DIFFRACTION OF A PLANE WAVE BY TWO IDEAL STRIPS

by Andrey V. Shanin

(*Department of Physics, Acoustics, Moscow State University, Leninskie Gory, Moscow 119992, Russia*)

Summary

The problem of scattering of a plane wave by two strips lying in one plane and having ideal boundary conditions is studied. The following exact results are obtained. 1) The embedding formula is derived. This formula enables to express the far-field diagram, depending on two variables (the angle of incidence and the angle of scattering) as the combination of 4 functions depending on one variable. 2) The ordinary differential equation with respect to the spectral variable is derived for the components of the far-field diagram. 3) The evolution equations describing the dependance of the far-field diagram on the parameters of the problem (such as the coordinates of the edges of the scatterer) are derived. The results listed above are obtained by applying two independent approaches: the Wiener-Hopf functional equations approach and the diffraction (Schwarzschild's) series approach.

1. Introduction

To our knowledge, it was Sommerfeld who first tried to solve the problem of diffraction by several strips. His method of solving the classical problem of diffraction by a half-plane (**1**) seemed to possess the capability of generalization on the case of any problem that can be reformulated in terms of the Riemann surfaces introduced by Sommerfeld. However, even the problem of diffraction by a single strip appeared to be very difficult. Altough the solution of this problem in the form of the Fourier series in Mathieu functions is known since 1908 (**2**), a significant number of researchers tried to find a rigorous closed-form solution analogous to Sommerfeld's solution of a half-line problem. A review of *incorrect* papers devoted to the strip problem can be found in (**3**). The first success was achieved in 1956 (**4**). Then in (**5**) a significant progress had been reported, namely the embedding formula and the evolution equation had been derived. Later, these results were re-derived by using a completely different technique close to the Wiener-Hopf method (**6**). The same results were reformulated using the method of diffraction series in (**7**)

The history of the several strips problem is less dramatic. Only two papers containing the rigorous results are known to us (**8**, **9**). In the first of them the series reminiscent of Mathieu one is constructed. However, the practical benefits of this series are not clear. In the second one the embedding formula for several strips is constructed in a rather sophisticated manner. A number of approximate results utilising the modified Wiener-Hopf approach and / or the ray ideas are known (see e.g. (**10**, **11**)).

The purpose of the current paper is to establish the rigorous analytical properties of the scattered field for the several strips case, which are analogous to the properties of the field in the single strip case. These properties (they are formulated as Theorems 1, 2 and 3) are valid for any geometrical size of the scatterer compared to the wavelength. Altough these theorems contain neither the explicit form of the solution, nor the simple recipe for its calculating, we demonstrate the possibility to use them in a combination with standard approximate methods.

For our consideration we choose the case of diffraction by two strips, but all results can be easily generalized to the case of the arbitrary number of the strips.

The structure of the paper is as follows. Sections 2 and 3 are devoted to the functional equations approach based on Wiener-Hopf method approach. In Section 2 the problem of diffraction of a plane wave by two strips is formulated and some preliminary steps are performed. Namely, the set of the auxiliary diffraction problems is introduced, and each problem is reformulated using the language of the Wiener-Hopf approach, i.e. the unknown functions in the spectral domain are introduced and the functional equations are derived. In Section 3 the main rigorous results, namely the embedding formula, the spectral equation and the evolution equation are obtained. These results are formulated as Theorems 1, 2 and 3, respectively. All theorems are based on the statement of Lemma 1, which is formulated and proved in the beginning of the section. The content of Section 3 can be considered as the generalization of the Wiener-Hopf approach.

In Sections 4 and 5 the relation between the new method and the iteration technique is established. For this purpose in Section 4 we write the solution of the diffraction problem in the form of the diffraction series, i.e. consider the diffraction process as the sequence of the simple acts of diffraction by the edges of the scatterer. In Section 5 we again derive the embedding formula, spectral equation and the evolution equation. These results are formulated as Theorems 1', 2' and 3'. Such parallel structure enables us to justify the main results from two different points of view. Moreover, when the diffraction series approach is used, the unknown numerical parameters of the coefficients of the spectral and evolution equations become expressed in the form of the asymptotic series. This makes it possible to use these equations for practical calculations.

Section 6 contains an example of calculations utilizing the presented theory.

2. Wiener-Hopf approach. Preliminary steps

2.1 *Formulation of diffraction problem with plane wave incidence*

Let the 2D Helmholtz equation

$$\Delta u + k_0^2 u = 0 \tag{2.1}$$

be valid on the (x, y) plane. The cross-sections of two strips coincide with the segments (a_1, a_2) and (a_3, a_4) of the x-axis (see Fig. 1). We assume that the time dependance everywhere has the form $e^{-\mathrm{i}\omega t}$. We assume that k_0 has a small positive imaginary part corresponding to small dissipation in the media.

Let the Dirichlet boundary conditions

$$u = 0 \tag{2.2}$$

be valid on the strips.

The incident field is a plane wave coming from the upper half-plane

$$u^{\mathrm{in}} = e^{-\mathrm{i}k_* x - \mathrm{i}\sqrt{k_0^2 - k_*^2}\, y}, \tag{2.3}$$

where $k_* = k_0 \cos\psi$; ψ is the angle of incidence.

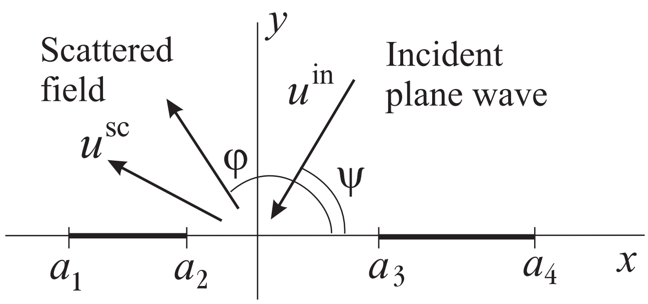

Fig. 1 Geometry of the problem

Due to the obvious symmetry of the scattered field, a problem with mixed boundary conditions can be formulated for it:

$$u^{\rm sc}(x, \pm 0) = -e^{-{\rm i}k_* x} \qquad \text{for} \quad x \in (a_1, a_2) \cup (a_3, a_4), \tag{2.4}$$

$$\frac{\partial}{\partial y} u^{\rm sc}(x, 0) = 0 \qquad \text{for} \quad x \in (-\infty, a_1) \cup (a_2, a_3) \cup (a_4, \infty). \tag{2.5}$$

It is clear that one can consider the field only in the half-plane $y > 0$.

The radiation condition is satisfied at infinity. The scattered field should not contain the components coming from infinity or growing at infinity. These conditions will be taken into account when constructing the integral representation of the scattered field.

Meixner's edge conditions should be taken into account. Here we demand that the total field should have the asymptotic behaviour near the edges similar to that for the static problem, namely

$$u \sim r^{1/2}, \tag{2.6}$$

where r is the distance between the observation point and the nearest edge. The normal derivative on the strips, therefore, behaves like $r^{-1/2}$.

Without the loss of generality, we suppose that $\psi \in (0, \pi/2)$. This corresponds to ${\rm Im}[k_*] > 0$.

2.2 *Formulation of auxiliary diffraction problems*

We find it necessary to introduce here a set of *auxiliary* diffraction problems. The solution of the initial problem can be expressed through the solution of the auxiliary problems via the embedding formula, while the other results, namely the spectral equation and the evolution equation look much simpler being formulated for the auxiliary solutions.

Four auxiliary diffraction problems are formulated; their solutions are denoted by $u^m(x, y)$, $m = 1 \dots 4$. These functions are defined as the result of the following limiting procedure. Let u^m_ϵ be solutions of the Helmholtz equations

$$\Delta u^m_\epsilon + k_0^2 u^m_\epsilon = (\pi/\epsilon)^{1/2}\, \delta(x - a_m - (-1)^m \epsilon)\, \delta(y) \tag{2.7}$$

Here ϵ is a small positive value, δ is the Dirac's delta-function.

The functions u_ϵ^m satisfy the homogeneous boundary conditions

$$u_\epsilon^m = 0 \qquad \text{for} \quad x \in (a_1, a_2) \cup (a_3, a_4), \tag{2.8}$$

$$\frac{\partial}{\partial y} u_\epsilon^m = 0 \qquad \text{for} \quad x \in (-\infty, a_1) \cup (a_2, a_3) \cup (a_4, \infty), \quad x \neq a_m + (-1)^m \epsilon, \tag{2.9}$$

Meixner's and radiation conditions.

