ \bar{\mathbb{C}}^+,$

where $\mathbb{C}$ represents the set of complex numbers. $\bar{\mathbb{C}}^+ = \{s|s \in \mathbb{C}, Re(s) \geq 0\}$ is the closed right half complex plane.

The conditions $(H1)$ and $(H2)$ are generalization of the conditions of the impulse observability and detectability properties of square descriptor systems to the descriptor systems (1), respectively. Since rank $E^* = n_0$, there exists a nonsingular matrix P such that $PE^* = \begin{bmatrix} E \\ 0 \end{bmatrix}$, $PA^* = \begin{bmatrix} A \\ A_1 \end{bmatrix}$, $PB^* = \begin{bmatrix} B \\ B_1 \end{bmatrix}$, $PF^* = \begin{bmatrix} F \\ F_1 \end{bmatrix}$, and system (1) is restricted system equivalent to following system (2).

$$E\dot{x} = Ax + Bu + Fv, \tag{2a}$$
$$y = Cx + Gv, \tag{2b}$$

where $E \in \mathbb{R}^{n_0 \times n}$, $A \in \mathbb{R}^{n_0 \times n}$, $B \in \mathbb{R}^{n_0 \times k}$, $F \in \mathbb{R}^{n_0 \times q}$, $C = \begin{bmatrix} A_1 \\ C^* \end{bmatrix} \in \mathbb{R}^{t \times n}$, $G = \begin{bmatrix} F_1 \\ G^* \end{bmatrix} \in \mathbb{R}^{t \times q}$ are constant matrices and $y = \begin{bmatrix} -B_1 u \\ y^* \end{bmatrix} \in \mathbb{R}^t$ with $t = p + m - n_0$.

Since rank $G := q_1 \leq q$, there exists two nonsingular matrices U and V such that $UGV = \begin{bmatrix} I_{q_1} & 0 \\ 0 & 0 \end{bmatrix}$. Now, system (2) can be written as follows.

$$\begin{aligned} E\dot{x} &= \Phi x + Bu + F_{11} y_1 + F_{12} v_2, & \text{(3a)} \\ y_2 &= C_{12} x, & \text{(3b)} \end{aligned}$$

with

$$y_1 = C_{11} x + v_1, \tag{4}$$

where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Uy$, $\begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = UC$, $v = V \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $FV = \begin{bmatrix} F_{11} & F_{12} \end{bmatrix}$, and $\Phi = A - F_{11} C_{11}$.

Under the assumptions $(H1)$ and $(H2)$ on the system (1), descriptor system (3) satisfies the following condition

(H3) rank $\begin{bmatrix} E & F_{12} \\ C_{12} & 0 \end{bmatrix} = n + q - q_1$.

(H4) rank $\begin{bmatrix} \Phi - \lambda E & F_{12} \\ C_{12} & 0 \end{bmatrix} = n + q - q_1 \ \forall \ \lambda \in \bar{\mathbb{C}}^+$.

Till now, in this section, all the definitions and used transformations are taken from [33]. Our main results, as described in the next two sections, also require the following lemma.

Lemma 1. *Let any matrix pair $(\mathcal{E}, \mathcal{C})$, where $\mathcal{E} \in \mathbb{R}^{m_1 \times n_1}$ with $m_1 \leq n_1$ and $\mathcal{C} \in \mathbb{R}^{p_1 \times n_1}$ satisfies following condition*

$$\mathit{rank} \begin{bmatrix} \mathcal{E} \\ \mathcal{C} \end{bmatrix} = n_1.$$

Then there exists a full column matrix R such that

$$\mathit{rank} \begin{bmatrix} I - R\mathcal{E} \\ \mathcal{C} \end{bmatrix} = \mathit{rank}(\mathcal{C})$$

Proof of the above Lemma can be found in [36]. This matrix R is not unique, one numerically reliable algorithm to find matrix R is given in the Appendix A. Now we will design observers for the system (3), which is also an observer for the system (2) and the system (1).