The functions u^m are defined as the limits of the corresponding functions u_ϵ^m:

$$u^m = \lim_{\epsilon \to 0} u_\epsilon^m. \tag{2.10}$$

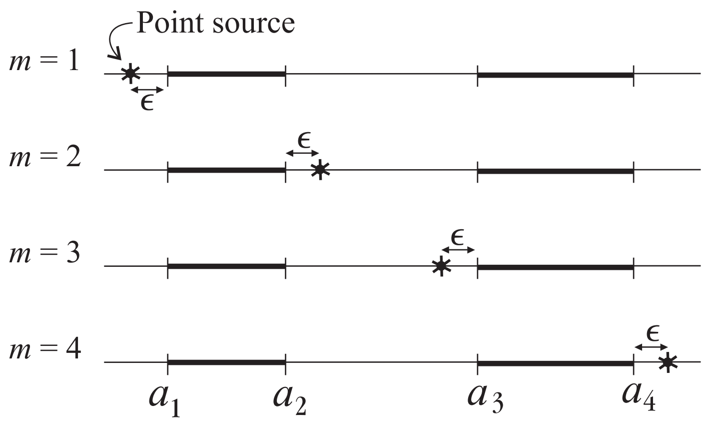

Fig. 2 To the formulation of the auxiliary diffraction problem

The functions u_ϵ^m admit an obvious physical interpretation. They are the diffraction fields caused by the point source located near one of the edges (see Fig. 2) and having the strength equal to $\sqrt{\pi/\epsilon}$. One can check by studying the local behaviour of u_ϵ^m that the limits (2.10) exist and they are non-zero. Moreover, the detailed study provide the following asymptotic estimations of the functions u^m near the edges a_n:

$$u^m(\rho_n, \theta_n) = -\frac{\delta_{m,n}}{\sqrt{\pi}} \rho_n^{-1/2} \sin\frac{\theta_n}{2} + \frac{2C_n^m}{\sqrt{\pi}} \rho_n^{1/2} \sin\frac{\theta_n}{2} + O(\rho_n^{3/2}), \tag{2.11}$$

where C_n^m are some unknown coefficients; $\delta_{m,n}$ is the Kronecker's delta; (ρ_n, θ_n) are the local cylindrical coordinates introduced in Fig. 3.

Note that the expansions for u^m contain the terms not obeying Meixner's condition. The reason for this is, obviously, the presence of the source near the corresponding edge.

There is a possibility to introduce the auxiliary solutions formally without the limiting procedure. Namely, one can seek u^m as the functions satisfying the homogeneous Helmholtz equation, homogeneous boundary conditions, radiation condition, and having the asymptotic behaviour of the form (2.11) at the edges.

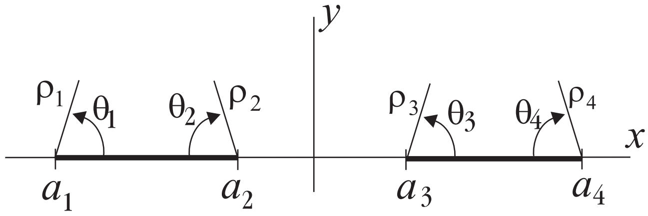

Fig. 3 Local polar coordinates

2.3 *Unknown functions in spectral domain. Derivation of functional equations*
Introduce the following functions:

$$\begin{aligned}
U_0(k) &= \int_{-\infty}^{a_1} u^{\rm sc}(x,0)e^{{\rm i}kx}dx - \frac{{\rm i}e^{{\rm i}(k-k_*)a_1}}{k-k_*}, \\
U_1(k) &= \frac{\rm i}{\sqrt{k_0^2-k^2}} \int_{a_1}^{a_2} \frac{\partial u^{\rm sc}}{\partial y}(x,+0)e^{{\rm i}kx}dx. \\
U_2(k) &= \int_{a_2}^{a_3} u^{\rm sc}(x,0)e^{{\rm i}kx}dx + \frac{{\rm i}e^{{\rm i}(k-k_*)a_2}}{k-k_*} - \frac{{\rm i}e^{{\rm i}(k-k_*)a_3}}{k-k_*}, \\
U_3(k) &= \frac{\rm i}{\sqrt{k_0^2-k^2}} \int_{a_3}^{a_4} \frac{\partial u^{\rm sc}}{\partial y}(x,+0)e^{{\rm i}kx}dx. \\
U_4(k) &= \int_{a_4}^{\infty} u^{\rm sc}(x,0)e^{{\rm i}kx}dx + \frac{{\rm i}e^{{\rm i}(k-k_*)a_4}}{k-k_*}.
\end{aligned} \tag{2.12}$$

Taking into account the boundary conditions (2.4) and (2.5), we conclude that

$$U_0(k) + U_2(k) + U_4(k) = \int_{-\infty}^{\infty} u^{\rm sc}(x,0)e^{{\rm i}kx}dx, \tag{2.13}$$

$$-{\rm i}\sqrt{k_0^2-k^2}\,(U_1(k)+U_3(k)) = \int_{-\infty}^{\infty} \frac{\partial}{\partial y}u^{\rm sc}(x,+0)e^{{\rm i}kx}dx. \tag{2.14}$$

Note that due to the radiation condition, the Fourier images of the field on the x-axis and its normal derivative are connected via the relation

$${\rm i}\sqrt{k_0^2-k^2}\int_{-\infty}^{\infty} u^{\rm sc}(x,0)e^{{\rm i}kx}dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial y}u^{\rm sc}(x,+0)e^{{\rm i}kx}dx. \tag{2.15}$$

Here the branch of the square root is chosen as follows. The square root is equal to k_0 at $k = 0$, and it is continuous along the contour passing below the branch point $k = k_0$ and above the point $k = -k_0$.

Substituting (2.13) and (2.14) into (2.15), one can write the following *functional equation:*

$$U_0(k) + U_1(k) + U_2(k) + U_3(k) + U_4(k) = 0 \qquad (2.16)$$

for all real k.

The functional equation should be considered together with the restrictions imposed on the behaviour of the unknown functions in the complex plane k. As it follows from the definition (2.12), the spectral functions satisfy the conditions:

- the function U_0 is regular in the lower half-plane k;
- the function U_4 is regular in the upper half-plane with the exception of the point $k = k_*$, where there is a pole with the known residue;
- the functions U_2, $\sqrt{k_0^2 - k^2}U_1$, $\sqrt{k_0^2 - k^2}U_3$ are entire on the whole k plane.

Note that k_0 belongs to the upper half-plane and $-k_0$ belongs to the lower half-plane. It means that the properties mentioned above bring some useful information concerning the behaviour of the unknown functions at the branch points.

Perform the estimation of growth of the unknown functions at infinity. Let the following estimations of the total field near the edges be valid:

$$u(\rho_m, \theta_m) = \frac{2}{\sqrt{\pi}} \left[C_m \rho_m^{1/2} \sin \frac{\theta_m}{2} + O(\rho_m^{3/2}) \right], \qquad m = 1 \dots 4, \qquad (2.17)$$

where C_m are some unknown constants; ρ_m and θ_m are the local polar coordinates introduced in Fig. 3. The asymptotics (2.17) are taken from the exact solution of the Sommerfeld half-line problem.

Using (2.17) one can estimate the growth of the unknown functions U_m. Namely, the asymptotics for the upper half-plane ($0 < \mathrm{Arg}[k] < \pi$), $|k| \to \infty$ are the following:

$$\begin{aligned}
U_1(k) &= e^{\mathrm{i}a_1 k} \left[\mathrm{i}C_1 (-\mathrm{i}k)^{-1/2} (k_0^2 - k^2)^{-1/2} + O(|k|^{-5/2}) \right], \\
U_2(k) &= e^{\mathrm{i}a_2 k} \left[C_2 (-\mathrm{i}k)^{-3/2} + O(|k|^{-5/2}) \right], \\
U_3(k) &= e^{\mathrm{i}a_3 k} \left[\mathrm{i}C_3 (-\mathrm{i}k)^{-1/2} (k_0^2 - k^2)^{-1/2} + O(|k|^{-5/2}) \right], \\
U_4(k) &= e^{\mathrm{i}a_4 k} \left[C_4 (-\mathrm{i}k)^{-3/2} + O(|k|^{-5/2}) \right].
\end{aligned} \qquad (2.18)$$

The asymptotics for the lower half-plane ($-\pi < \mathrm{Arg}[k] < 0$), $|k| \to \infty$ are the following:

$$\begin{aligned}
U_0(k) &= e^{\mathrm{i}a_1 k} \left[C_1 (\mathrm{i}k)^{-3/2} + O(|k|^{-5/2}) \right], \\
U_1(k) &= e^{\mathrm{i}a_2 k} \left[\mathrm{i}C_2 (\mathrm{i}k)^{-1/2} (k_0^2 - k^2)^{-1/2} + O(|k|^{-5/2}) \right], \\
U_2(k) &= e^{\mathrm{i}a_3 k} \left[C_3 (\mathrm{i}k)^{-3/2} + O(|k|^{-5/2}) \right], \\
U_3(k) &= e^{\mathrm{i}a_4 k} \left[\mathrm{i}C_4 (\mathrm{i}k)^{-1/2} (k_0^2 - k^2)^{-1/2} + O(|k|^{-5/2}) \right].
\end{aligned} \qquad (2.19)$$

Here and below we assume that the function k^α is regular on the plane k cut along the real negative half-axis. If α is real, then the values of the function for real positive k are real positive.