3. Full-order observer design

Let the proposed full-order observer is of the form

$$\begin{aligned} \dot{z} &= Nz + TBu + TF_{11} y_1 + L y_2, & \text{(5a)} \\ \hat{x} &= z + M y_2, & \text{(5b)} \end{aligned}$$

where $z \in \mathbb{R}^n$. Problem is to find matrices N, L, T, and M of compatible dimensions such that $\hat{x}(t) \to x(t)$ as $t \to \infty$ for arbitrary initial conditions $x(0)$ and $z(0)$.

Theorem 1. *Under the assumptions (H3) and (H4), there exists matrices N, L, T and M of compatible dimensions such that the system (5) is an observer for the system (3).*

Proof. From systems (5) and (3) the error vector

$$\begin{aligned} e &= x - \hat{x} \\ &= x - z - MC_{12}x \\ &= (I_n - MC_{12})x - z \\ &= TEx - z \end{aligned} \tag{6}$$

gives the dynamics

$$\begin{aligned} \dot{e} &= TE\dot{x} - \dot{z} \\ &= T\Phi x + TBu + TF_{11}y_1 + TF_{12}v_2 - (Nz + TBu + TF_{11}y_1 + LC_{12}x) \\ &= (T\Phi - LC_{12})x - N(TEx - e) \\ &= Ne + (T\Phi - LC_{12} - NTE)x \\ &= Ne + (T\Phi - LC_{12} - N + NMC_{12})x \\ &= Ne. \end{aligned} \tag{7}$$

In the construction of equations (6) and (7), we have assumed the existence of matrices M, K, N, and L of compatible orders such that

$$\begin{bmatrix} T & M \end{bmatrix} \begin{bmatrix} E & F_{12} \\ C_{12} & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix}, \tag{8}$$

$$K = L - NM, \tag{9}$$

$$N = T\Phi - KC_{12}. \tag{10}$$

Now, the problem of designing the state observer (5) is converted into the design of the matrices $M,\ K,\ N$ and L such that the equations (8)-(10) are satisfied with the stability property of the matrix N, because the error dynamics (7) is stable if and only if the system (5) is an observer for the system (3). Assumption (H3) ensures the solvability of equation (8), but we have to find T such that matrix pair $(T\Phi,\ C_{12})$ is detectable, because detectability of matrix pair $(T\Phi, C_{12})$ implies the stability of matrix N. Now, first we shall prove the claim that (H3) implies the solution of equation (8).

Since rank of $F_{12} \in \mathbb{R}^{n_0 \times (q-q_1)}$ is $q - q_1$, we can find a full row rank matrix $T_0 \in \mathbb{R}^{(n_0+q_1-q)\times n_0}$ such that $T_0F_{12} = 0$. Now, it is clear that

$$\begin{bmatrix} T_0 & 0 \\ 0 & I_{t-q_1} \end{bmatrix} \begin{bmatrix} E & F_{12} \\ C_{12} & 0 \end{bmatrix} = \begin{bmatrix} T_0E & 0 \\ C_{12} & 0 \end{bmatrix}. \tag{11}$$

But, assumption (H3) implies

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} T_0 & 0 \\ 0 & I_{t-q_1} \end{bmatrix} \begin{bmatrix} E & F_{12} \\ C_{12} & 0 \end{bmatrix} &\geq (n_0 - q + q_1 + t - q_1) + (n + q - q_1) - (n_0 + t - q_1) \\ &= n, \end{aligned} \tag{12}$$

which implies (together with equation (11)) that

$$\operatorname{rank} \begin{bmatrix} T_0E \\ C_{12} \end{bmatrix} = n. \tag{13}$$