This fact is not easy to prove, but the equation (2.16) and the analytic restrictions listed above are enough to determine the unknown functions. We shall call the set of the functional equation, conditions of analyticity and the estimations of growth a *functional problem.*

Formulate also the functional problems for the auxiliary diffraction problems. For each $m = 1 \dots 4$ introduce the unknown spectral functions

$$\begin{aligned}
U_0^m(k) &= \int_{-\infty}^{a_1} u^m(x,0)e^{\mathrm{i}kx}dx, \\
U_1^m(k) &= \frac{\mathrm{i}}{\sqrt{k_0^2-k^2}} \int_{a_1}^{a_2} \frac{\partial u^m}{\partial y}(x,+0)e^{\mathrm{i}kx}dx, \\
U_2^m(k) &= \int_{a_2}^{a_3} u^m(x,0)e^{\mathrm{i}kx}dx, \\
U_3^m(k) &= \frac{\mathrm{i}}{\sqrt{k_0^2-k^2}} \int_{a_3}^{a_4} \frac{\partial u^m}{\partial y}(x,+0)e^{\mathrm{i}kx}dx, \\
U_4^m(k) &= \int_{a_4}^{\infty} u^m(x,0)e^{\mathrm{i}kx}dx,
\end{aligned}$$

where the divergent integrals are implied to be properly regularized.

The following functional equations similar to (2.16) are valid for each m :

$$U_0^m(k) + U_1^m(k) + U_2^m(k) + U_3^m(k) + U_4^m(k) = 0. \tag{2.20}$$

Conditions of analyticity similar to those formulated in the previous subsection are valid:

- the functions U_0^m are regular in the lower half-plane;
- the functions U_4^m are regular in the upper half-plane;
- the functions U_2^m, $\sqrt{k_0^2-k^2}U_1^m$, $\sqrt{k_0^2-k^2}U_3^m$ are entire on the whole k plane.

Note that there is no singularity at $k = k_*$.

Using (2.11), derive a set of asymptotic estimations for the upper half-plane ($0 < \mathrm{Arg}[k] < \pi$):

$$\begin{aligned}
U_1^m(k) &= \mathrm{i}e^{\mathrm{i}a_1k}(k_0^2-k^2)^{-1/2}\left[\delta_{m,1}(-\mathrm{i}k)^{1/2} + C_1^m(-\mathrm{i}k)^{-1/2} + O(k^{-3/2})\right], \\
U_2^m(k) &= e^{\mathrm{i}a_2k}\left[-\delta_{m,2}(-ik)^{-1/2} + C_2^m(-\mathrm{i}k)^{-3/2} + O(k^{-5/2})\right], \\
U_3^m(k) &= \mathrm{i}e^{\mathrm{i}a_3k}(k_0^2-k^2)^{-1/2}\left[\delta_{m,3}(-\mathrm{i}k)^{1/2} + C_3^m(-\mathrm{i}k)^{-1/2} + O(k^{-3/2})\right], \\
U_4^m(k) &= e^{\mathrm{i}a_4k}\left[-\delta_{m,4}(-\mathrm{i}k)^{-1/2} + C_4^m(-\mathrm{i}k)^{-3/2} + O(k^{-5/2})\right];
\end{aligned} \tag{2.21}$$

and for the lower half-plane ($-\pi < \mathrm{Arg}[k] < 0$):

$$
\begin{aligned}
U_0^m(k) &= e^{\mathrm{i}a_1 k}\left[-\delta_{m,1}(\mathrm{i}k)^{-1/2} + C_1^m(\mathrm{i}k)^{-3/2} + O(k^{-5/2})\right], \\
U_1^m(k) &= \mathrm{i}e^{\mathrm{i}a_2 k}(k_0^2 - k^2)^{-1/2}\left[\delta_{m,2}(\mathrm{i}k)^{1/2} + C_2^m(\mathrm{i}k)^{-1/2} + O(k^{-3/2})\right], \\
U_2^m(k) &= e^{\mathrm{i}a_3 k}\left[-\delta_{m,3}(\mathrm{i}k)^{-1/2} + C_3^m(\mathrm{i}k)^{-3/2} + O(k^{-5/2})\right], \\
U_3^m(k) &= \mathrm{i}e^{\mathrm{i}a_4 k}(k_0^2 - k^2)^{-1/2}\left[\delta_{m,4}(\mathrm{i}k)^{1/2} + C_4^m(\mathrm{i}k)^{-1/2} + O(k^{-3/2})\right].
\end{aligned}
\tag{2.22}
$$

The functional equations (2.20), analiticity retrictions and the estimations of growth form the set of four functional problems for the auxiliary spectral functions U_n^m.

Using the inverse Fourier transform and taking into account (2.14) one can reconstruct the scattered field in the half-plane $y > 0$:

$$
u^{\mathrm{sc}}(x, y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}(U_1(k) + U_3(k))e^{\mathrm{i}kx + \mathrm{i}\sqrt{k_0^2 - k^2}y}dk. \tag{2.23}
$$

The spectral functions are naturally connected with the directivities of wave fields. Introduce the functions

$$
S(k, k_*) \equiv \sqrt{k_0^2 - k^2}\,(U_1(k) + U_3(k)), \qquad S^m(k) \equiv \sqrt{k_0^2 - k^2}\,(U_1^m(k) + U_3^m(k)) \tag{2.24}
$$

Using a standard technique, one can prove that the leading term of the wave fields in a far-field zone can be represented as cylindrical waves:

$$
u^{\mathrm{sc}}(R, \varphi) \approx -\frac{e^{\mathrm{i}k_0 R - \mathrm{i}\pi/4}}{\sqrt{2\pi k_0 R}}S(-k_0\cos\varphi, k_*), \qquad u^m(R, \varphi) \approx -\frac{e^{\mathrm{i}k_0 R - \mathrm{i}\pi/4}}{\sqrt{2\pi k_0 R}}S(-k_0\cos\varphi), \tag{2.25}
$$

where R is the (large) distance between the origin and the observation point; φ is the observation angle defined in Fig. 1.

3. Main theorems for functional problems

3.1 *A property of Wronsky-type determinants*

LEMMA 1. Let the functions $V_n^m(k)$, $m = 1\ldots4$, $n = 0\ldots4$, of the complex variable k have the following properties:

1.all functions are analytic on the real axis;

2.the functional equations

$$
V_0^m(k) + V_1^m(k) + V_2^m(k) + V_3^m(k) + V_4^m(k) = 0 \tag{3.1}
$$

are valid for all real k and $m = 1\ldots4$;

3.the functions V_0^m are regular in the lower half-plane; the functions V_4^m are regular in the upper half-plane; the functions V_2^m, $\sqrt{k_0^2 - k^2}V_1^m$, $\sqrt{k_0^2 - k^2}V_3^m$ are entire on the whole k plane.

4.the following asymptotic estimations are valid in the upper half-plane:

$$\begin{aligned} V_1^m(k) &= \mathrm{i}e^{\mathrm{i}a_1 k}(k_0^2-k^2)^{-1/2}\left[E_1^m(-\mathrm{i}k)^{1/2}+O(k^{-1/2})\right], \\ V_2^m(k) &= e^{\mathrm{i}a_2 k}\left[-E_2^m(-\mathrm{i}k)^{-1/2}+O(k^{-3/2})\right], \\ V_3^m(k) &= \mathrm{i}e^{\mathrm{i}a_3 k}(k_0^2-k^2)^{-1/2}\left[E_3^m(-\mathrm{i}k)^{1/2}+O(k^{-1/2})\right], \\ V_4^m(k) &= e^{\mathrm{i}a_4 k}\left[-A_4^m(-\mathrm{i}k)^{-1/2}+O(k^{-3/2})\right], \end{aligned} \tag{3.2}$$

and the following asymptotic estimations are valid in the lower half-plane:

$$\begin{aligned} V_0^m(k) &= e^{\mathrm{i}a_1 k}\left[-E_1^m(\mathrm{i}k)^{-1/2}+O(k^{-3/2})\right], \\ V_1^m(k) &= \mathrm{i}e^{\mathrm{i}a_2 k}(k_0^2-k^2)^{-1/2}\left[E_2^m(\mathrm{i}k)^{1/2}+O(k^{-1/2})\right], \\ V_2^m(k) &= e^{\mathrm{i}a_3 k}\left[-E_3^m(\mathrm{i}k)^{-1/2}+O(k^{-3/2})\right], \\ V_3^m(k) &= \mathrm{i}e^{\mathrm{i}a_4 k}(k_0^2-k^2)^{-1/2}\left[E_4^m(\mathrm{i}k)^{1/2}+O(k^{-1/2})\right] \end{aligned} \tag{3.3}$$

for some constants E_n^m.