Now applying the Lemma 1 on matrix pair $(T_0E,\ C_{12})$, we can find a full column rank matrix R such that

$$\operatorname{rank} \begin{bmatrix} I - RT_0E \\ C_{12} \end{bmatrix} = \operatorname{rank}(C_{12}), \tag{14}$$

which implies the existence of a matrix M such that $I - TE = MC_{12}$, where $T = RT_0$. Thus, as constructed matrices T and M satisfy the equation (8). Now it remains to prove that for constructed T, matrix pair $(T\Phi, C_{12})$ is detectable. It is straightforward that $\forall\ \lambda\ \in\ \bar{\mathbb{C}}^+$,

$$\begin{bmatrix} T_0 & 0 \\ 0 & I_{(t-q_1)} \end{bmatrix} \begin{bmatrix} \Phi - \lambda E & F_{12} \\ C_{12} & 0 \end{bmatrix} = \begin{bmatrix} T_0\Phi - \lambda T_0 E & 0 \\ C_{12} & 0 \end{bmatrix}. \tag{15}$$

But, assumption (H4) implies

$$\begin{aligned} \text{rank} \begin{bmatrix} T_0 & 0 \\ 0 & I_{(t-q_1)} \end{bmatrix} \begin{bmatrix} \Phi - \lambda E & F_{12} \\ C_{12} & 0 \end{bmatrix} &\geq (n_0 - q + q_1 + t - q_1) + (n + q - q_1) - (n_0 + t - q_1) \\ &= n. \end{aligned} \tag{16}$$

Thus, we have,

$$\begin{aligned} n &= \text{rank} \begin{bmatrix} T_0\Phi - \lambda T_0 E \\ C_{12} \end{bmatrix} \qquad (17) \\ &= \text{rank} \begin{bmatrix} T\Phi - \lambda TE \\ C_{12} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} T\Phi - \lambda(I - MC_{12}) \\ C_{12} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} I & \lambda M \\ 0 & I \end{bmatrix} \begin{bmatrix} T\Phi - \lambda I \\ C_{12} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} T\Phi - \lambda I \\ C_{12} \end{bmatrix}. \end{aligned}$$

Hence the matrix pair $(T\Phi, C_{12})$ is detectable. □

Now, we summarize the procedure explained in the above theorem by writing the following algorithm.
Algorithm for Full-order observer for the system (1)

1. Reduce the given descriptor system (1) in the form of system (3) as described in Section 2.
2. Find the left null matrix T_0 for matrix F_{12}.
3. Find matrix R for matrix pair (T_0E, C_{12}) by algorithm given in the Appendix.
4. Solve matrix equation $MC = I - TE$ for unknown matrix M, where $T = RT_0$.
5. Find matrix K by pole placement or LMI approach such that $N = T\Phi - KC$ is a stable matrix.
6. Calculate $L = K + NM$.

4. Reduced order observer design

Let the proposed observer is of the form

$$\dot{z} = \bar{N}z + T_0Bu + T_0F_{11}y_1 + \bar{L}y_2, \tag{18a}$$
$$\hat{x} = Rz + My_2, \tag{18b}$$

where $z \in \mathbb{R}^{n \times (n_0+q_1-q)}$, and the matrices T_0, R and M are the same as mentioned in the previous Section. Problem is to find matrices $\bar{N}$ and $\bar{L}$ of compatible dimensions such that $\hat{x}(t) \to x(t)$ as $t \to \infty$ for arbitrary initial conditions $x(0)$ and $z(0)$.

Theorem 2. *Under the assumptions (H3) and (H4), there exists matrices $\bar{N}$ and $\bar{L}$ of compatible dimensions such that the system (18) is an observer for the system (3).*

Proof. Assume the existence of matrices $\bar{N}$, $\bar{K}$ and $\bar{L}$ such that

$$\bar{N} = T_0\Phi R - \bar{K}C_{12}R \tag{19}$$

$$\bar{L} = T_0\Phi M + \bar{K} - \bar{K}C_{12}M. \tag{20}$$

From systems (18) and (3) the error vector

$$\begin{aligned} e &= x - \hat{x} \\ &= x - Rz - MC_{12}x \\ &= (I_n - MC_{12})x - Rz \\ &= R(T_0Ex - z). \end{aligned} \tag{21}$$

Assume, $e_1 = T_0Ex - z$. Then,

$$\begin{aligned} \dot{e_1} &= T_0E\dot{x} - \dot{z} \\ &= T_0\Phi x + T_0Bu + T_0F_{11}y_1 + T_0F_{12}v_2 - (\bar{N}z + T_0Bu + T_0F_{11}y_1 + \bar{L}C_{12}x) \\ &= (T_0\Phi - \bar{L}C_{12})x - \bar{N}(T_0Ex - e_1) \\ &= \bar{N}e_1 + (T_0\Phi - \bar{L}C_{12} - \bar{N}T_0E)x \\ &= \bar{N}e_1. \end{aligned} \tag{22}$$

Equations (19) and (20) imply that

$$T_0\Phi - \bar{L}C_{12} - \bar{N}T_0E = 0,$$

and from (21) and (22), we can write

$$e = R\exp(\bar{N}t)(T_0Ex(0) - z(0)). \tag{23}$$

It can be seen from the equation (23) that error is asymptotically stable if and only if matrix $\bar{N}$ is a stable matrix. It implies from equation (19), that detectability of matrix pair $(T_0\Phi R, C_{12}R)$ is required to construct stable matrix $\bar{N}$. It is clear that $RT_0E = I_n - MC_{12}$ implies

$$T_0E = R^+(I_n - MC_{12}), \tag{24}$$

where, R^+ is any left inverse of matrix R. Now, from (17), for each $\lambda \in \bar{\mathbb{C}}^+$

$$\begin{aligned} &\text{rank}\begin{bmatrix} T_0\Phi - \lambda T_0E \\ C_{12} \end{bmatrix} = n \\ \Rightarrow &\text{rank}\begin{bmatrix} T_0\Phi - \lambda R^+(I_n - MC_{12}) \\ C_{12} \end{bmatrix} = n \\ \Rightarrow &\text{rank}\begin{bmatrix} T_0\Phi R - \lambda R^+(I_n - MC_{12})R \\ C_{12}R \end{bmatrix} = n_0 + q_1 - q \\ \Rightarrow &\text{rank}\begin{bmatrix} T_0\Phi R - \lambda I_{n_0+q_1-q} \\ C_{12}R \end{bmatrix} = n_0 + q_1 - q \end{aligned} \tag{25}$$

which implies that the matrix pair $(T_0\Phi R, C_{12}R)$ is detectable. □

Remark 1. In [33] (see equations (5e), (8), (20), and (21)) a full row rank matrix T is constructed in such a way that $TF_{12} = 0$, $\begin{bmatrix} TE \\ C_{12} \end{bmatrix}$ is nonsingular, and the matrix pair (Ω, Γ) is detectable. In the proposed technique, we have also designed one detectable matrix pair, but the complexity of the whole work is reduced in two steps. First, a full row rank matrix T_0 (left null matrix of F_{12}) is used to eliminate F_{12}. Secondly, full column rank property of matrix R easily concludes the detectability of full- and reduced-order normal matrix pairs $(T\Phi, C_{12})$ and $(T_0\Phi R, C_{12}R)$, respectively.