Then

$$\begin{vmatrix} V_1^1 & V_1^2 & V_1^3 & V_1^4 \\ V_2^1 & V_2^2 & V_2^3 & V_2^4 \\ V_3^1 & V_3^2 & V_3^3 & V_3^4 \\ V_4^1 & V_4^2 & V_4^3 & V_4^4 \end{vmatrix} = \frac{\gamma(k)}{k^2-k_0^2}\begin{vmatrix} E_1^1 & E_1^2 & E_1^3 & E_1^4 \\ E_2^1 & E_2^2 & E_2^3 & E_2^4 \\ E_3^1 & E_3^2 & E_3^3 & E_3^4 \\ E_4^1 & E_4^2 & E_4^3 & E_4^4 \end{vmatrix}, \tag{3.4}$$

where $\gamma(k) = \exp\{\mathrm{i}k(a_1+a_2+a_3+a_4)\}$.

Proof. Denote the determinants in the left and in the right of (3.4) by the symbols $|V(k)|$ and $|E|$, respectively. Study the properties of the determinant $|V(k)|$ in the upper half-plane. Multiply the raws of the determinant by the functions $e^{-\mathrm{i}a_1 k}(k_0^2-k^2)^{1/2}$, $e^{-\mathrm{i}a_2 k}$, $e^{-\mathrm{i}a_3 k}(k_0^2-k^2)^{1/2}$, $e^{-\mathrm{i}a_4 k}$, respectively. As the result, we obtain the determinant $\gamma^{-1}(k_0^2-k^2)|V(k)|$. All its elements are regular functions in the upper half-plane. The determinant grows no faster than a constant at infinity there. This constant can be found by studying the asymptotics of the elements:

$$\gamma^{-1}(k_0^2-k^2)|V(k)| \sim -|E| + O(k^{-1}) \qquad \text{as} \quad |k|\to\infty. \tag{3.5}$$

Now study the behaviour of $|V(k)|$ in the lower half-plane. Using the functional equations (3.1) and the general properties of the determinants, we can write $|V(k)|$ in the form

$$|V(k)| = \begin{vmatrix} V_0^1 & V_0^2 & V_0^3 & V_0^4 \\ V_1^1 & V_1^2 & V_1^3 & V_1^4 \\ V_2^1 & V_2^2 & V_2^3 & V_2^4 \\ V_3^1 & V_3^2 & V_3^3 & V_3^4 \end{vmatrix}. \tag{3.6}$$

This representation can be continued into the lower half-plane. Multiply the raws of the determinant by the functions $e^{-ia_1 k}$, $e^{-ia_2 k}(k_0^2 - k^2)^{1/2}$, $e^{-ia_3 k}$, $e^{-ia_4 k}(k_0^2 - k^2)^{1/2}$, respectively. Note that all elements of the resulting determinant are regular in the lower half-plane and the determinant grows no faster than a constant there.

Thus, the function $\gamma^{-1}(k_0^2 - k^2)|V(k)|$ is analytic on the whole k-plane. Moreover, the estimation (3.5) is valid both in the lower and in the upper half-plane. The identity (3.4) follows then from the Liouville's theorem ■

The determinant $|V(k)|$ can be called a *Wronsky-type determinant,* since the properties of the functions V_n^m are reminiscent of the properties of solutions for ordinary differential equations.

Instead of considering the set of 20 functions V_n^m in Lemma 1, one can consider 16 functions, say V_n^m for $m, n = 1 \dots 4$, and define the rest 4 functions V_0^m using the functional equations (3.1). In this case it is possible to say that a determinant itself satisfies the conditions of Lemma 1. Note also that the conditions of Lemma 1 can be checked separately for each column of a determinant.

3.2 *Embedding formula*

THEOREM 1. The following *embedding formula* is valid:

$$U_n(k, k_*) = \frac{\sqrt{k_0^2 - k_*^2}}{k - k_*} \sum_{m=1}^{4} (-1)^{m-1} U_n^m(k) \left(U_1^m(-k_*) + U_3^m(-k_*) \right), \tag{3.7}$$

for $n = 0 \dots 4$.

COROLLARY. Using (3.7) and (2.25) one can write the embedding formula for the directivities:

$$S(k, k_*) = \frac{1}{k - k_*} \sum_{m=1}^{4} (-1)^{m-1} S^m(-k_*) S^m(k). \tag{3.8}$$

Proof. Let us find the functions $Q_m(k, k_*)$, such that

$$U_n(k, k_*) = \sum_{m=1}^{4} Q_m(k, k_*) U_n^m(k) \qquad \text{for} \quad n = 1 \dots 4. \tag{3.9}$$

Note that due to the functional equations (2.16) and (2.20), a similar relation should be valid for the rest spectral functions:

$$U_0(k, k_*) = \sum_{m=1}^{4} Q_m(k, k_*) U_0^m(k). \tag{3.10}$$

The equations (3.9) can be solved as a system of linear algebraic equations with respect to Q_m using the Cramer's rule:

$$Q_1 = \frac{D_1}{D}, \qquad Q_2 = \frac{D_2}{D}, \qquad Q_3 = \frac{D_3}{D}, \qquad Q_4 = \frac{D_4}{D}, \tag{3.11}$$

where $D_1 \dots D_4$ and D are Wronsky-type determinants:

$$D = \begin{vmatrix} U_1^1 & U_1^2 & U_1^3 & U_1^4 \\ U_2^1 & U_2^2 & U_2^3 & U_2^4 \\ U_3^1 & U_3^2 & U_3^3 & U_3^4 \\ U_4^1 & U_4^2 & U_4^3 & U_4^4 \end{vmatrix}, \quad D_1 = \begin{vmatrix} U_1 & U_1^2 & U_1^3 & U_1^4 \\ U_2 & U_2^2 & U_2^3 & U_2^4 \\ U_3 & U_3^2 & U_3^3 & U_3^4 \\ U_4 & U_4^2 & U_4^3 & U_4^4 \end{vmatrix}, \quad D_2 = \begin{vmatrix} U_1^1 & U_1 & U_1^3 & U_1^4 \\ U_2^1 & U_2 & U_2^3 & U_2^4 \\ U_3^1 & U_3 & U_3^3 & U_3^4 \\ U_4^1 & U_4 & U_4^3 & U_4^4 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} U_1^1 & U_1^2 & U_1 & U_1^4 \\ U_2^1 & U_2^2 & U_2 & U_2^4 \\ U_3^1 & U_3^2 & U_3 & U_3^4 \\ U_4^1 & U_4^2 & U_4 & U_4^4 \end{vmatrix}, \quad D_4 = \begin{vmatrix} U_1^1 & U_1^2 & U_1^3 & U_1 \\ U_2^1 & U_2^2 & U_2^3 & U_2 \\ U_3^1 & U_3^2 & U_3^3 & U_3 \\ U_4^1 & U_4^2 & U_4^3 & U_4 \end{vmatrix}. \tag{3.12}$$

Fix the variable k_* and consider determinants D, $D_1 \dots D_4$ as the functions of the variable k. The determinant D satisfies the conditions of Lemma 1. Studying the asymptotics of its elements, we conclude that

$$D(k) = \frac{\gamma(k)}{k^2 - k_0^2}. \tag{3.13}$$

The determinants D_m do not satisfy the conditions of Lemma 1, since their elements have simple poles at $k = k_*$. Multiply the m-th column of the determinant D_m by $(k - k_*)$. The resulting determinant satisfies the conditions of Lemma 1. Studying the asymptotics of its elements, we conclude that

$$D_m(k) = \frac{\mathrm{i}(-1)^{m-1} C_m \gamma(k)}{(k^2 - k_0^2)(k - k_*)} \qquad m = 1 \dots 4. \tag{3.14}$$

(the constants C_m are taken from (2.18)), and therefore

$$U_n(k, k_*) = \frac{\mathrm{i}}{k - k_*} \sum_{m=1}^{4} (-1)^{m-1} C_m(k_*) U_n^m(k),, \qquad n = 0 \dots 4, \tag{3.15}$$

This equation is the *weak form* of the embedding formula, since the functions $C_m(k_*)$ remain unknown.