Remark 2. In the proposed technique, the order of reduced-order observer is $n_0 + q_1 - q$, which is $\text{rank}\begin{bmatrix} E^* & F^* \\ 0 & G^* \end{bmatrix} - \text{rank}\begin{bmatrix} F^* \\ G^* \end{bmatrix}$ in terms of the given system coefficient matrices. The order of the observer given by [33] is $n - \text{rank}(C_{12})$, which is $n + \text{rank}\begin{bmatrix} E^* & F^* \\ 0 & G^* \end{bmatrix} - \text{rank}\begin{bmatrix} E^* & A^* & F^* \\ 0 & C^* & G^* \end{bmatrix}$. It is clear from the equation (13) that $n_0 + q_1 - q \geq n - rank(C_{12})$. Here equality holds if C_{12} is of full row rank and matrix $\begin{bmatrix} T_0 E \\ C_{12} \end{bmatrix}$ is nonsingular. Note that, in case of proportional integral observers, the order is always higher than the number of states. Hence, in [34] (if we do not consider the faults) the order of observer is $n + t$ that is addition of states and outputs.

Remark 3. After estimating states by full- or reduced-order observers, unknown inputs can be easily estimated by equations (3a) and (4).

Now, we again summarize the procedure for designing the reduced-order observer in the following algorithm.

Algorithm for Reduced-order observer for the system (1)

1. Repeat steps 1-4 of full-order observer design algorithm.
2. Find matrix $\bar{K}$ by pole placement or LMI approach such that $\bar{N} = T_0 \Phi R - \bar{K} C_{12} R$ is a stable matrix.
3. Calculate $\bar{L} = T_0 \Phi M + \bar{K} - \bar{K} C_{12} M$.

5. Numerical Examples

Example 1. Consider the system (1) as described by the following matrices:

$$E^* = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^* = \begin{bmatrix} -1 & 1 & 0 & 8 \\ -1 & 1 & 0 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & 0 & 3 & -1 \end{bmatrix},\ B^* = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},\ F^* = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C^* = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 8 & 5 & 1 & 1 \end{bmatrix},\ G^* = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix},\ v = \begin{bmatrix} t^2 \\ cos(t) \end{bmatrix} \text{ and } u = sin(t).$$

As explained in Section 2, we calculate

$$E = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1 & 1 & 0 & 8 \\ -1 & 1 & 0 & 1 \\ 2 & -1 & -1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ F = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & 0 & 2 & 3 \\ 8 & 5 & 1 & 1 \end{bmatrix},\ G = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix},\ U = \begin{bmatrix} 0.5774 & -0.5774 & 0.5774 \\ -0.5774 & 0.2113 & 0.7887 \\ 0.5774 & 0.7887 & 0.2113 \end{bmatrix}, \text{ and } V = \begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}.$$

Matrices U and V give

$$C_{11} = \begin{bmatrix} 4.6188 & 2.8868 & 1.1547 & -1.7321 \end{bmatrix},\ C_{12} = \begin{bmatrix} 5.9434 & 3.9434 & -0.5207 & 2 \\ 3.0566 & 1.0566 & 3.5207 & 2 \end{bmatrix},$$

$$F_{11} = \begin{bmatrix} -0.5774 \\ 0 \\ 0.5774 \end{bmatrix},\ F_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \Phi = \begin{bmatrix} 1.6667 & 2.6667 & 0.6667 & 7 \\ -1 & 1 & 0 & 1 \\ -0.6667 & -2.6667 & -1.6667 & 2 \end{bmatrix}$$

and $T_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now, by algorithm given in the Appendix A, we calculate $R = \begin{bmatrix} 1 & 1.4 \\ -1 & 0 \\ 0 & 1 \\ -1 & -3.9 \end{bmatrix}$.

(a) Full-order observer: Considering the algorithm for full-order observer, we get $T = \begin{bmatrix} 0 & 1 & 1.4 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -3.9 \end{bmatrix}$,
$M = \begin{bmatrix} 0.3464 & -0.3464 \\ 0 & 0 \\ 0 & 0 \\ -0.5294 & 1.0294 \end{bmatrix}$, $K = \begin{bmatrix} -0.2324 & 0.0548 \\ -0.0002 & 0.1487 \\ -0.3151 & -0.0587 \\ 0.3296 & 0.7205 \end{bmatrix}$, $N = \begin{bmatrix} -0.7195 & -1.8747 & -2.6474 & 4.1552 \\ 0.5470 & -1.1561 & -0.5236 & -1.2969 \\ 1.3855 & -1.3621 & -1.6241 & 2.7476 \\ -0.5614 & 7.3388 & 4.1351 & -10.9002 \end{bmatrix}$,
and $L = \begin{bmatrix} -2.6815 & 4.5815 \\ 0.8758 & -1.3758 \\ -1.2898 & 2.2898 \\ 5.9060 & -10.3060 \end{bmatrix}$.
If we take initial condition of descriptor system and observer as $x(0) = \begin{bmatrix} -1 & 1 & 2 & 5 \end{bmatrix}^T$ and $z(0) = \begin{bmatrix} 10 & 12 & 15 & 8 \end{bmatrix}^T$, respectively, simulation results are plotted in Figure 1, which reveals that the estimated values of the states follow the true states well.

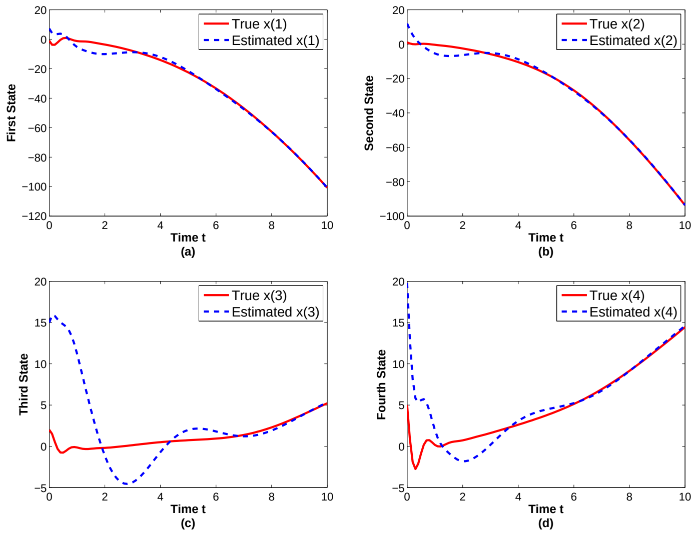

Figure 1. Plot of true and estimated values of states by full-order observer in Example 1

(b) Reduced order observer: Using the algorithm for reduced-order observer, we get $\bar{N} = \begin{bmatrix} -3 & -5.3 \\ 0 & -10.4 \end{bmatrix}$ and $\bar{L} = \begin{bmatrix} -0.8758 & 1.3758 \\ -1.2898 & 2.2898 \end{bmatrix}$. Taking $x(0) = \begin{bmatrix} -1 & 1 & 2 & 5 \end{bmatrix}^T$ and $z(0) = [10\ 12]^T$, simulation results are plotted in Figure 2.

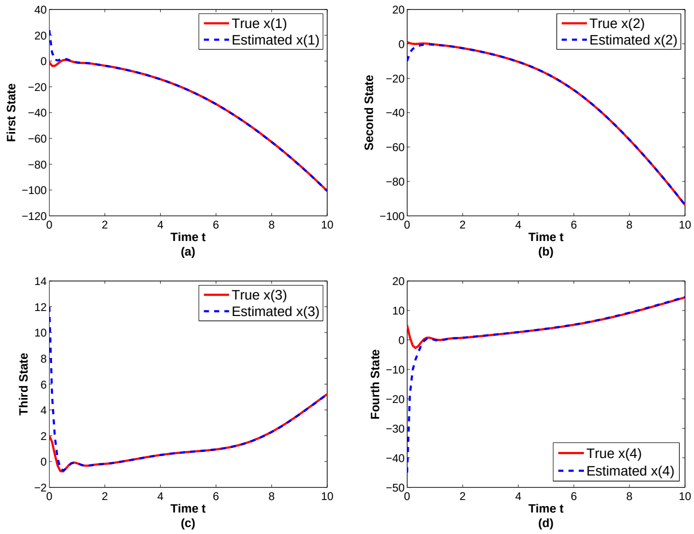

Figure 2. Plot of true and estimated values of states by reduced-order observer in Example 1

This example shows the effectiveness of the proposed method because Figures 1 and 2 expose the convergence of estimated states to the true states in case of unknown input like t^2, which grows rapidly as time increases. In [34] simulation results have been plotted only for periodic unknown input $0.2\sin 5t$.