Express the values C_m in terms of the auxiliary functions U_n^m. Use the reciprocity principle for this. Let the scatterer be illuminated by a point source of the unit strength having the polar coordinates R, ψ (see Fig. 4a). The value R is large enough; the source is assumed to be located at the far-field zone. In this case the scattered field is almost equal to the field generated by plane-wave incidence. The asymptotic of the total field at the observation point located near the edge is the following:

$$u(\rho, \theta) \approx -\frac{\mathrm{i} C_m \sqrt{\rho}\, e^{\mathrm{i}k_0 R - \mathrm{i}\pi/4} \sin\theta/2}{\pi\sqrt{2k_0 R}}. \tag{3.16}$$

Consider the inverse situation, i.e. the source is located near the edge and the observation point is far from the scatterer (see Fig. 4b). The approximation for the field can be found using the second Green's formula. We omit the calculations (they are rather standard) and write the final result:

$$u(R, \psi) \approx -\frac{\sqrt{\epsilon}\, e^{\mathrm{i}k_0 R - \mathrm{i}\pi/4}}{\pi\sqrt{2k_0 R}} \sqrt{k_0^2 - k_*^2} (U_1^m(-k_*) + U_3^m(-k_*)), \tag{3.17}$$

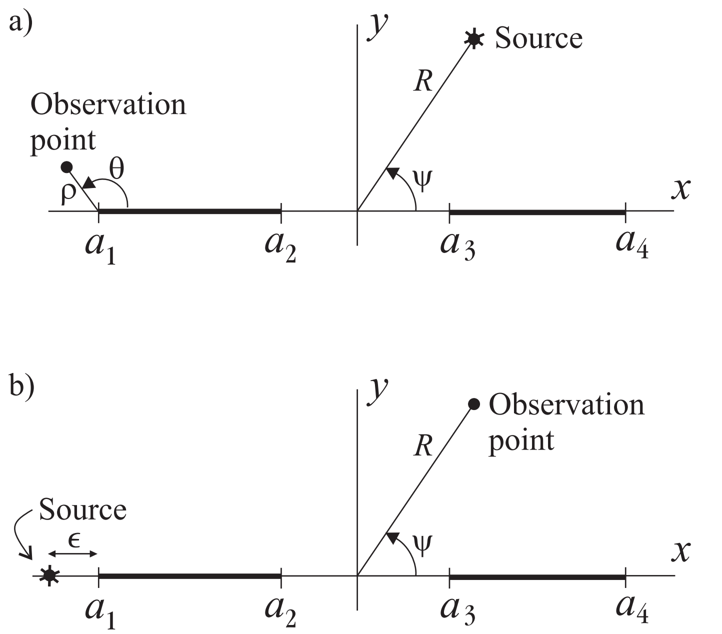

Fig. 4 To the application of the reciprocity principle

where again $k_* = k_0 \cos \psi$.

The reciprocity theorem states that the formulae (3.16) and (3.17) with $\theta = \pi$ and $\rho = \epsilon$ should give the same values. Taking the limit $R \to \infty$, $\epsilon \to 0$ we obtain the exact formula

$$C_m = -\mathrm{i}\sqrt{k_0^2 - k_*^2}(U_1^m(-k_*) + U_3^m(-k_*)). \tag{3.18}$$

Substituting (3.18) into (3.15), we obtain the *strong form* of the embedding formula (3.7) ■

It is obvious how the formula (3.7) can be generalized on the case of more than 2 complanar strips: the summation should be performed over all edges of the system.

According to the embedding formula, one needs to find the auxiliary functions depending on a *single* variable k, to have the possibility to calculate the far-field diagram for each pair (k, k_*).

Here we obtained the formula (3.18) using the reciprocity principle. The same formula can be derived using only the technique of Wronsky-type determinants. However, this consideration is rather sophisticated and we don't describe it here.

Earlier the embedding formula was obtained in (**5**) for a single strip and in (**9**) for several strips, so this result is basically not new. The main difference between the forms of embedding formulae is the choice of the auxiliary functions (or basis). All of them can be transformed one into another by linear transformations. The set of the auxiliary functions chosen in the present paper appears to be very convenient when the spectral equations are derived. All other sets lead to more complicated coefficients.

Compare our result with the one obtained in (**9**). Introduce the functions

$$H(k, k_*) = (k + k_*)S(k, -k_*),$$

which are proportional to those introduced in (**9**). The basis taken in (**9**) for the embedding formula was composed of the directivities for 4 different "reference" angles of incidence. Namely, let k_1, k_2, k_3, k_4 be the set of arbitrary reference angles not equal to each other. Then, as it is follows from the embedding formula (3.8),

$$H(k, k_*) = \sum_{j=1}^{4} \lambda_j(k_*) H(k, k_j), \tag{3.19}$$

where λ_m are defined from the equations

$$S^m(k_*) = \sum_{j=1}^{4} \lambda_j(k_*) S^m(k_j) \tag{3.20}$$

The formula (3.19) coincides with (3.49) from (**9**) up to the change $H(k, k_*) = H(k_*, k)$, which follows from the reciprocity principle. The equations for λ_j were formulated in (**9**) in a different manner:

$$H(k_m, k_*) = \sum_{j=1}^{4} \lambda_j(k_*) H(k_m, k_j).$$

However, this system follows directly from (3.19) and can be an alternative to (3.20).

3.3 *Spectral equation for the auxiliary functions*

The embedding formula expresses the unknown spectral functions U_n in terms of the auxiliary functions U_n^m. Unfortunately, there is no compact representation for U_n^m. However, in this section we prove that these functions obey an ordinary differential equation (ODE) with respect to the variable k; the coefficients of this equation are rather simple rational functions of k. Thus, the auxiliary functions $U_n^m(k)$ can be found numerically by solving the corresponding equation with appropriate initial conditions.

Introduce the matrix notation for the auxiliary functions:

$$\mathbf{U}(k) = \begin{pmatrix} U_1^1 & U_2^1 & U_3^1 & U_4^1 \\ U_1^2 & U_2^2 & U_3^2 & U_4^2 \\ U_1^3 & U_2^3 & U_3^3 & U_4^3 \\ U_1^4 & U_2^4 & U_3^4 & U_4^4 \end{pmatrix}.$$

Denote the differentiation with respect to k by prime.

THEOREM 2. The matrix $\mathbf{U}$ obeys the following *spectral equation:*

$$\mathbf{U}' = \mathbf{K}\mathbf{U}. \tag{3.21}$$

The matrix of the coefficients $\mathbf{K}$ has the form

$$\mathbf{K}(k) = \begin{pmatrix} \mathrm{i}a_1 & 0 & 0 & 0 \\ 0 & \mathrm{i}a_2 & 0 & 0 \\ 0 & 0 & \mathrm{i}a_3 & 0 \\ 0 & 0 & 0 & \mathrm{i}a_4 \end{pmatrix} + \frac{1}{k - k_0}\mathbf{K}^+ + \frac{1}{k + k_0}\mathbf{K}^-, \tag{3.22}$$

where $\mathbf{K}^+$ and $\mathbf{K}^-$ are some matrices not depending on k.

Proof. Consider formally the equation (3.21) as a system of linear algebraic equations with respect to the elements $K_{m,n}$ of the matrix $\mathbf{K}$. Solve this system formally using the Cramer's rule. The result is

$$K_{m,n} = D_{m,n}(k)/D(k), \tag{3.23}$$

where the determinant D is given by (3.12) and each determinant $D_{m,n}$ is obtained by replacing the n-th column of D by the derivative of the m-th column. For example,

$$D_{2,3} = \begin{vmatrix} U_1^1 & U_1^2 & (U_1^2)' & U_1^4 \\ U_2^1 & U_2^2 & (U_2^2)' & U_2^4 \\ U_3^1 & U_3^2 & (U_3^2)' & U_3^4 \\ U_4^1 & U_4^2 & (U_4^2)' & U_4^4 \end{vmatrix}.$$

All determinants $D_{m,n}$ belong to the Wronsky type, but the conditions of Lemma 1 are not fulfilled for the n-th column, since its elements have the singularities at the points $\pm k_0$ stronger than that is allowed by conditions of Lemma 1. However, this can be overcome by a simple trick. Denote the columns of the determinant D by $\mathbf{d}_m$, $m = 1 \dots 4$. The n-th column of the determinant $D_{m,n}$ has the form $\mathbf{d}'_m$. Construct the determinant $D^*_{m,n}$, whose n-th column has the form

$$(k^2 - k_0^2)\mathbf{d}'_m - (\mathrm{i}a_m k^2 - k/2)\mathbf{d}_m - \sum_{l=1}^{4}(-1)^{l-1}(a_m - a_l)C_l^m k\, \mathbf{d}_l$$

and 3 other columns are the same as of $D_{m,n}$. By studying the singularities and the behaviour at infinity one can check that the new column satisfies the conditions of Lemma 1, therefore

$$D^*_{m,n} = \mathrm{const} \times D(k).$$

On the other hand, by construction

$$D^*_{m,n} = (k^2 - k_0^2)D_{m,n} - [(\mathrm{i}a_m k^2 - k/2)\delta_{m,n} + (-1)^{n-1}(a_m - a_n)C_n^m k]D.$$