Example 2. Consider the system (1) as described by the following matrices:

$$E^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A^* = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},\ F^* = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ G^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \text{and } u = \begin{bmatrix} cos(t) \\ 3t \end{bmatrix}.$$

Steps 1-4 of algorithm for full-order observer design give

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ F = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ G = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ U = I_3, \text{ and } V = 1,\ C_{11} = \text{Empty matrix},$$

$$C_{12} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ F_{11} = \text{Empty matrix},\ F_{12} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},\ \Phi = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix},$$

$$T_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1.3764 \\ 0 & 0.8507 \end{bmatrix},\ T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1.3764 \\ 0 & 0 & 0.8507 \end{bmatrix},\ \text{and}$$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -2.3764 & 0 & 2.3764 \\ 1.8507 & 0 & -0.8507 \end{bmatrix}.$$

(a) Full-order observer:

Assuming unknown input $v = 2t^{3/2}$, steps 5-6 of algorithm for full-order observer design give

$$K = \begin{bmatrix} -30.2210 & 0.5 & -32.6066 \\ -1.6054 & -2.4726 & 1.5516 \\ -2.7414 & 32.2816 & 2.2188 \\ 1.4161 & 63.0285 & -0.7723 \end{bmatrix}, \; N = \begin{bmatrix} -0.5 & 0 & 32.6066 & 62.8276 \\ 1.4726 & 0 & -1.5516 & 1.0538 \\ -32.2816 & 1.3764 & -0.8425 & 0.5226 \\ -63.0285 & -0.8507 & -0.0783 & -0.6437 \end{bmatrix},$$

and $L = \begin{bmatrix} 8.5652 & 0 & -8.5652 \\ 4.0321 & -1 & -3.0321 \\ 0.2277 & 0 & -0.2277 \\ 0.4109 & 0 & -0.4109 \end{bmatrix}$. Taking $x(0) = \begin{bmatrix} -1 & 1 & 2 & -1 \end{bmatrix}^T$ and $z(0) = \begin{bmatrix} 10 & 12 & 15 & 8 \end{bmatrix}^T$, simulation results are plotted in Figure 3, which reveals that the estimated values of the states follow the true states well.

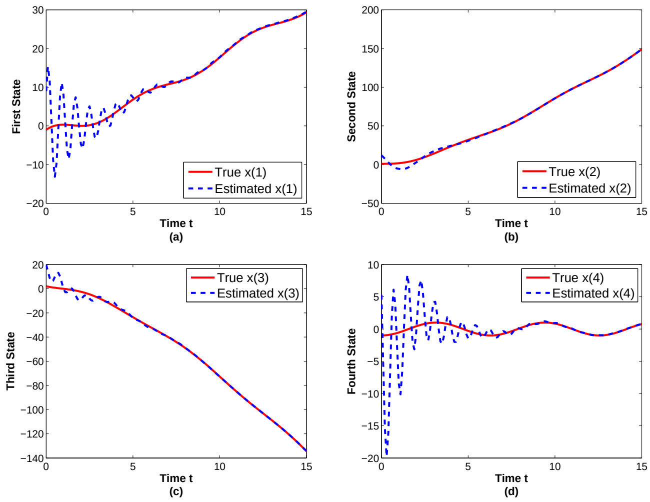

Figure 3. Plot of true and estimated values of states by full-order observer in Example 2

(b) Reduced order observer:-
Considering unknown input $v = 2sin(5t)$, steps 2-3 of algorithm for reduced-order observer design give $\bar{K} = \begin{bmatrix} -0.0398 & 0 & 0.8142 \\ 2.1925 & 0 & -0.8537 \end{bmatrix}$, $\bar{N} = \begin{bmatrix} 0 & 1.3125 \\ -1 & -0.9375 \end{bmatrix}$ and $\bar{L} = \begin{bmatrix} 2.3125 & -1 & -1.3125 \\ 0.0625 & 0 & -0.0625 \end{bmatrix}$. If we take $x(0) = \begin{bmatrix} -1 & 1 & 2 & -1 \end{bmatrix}^T$ and $z(0) = [10 \; 12]^T$, simulation results are plotted in Figure 4.

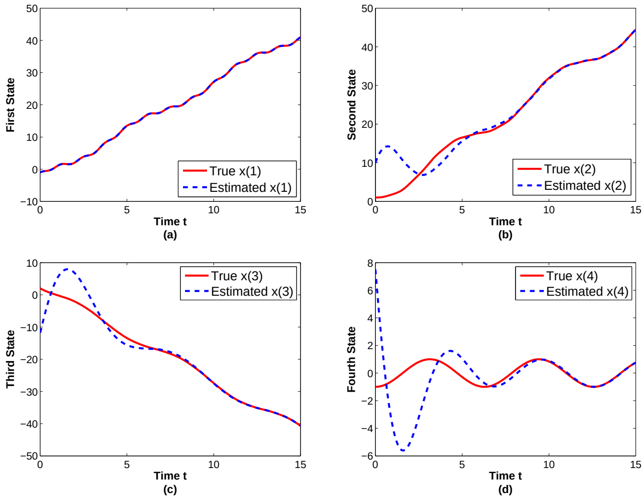

Figure 4. Plot of true and estimated values of states by reduced-order observer in Example 2

This example has been already discussed in [33] without the simulation results and the mathematical description of unknown inputs. Here, Figures 3 and 4 show the estimation performances of the proposed full- and reduced-order observers with different types of unknown inputs.

6. Conclusions

In this paper, methods have been developed to design full- and reduced-order state observers for rectangular descriptor system with unknown inputs. Observer design problem is converted into the solution of some matrix equations. Beauty of the this work lies in the fact that same matrices have been used to design the full- and reduced-order observers. The extension of this work to design observers with more reduced order is under construction.

Appendix A

Algorithm to find a matrix R used in the Lemma 1

1. Determine
$r_1 :=$ rank of matrix $\mathcal{C}$
$m_1 \times n_1$:=order of matrix $\mathcal{E}$, with $m_1 \leq n_1$.

2. Check rank $\begin{bmatrix} \mathcal{E} \\ \mathcal{C} \end{bmatrix} = n_1$, then go to steps 3-8.

3. Carry out the singular value decomposition (SVD) of matrix $\mathcal{C} = U_1 \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^T$.

4. Calculate $P = V_1 \begin{bmatrix} D_1^{-1} & 0 \\ 0 & I_{n_1-r_1} \end{bmatrix}$.

5. Calculate $\mathcal{E}_2 = \mathcal{E}P \begin{bmatrix} 0_{r_1 \times (n_1-r_1)} \\ I_{n_1-r_1} \end{bmatrix}$.

6. Carry out the SVD of matrix $\mathcal{E}_2 = U_2 \begin{bmatrix} D_2 \\ 0 \end{bmatrix} V_2^T$.

7. Calculate $R_0 = \begin{bmatrix} 0 & I_{m_1+r_1-n_1} \\ V_2 D_2^{-1} & 0 \end{bmatrix} U_2^T$.

8. Calculate $R = P \begin{bmatrix} 0_{(n_1-m_1)\times m_1} \\ R_0 \end{bmatrix}$.

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