Comparing these two forms and substituting the result into (3.23), we obtain the formula (3.22) with the additional relation

$$K^+_{m,n} + K^-_{m,n} = -\frac{\delta_{m,n}}{2} + (-1)^{n-1}(a_m - a_n)C_n^m, \tag{3.24}$$

where $K^\pm_{m,n}$ are the elements of the matrices $\mathbf{K}^\pm$ ■

The coefficients of the equation (3.21) are known up to 32 constant parameters. Using the analyticity and growth restrictions for the functions U_n^m, one can formulate the eigenvalue problem with global monodromy restrictions for finding the unknown constants. One can show that the number of links is equal to the number of unknown constants (a similar result was obtained in (**6**)). The analyticity and growth restrictions can be used to specify the functions U_n^m among the solutions of the equation (3.21). However, this method seems to be too complicated for practical calculations. In (**7**) a simple and effective approximate technique based on the diffraction series is developed for a single strip problem. A similar technique will be developed for the case of several strips for practical calculations in Sections 4,5.

Equation (3.21) with the coefficient (3.22) is the most important equation related to the problem of diffraction by several strips. Two main reasons exist for saying that this equation is a generalization of the Wiener-Hopf method. First, we found the functions, to which the Liouville theorem is applicable. These functions are the Wronsky-type determinants D and $D_{m,n}$. The application of the Liouville theorem is the key point of the Wiener-Hopf method. Second, the method developed here can be applied to the classical Sommerfeld problem of diffraction by an ideal half-plane. It leads to a degenerated ODE of order 1. The coefficient K in this case is equal to the logarithmic derivative of the unknown spectral function. The solution coincides with that obtained by the Wiener-Hopf method.

3.4 *Evolution equations*

THEOREM 3. The following *evolution equations* describe the dependance of U_n^m on the edge coordinates a_m, $m = 1 \dots 4$:

$$\frac{\partial}{\partial a_j}\mathbf{U} = \mathbf{A}^j \mathbf{U}, \tag{3.25}$$

where the elements of the matrices $\mathbf{A}^j$ have the form

$$A^j_{m,n}(k) = \mathrm{i}k\delta_{j,m}\delta_{m,n} + (-1)^{n-1} C_n^m (\delta_{j,m} - \delta_{j,n}). \tag{3.26}$$

Proof. The equations (3.25) can be formally solved with respect to the matrices $\mathbf{A}^j$. Each element $A^j_{m,n}$ of the matrix $\mathbf{A}^j$ can be represented as a ratio of the determinants

$$A^j_{m,n} = D^j_{m,n}(k)/D(k),$$

where $D^j_{m,n}$ is the determinant obtained from D by replacing the n-th column by the derivative of the m-th column with respect to a_j. For example,

$$D^j_{m,n} = \begin{vmatrix} U_1^1 & U_1^2 & (U_1^2)_{,a_j} & U_1^4 \\ U_2^1 & U_2^2 & (U_2^2)_{,a_j} & U_2^4 \\ U_3^1 & U_3^2 & (U_3^2)_{,a_j} & U_3^4 \\ U_4^1 & U_4^2 & (U_4^2)_{,a_j} & U_4^4 \end{vmatrix}.$$

The column $(\mathbf{d}_m)_{,a_j}$ can violate the conditions of Lemma 1 due to the growth at infinity. Replace the n-th column of $D^j_{m,n}$ by the column

$$(\mathbf{d}_m)_{,a_j} - \mathrm{i}k\delta_{j,m}\delta_{m,n}\mathbf{d}_m.$$

The resulting determinant is equal to

$$D^j_{m,n} - \mathrm{i}k\delta_{j,m}\delta_{m,n} D$$

and it satisfies the conditions of Lemma 1. Studying the leading terms of the asymptotics of its elements at infinity, we obtain the relation (3.26) ■

COROLLARY. The following nonlinear differential equations are valid for $K^{\pm}_{m,n}$:

$$\frac{\partial K^{\pm}_{m,n}}{\partial a_j} = \pm \mathrm{i} k_0 K^{\pm}_{m,n}(\delta_{m,j} - \delta_{n,j}) + \frac{(K^+_{m,j} + K^-_{m,j})K^{\pm}_{j,n}}{a_j - a_m} + \frac{(K^+_{j,n} + K^-_{j,n})K^{\pm}_{m,j}}{a_n - a_j} +$$

$$\sum_{l \neq j} \frac{(K^+_{j,l} + K^-_{j,l})K^{\pm}_{l,n}}{a_j - a_l} + \sum_{l \neq j} \frac{(K^+_{l,j} + K^-_{l,j})K^{\pm}_{m,l}}{a_l - a_j}. \tag{3.27}$$

Here we assume that the second term in the r.-h.s. of (3.27) is equal to zero if $m = j$, and the third term is equal to zero if $n = j$.

Proof. Differentiate $\mathbf{U}$ with respect to k and a_j in two different ways:

$$\frac{\partial^2 \mathbf{U}}{\partial k \partial a_j} = \left(\frac{\partial \mathbf{K}}{\partial a_j} + \mathbf{K}\mathbf{A}^j \right) \mathbf{U} = \frac{\partial^2 \mathbf{U}}{\partial a_j \partial k} = \left(\frac{\partial \mathbf{A}^j}{\partial k} + \mathbf{A}^j \mathbf{K} \right) \mathbf{U}.$$

Multiply this equation by $\mathbf{U}^{-1}$. Note that the determinant of this matrix is not zero almost everywhere. This yields

$$\frac{\partial \mathbf{K}}{\partial a_j} = \frac{\partial \mathbf{A}^j}{\partial k} + [\mathbf{A}^j, \mathbf{K}]. \tag{3.28}$$

Note that using the relation (3.24) the element of the matrices $\mathbf{A}^j$ can be expressed trought the elements of $\mathbf{K}^{\pm}$:

$$A^j_{m,n}(k) = \mathrm{i} k \delta_{j,m} \delta_{m,n} + \frac{(K^+_{m,n} + K^-_{m,n})(\delta_{j,n} - \delta_{j,m})}{a_n - a_m}, \tag{3.29}$$

where the second term is equal to zero if $m = n$.

Substitute (3.29) into (3.28). Expand the result into the partial fraction, i.e. rewrite (3.28) in the form

$$(\dots)k^1 + (\dots)k^0 + (\dots)(k - k_0)^{-1} + (\dots)(k + k_0)^{-1} = 0,$$

where the expressions in the parentheses do not depend k (in fact, they are some ugly expressions containing a_m and $K^{\pm}_{m,n}$). Each expression should be equal to zero separately. One can check directly that the first two terms are equal to zero identically. The second two terms yield the equation (3.27) ∎

4. Diffraction series approach. Preliminary steps

The theorems proved in Section 3 are difficult to be used for finding the diffraction field because the coefficients of the spectral and evolution equations contain unknown parameters. These paprameters can be found by solving a complicated eigenvalue problem. Here we develop another technique enabling us to derive the coefficients in the form of the asymptotic series. We use the diffraction (or Schwarzschild's) series for this.

In this section we assume that k_0 has a positive imaginary part enough to establish the convergence of all series below. It is not an unusual procedure in the diffraction theory to prove some theorem for the imaginary k_0 and then to continue these results analytically onto the real axis. Moreover, the convergence of the most important series was established by Schwarzschild even for real k_0.

We remind that $0 < \psi < \pi/2$ and $\mathrm{Im}[k_*] > 0$.

4.1 *Solution of the diffraction problem in terms of the diffraction series. Diffraction indices and diffraction terms*

Return to the diffraction problem formulated in Subsection 2.1. Consider the diffraction process as a series of successive acts of diffraction by the edges of the structure. Each act of diffraction is described by the Sommerfeld theory, i.e. it is treated as the diffraction by an ideal half-line. We assume that two successive acts of diffraction can occur only at the adjacent edges, i.e. the points a_m and a_n, where $n = m \pm 1$.

Introduce the *diffraction indices* denoting the sequences of diffraction acts. Let a diffraction index be a sequence of symbols from the set $\{1, 2, 3, 4\}$, in which the difference between adjacent symbols is equal to ± 1. The index 1232123 is valid, and the index 124 is invalid. The first symbol from the left corresponds to the first act of diffraction, the last symbol corresponds to the last act. The main parameter of a diffraction index is its *order*, namely, its length minus 1. For example, the order of the index 1234 is equal to 3. We shall denote the diffraction indexes by small Greek letters, and the elements of the indexes (the symbols from the set $\{1, 2, 3, 4\}$) by small Latin letters.

Represent the scattered field $u^{\rm sc}(x, y)$ for $y > 0$ in the form

$$u^{\rm sc} = u_4 + u_{43} + u_{432} + u_{434} + \dots \tag{4.1}$$

where the diffraction terms $u_\alpha(x, y)$ obey the Helmholtz equation, Sommerfeld radiation conditions, Meixner edge conditions (i.e. the energy is integrable near the edges) and some boundary conditions.

The boundary conditions are connected with the recursive procedure of calculation of the diffraction terms. The first term having the order zero is specified by the following boundary conditions:

$$u_4(x, 0) = -e^{-{\rm i}k_* x} \quad \text{for} \quad x < a_4, \qquad \frac{\partial u_4}{\partial y}(x, 0) = 0 \quad \text{for} \quad x > a_4, \tag{4.2}$$

For each other term having the index α define the preceding term, the index for which is obtained by truncating α by one symbol from the right. The diffraction terms are calculated recursively; each diffraction term is the solution of a correspondent Sommerfeld half-line problem. The boundary conditions for each term are inhomogeneous and the right-hand side is presented by either the field or its normal derivative of the preceding term.

Note that there are 6 types of Sommerfeld problems, since here we distinguish the cases when the incidence wave approaches the edge from right and from left, e.g., the recursive relations, say, for the terms $u_{\alpha 32}$ and $u_{\alpha 12}$ have different form.

There is only one type of the Sommerfeld problem related to edge a_1. Recursive boundary conditions are the following:

$$u_{\alpha 1}(x, 0) = 0 \quad \text{for} \quad x > a_1, \qquad \frac{\partial u_{\alpha 1}}{\partial y}(x, 0) = -\frac{\partial u_\alpha}{\partial y}(x, 0) \quad \text{for} \quad x < a_1.$$

There are two cases of diffraction by the edge a_2. Namely, the incident wave can come from the edge a_1:

$$u_{\alpha 12}(x, 0) = 0 \quad \text{for} \quad x < a_2, \qquad \frac{\partial u_{\alpha 12}}{\partial y}(x, 0) = -\frac{\partial u_{\alpha 1}}{\partial y}(x, 0) \quad \text{for} \quad x > a_2$$

or from the edge a_3:

$$u_{\alpha 32}(x,0) = -u_{\alpha 3}(x,0) \quad \text{for} \quad x < a_2, \qquad \frac{\partial u_{\alpha 32}}{\partial y}(x,0) = 0 \quad \text{for} \quad x > a_2.$$

Analogously,

$$u_{\alpha 23}(x,0) = -u_{\alpha 2}(x,0) \quad \text{for} \quad x > a_3, \qquad \frac{\partial u_{\alpha 23}}{\partial y}(x,0) = 0 \quad \text{for} \quad x < a_3,$$

$$u_{\alpha 43}(x,0) = 0 \quad \text{for} \quad x > a_3, \qquad \frac{\partial u_{\alpha 43}}{\partial y}(x,0) = -\frac{\partial u_{\alpha 4}}{\partial y}(x,0) \quad \text{for} \quad x < a_3,$$

$$u_{\alpha 4}(x,0) = 0 \quad \text{for} \quad x < a_4, \qquad \frac{\partial u_{\alpha 4}}{\partial y}(x,0) = -\frac{\partial u_{\alpha}}{\partial y}(x,0) \quad \text{for} \quad x > a_2.$$

Everywhere α denotes the preceding symbols of the corresponding diffraction index.

One can check directly that the sum of all diffraction terms specified by the boundary conditions listed above satisfies the boundary conditions (2.4), (2.5).

4.2 *Explicit formulae for diffraction terms*

Use the Wiener-Hopf technique to find the diffraction terms. Introduce the following notations for the spectra of the diffraction terms:

$$u_\alpha(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_\alpha(k) e^{-\mathrm{i}kx + \mathrm{i}\sqrt{k_0^2 - k^2}\, y} dk. \tag{4.3}$$

Comparing (4.3) with (4.1), we conclude that the angular spectrum of the scattered field is the sum of all terms W_α with all valid diffraction indexes α starting with 4. We shall denote this sum by W_{4-}:

$$W_{4-} = W_4 + W_{43} + W_{432} + W_{434} + \dots$$

Thus,

$$u^{\mathrm{sc}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_{4-}(k) e^{-\mathrm{i}kx + \mathrm{i}\sqrt{k_0^2 - k^2}\, y} dk \tag{4.4}$$

Note that

$$W_{4-}(k) = -(U_1(k) + U_3(k)),$$

where U_1 and U_3 are the spectral functions introduced in Section 3. The dependance of W_{4-} on k_* is implied. In some places below we shall indicate this dependence explicitly.

Using the standard procedure of the Wiener-Hopf method (**12**), we can represent the diffraction terms in recursive form:

$$W_4(k) = \frac{\mathrm{i}}{(k - k_*)} \frac{\beta_4(k)}{\beta_4(k_*)}, \tag{4.5}$$

$$W_{\alpha m} = -\beta_m F_\varepsilon[\beta_m^{-1} W_\alpha], \tag{4.6}$$

where

$$\beta_1(k) = \frac{\sqrt{\mathrm{i}}\, e^{\mathrm{i}a_1 k}}{\sqrt{k_0 - k}}, \quad \beta_2(k) = -\frac{\sqrt{\mathrm{i}}\, e^{\mathrm{i}a_2 k}}{\sqrt{k_0 + k}}, \quad \beta_3(k) = \frac{\sqrt{\mathrm{i}}\, e^{\mathrm{i}a_3 k}}{\sqrt{k_0 - k}}, \quad \beta_4(k) = -\frac{\sqrt{\mathrm{i}}\, e^{\mathrm{i}a_4 k}}{\sqrt{k_0 + k}}; \quad (4.7)$$

F_ε is either F_+ or F_-; the operators F_+ and F_- perform the additive decomposition of a function into the terms, regular and decaying in the upper and lower half-plane respectively:

$$\begin{aligned} F_+[V](k) &= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{V(\tau)}{\tau - k} d\tau \qquad \text{for } \mathrm{Im}[k] > 0, \\ F_-[V](k) &= -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{V(\tau)}{\tau - k} d\tau \qquad \text{for } \mathrm{Im}[k] < 0; \end{aligned} \quad (4.8)$$

the symbol ε here and below stands for "+" or "−". The sign "+" is chosen if m is equal to the last symbol of α plus 1, and the sign "−" is chosen if m is equal to the last symbol of α minus 1.

One can note that the expressions for the diffraction terms are rather complicated. The term of order n (i.e. with the index containing $n+1$ symbol) is represented by n integrals with Cauchy's kernel.

4.3 *Auxiliary terms G_α. Connections with the spectral functions introduced is Section 3*

Introduce a set of auxiliary functions G_α, depending only on the variable k and having the diffraction-styled indexes. The auxiliary functions are defined by the following recursive relations:

$$G_m(k) = \beta_m(k), \quad (4.9)$$

$$G_{\alpha m} = \beta_m F_\varepsilon \left[\beta_m^{-1} G_\alpha \right]. \quad (4.10)$$

one can note that the definition of functions G is close to (4.6), the exception is the first term and the sign.

Establish the connections between the spectral functions introduced above and in Section 2. Compare the boundary conditions for the terms u_α with the definition of the functions U_n. One can easily prove that

$$U_0 = W_{4-1}, \qquad U_1 = -(W_{4-1} + W_{4-2}), \qquad U_2 = W_{4-2} + W_{4-3},$$
$$U_3 = -(W_{4-3} + W_{4-4}), \qquad U_4 = W_{4-4}, \quad (4.11)$$

where W_{m-n} is the infinite sum sum of all diffraction terms W_α with α starting with m and ending with n. Obviously,

$$W_{4-} = W_{4-1} + W_{4-2} + W_{4-3} + W_{4-4}.$$

Analogously, taking into account the edge behaviour of the functions u^m and comparing it with the definition of G_α, one concludes that

$$U_0^m = -G_{m-1}, \qquad U_1^m = G_{m-1} - G_{m-2}, \qquad U_2^m = G_{m-2} - G_{m-3},$$
$$U_3^m = G_{m-3} - G_{m-4}, \qquad U_4^m = G_{m-4}, \quad (4.12)$$

where G_{m-n} is the sum of all values G_α with the index α starting with m and ending with n.

5. Main results for diffraction series

5.1 *Formulae for transforming diffraction terms and series*

LEMMA 2. Let the function $V(k)$ decay as $|k| \to \infty$ for real k.
a. For any ξ not lying on the real axis

$$F_{\pm}\left[\frac{V(\tau)}{\tau-\xi}\right](k) = \frac{F_{\pm}[V](k)}{k-\xi} + \frac{\mathcal{F}_{\pm}[V,\xi]}{k-\xi}, \tag{5.1}$$

where the functionals $\mathcal{F}_{\pm}$ are defined as follows:

$$\mathcal{F}_{\pm}[V,\xi] = \mp\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{V(\tau)}{\tau-\xi}d\tau. \tag{5.2}$$

b. If all integrals below exist, then

$$F_{\pm}[\tau V(\tau)](k) = kF_{\pm}[V](k) + \mathcal{C}_{\pm}[V], \tag{5.3}$$

where

$$\mathcal{C}_{\pm}[V] = \pm\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}V(\tau)d\tau.$$

Formula (a) follows from the obvious relation

$$\frac{1}{(\tau-\xi)(\tau-k)} = \frac{1}{k-\xi}\left[\frac{1}{\tau-k} - \frac{1}{\tau-\xi}\right];$$

formula (b) follows from the relation

$$\frac{\tau}{\tau-k} = \frac{k}{\tau-k} + 1.$$

One can note that the values $\mathcal{F}_{\pm}[V(k),\xi]$ and $\mathcal{C}_{\pm}$ do not depend on k. The functional $\mathcal{C}_{\pm}$ is related to the asymptotics of the function $F_{\pm}[V]$. Namely,

$$\mathcal{C}_{\pm}[V] = \lim_{|k|\to\infty}\left[\mp\frac{k}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\frac{V(\tau)}{\tau-k}d\tau,\right]$$

i.e. if $F_{+}[V](k)$ or $F_{-}[V](k)$ has the asymptotics $bk^{-1} + o(k^{-1})$ at infinity in the upper or lower half plane, respectively, then $\mathcal{C}_{\pm}[V] = -b$. The formula (5.3) can be treated as a particular case of (5.1) with $\xi = \infty$.

We remind also the elementary properties of the operators $F_{\pm}$:

$$V = F_{+}[V] + F_{-}[V] \tag{5.4}$$

and

$$(F_{\pm}[V])' = F_{\pm}[V'], \qquad \left(\,' = \frac{d}{dk}\right). \tag{5.5}$$

The property (5.4) follows from the Cauchi's theorem, and the formula (5.5) can be proved by integration by parts.

We are going to formulate also the property of the sums, whose form is typical for the series below. Introduce the following notations: Let $s(\alpha, n)$ be the diffraction index that consists of the first $n+1$ symbols of α; $t(\alpha, n)$ be the diffraction index consisting of the last $n+1$ symbols of α; $N(\alpha)$ be the order of the index α.

LEMMA 3. Let some values d_α, h_α and z_α have diffraction-style indices. Let z_{m-n}, d_{m-n}, h_{m-n} denote the series of values z_α (d_α, h_α, respectively) over all diffraction indices α starting with m and ending with n. Let the following formula be valid for each α:

$$z_\alpha = \sum_{m=0}^{n} d_{s(\alpha,m)} h_{t(\alpha,n-m)}, \qquad n = N(\alpha). \tag{5.6}$$

Then

$$z_{m-n} = \sum_{j=1}^{4} d_{m-j} h_{j-n} \tag{5.7}$$

if all series are absolutely convergent.

This lemma can be proved by regrouping the terms of the series in (5.7). Note that the structure of indices in (5.6) is reminiscent of convolution. The statement of Lemma 3 can be considered as an analog of the theorems related to convolution under Fourier transform.

5.2 *Embedding formula*

Introduce the functions $f_\alpha(k)$ as follows. Let f_α have diffraction indices α. Let f_α be not defined for the indices of zero order, and let for all other indices it be defined by the formula

$$f_{\alpha m}(k) = \mathcal{F}_\varepsilon[\beta_m^{-1} G_\alpha, k], \tag{5.8}$$

where the sign ε depends on m and the last symbol of α according to the rule introduced above.

Let f_{m-n} be the sum of all functions f_α, having the index of non-zero order starting with m and ending with n, e.g.

$$f_{1-1} = f_{121} + f_{12321} + f_{12121} + \dots$$

The series for f_{m-n} is asymptotic; its approximate value can be calculated by truncation. Let $\mathbf{f}(k)$ be the matrix having the elements f_{m-n} (m is the first index, n is the second one).

THEOREM 1'. The following relation is valid

$$W_{4-n} = \frac{\mathrm{i}(-1)^n}{\beta_4(k_*)(k-k_*)} \sum_{l=1}^{4} g_{4-m}(k_*) G_{m-n}(k), \tag{5.9}$$

where $g_{m-n}(k_*)$ are the elements of the matrix $\mathbf{g}(k_*)$:

$$\mathbf{g}(k_*) = (\mathbf{I} - \mathbf{f}(k_*))^{-1}. \tag{5.10}$$

The relation (5.9) is analogous to the the embedding formula (3.7).

Proof. Applying Lemma 2 (a), one can prove by induction the general formula

$$W_\alpha = \frac{\mathrm{i}(-1)^n}{\beta_4(k_*)(k-k_*)} \sum_{m=0}^{n} g_{s(\alpha,m)}(k_*) G_{t(\alpha,n-m)}(k), \qquad n = N(\alpha) \tag{5.11}$$

where the coefficients g are defined by the recursive relations:

$$g_m \equiv 1 \qquad \text{for} \quad m = 1 \dots 4 \tag{5.12}$$

$$g_{\alpha m}(k_*) = \sum_{j=0}^{n} g_{s(\alpha,j)}(k_*) \mathcal{F}_\varepsilon[\beta_m^{-1} G_{t(\alpha,n-j)}, k_*]. \qquad n = N(\alpha). \tag{5.13}$$

The induction is carried with respect to the order of the index α. Apply (5.11) to each term of the series for W_{4-n}. Using Lemma 3 obtain (5.9).

The expression (5.9) is equivalent to the embedding formula in the weak form (see (3.15)). Simplify the functions $g_{m-n}(k_*)$. Using the notations for f_α, rewrite (5.11) in the form

$$g_\alpha(k_*) = \sum_{j=0}^{n-1} g_{s(\alpha,j)}(k_*) f_{t(\alpha,n-j)}(k_*), \qquad n = N(\alpha). \tag{5.14}$$

Applying Lemma 3 obtain the following relation:

$$g_{m-n}(k_*) = \delta_{m,n} + \sum_{l=1}^{4} g_{m-l}(k_*) f_{l-n}(k_*), \tag{5.15}$$

Solving this equation with respect to the values g_{m-n} obtain the relation (5.10) ■

Further simplification of the coefficients g_{m-n} is possible in terms of the diffraction series approach, but here for simplicity we prefer to compare the equation (5.9) with (3.7) and write down the final result

$$g_{4-m}(k_*) = \tag{5.16}$$

$$-\mathrm{i}(-1)^{m-1}\beta_4(k_*)\sqrt{k_0^2-k_*^2}(G_{m-1}(-k_*) - G_{m-2}(-k_*) + G_{m-3}(-k_*) - G_{m-4}(-k_*)).$$

Note that equation (5.9) is *exact* (at least for the values k_0 providing the convergence of all series), although it has been obtained using the diffraction series technique.

5.3 *Spectral equation*

Let $\mathbf{G}(k)$ be a matrix 4×4, whose elements are the functions $G_{m-n}(k)$ (m corresponds to the first index, n corresponds to the second one). Let be

$$\mathbf{Y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

THEOREM 2'. The following equation is valid

$$\mathbf{G}' = \mathbf{K}\mathbf{G} \tag{5.17}$$

The matrix of the coefficients $\mathbf{K}$ has the form (3.22) with $\mathbf{K}^{\pm}$ equal to

$$\begin{aligned} \mathbf{K}^{+} &= -\frac{1}{2}(\mathbf{I} - \mathbf{f}(k_0))\mathbf{Y}(\mathbf{I} - \mathbf{f}(k_0))^{-1}, \\ \mathbf{K}^{-} &= -\frac{1}{2}(\mathbf{I} - \mathbf{f}(-k_0))(\mathbf{I} - \mathbf{Y})(\mathbf{I} - \mathbf{f}(-k_0))^{-1}, \end{aligned} \tag{5.18}$$

The equation (5.17) is equivalent to the spectral equation (3.21); the matrix $\mathbf{K}$ in (5.17) is equal to the matrix $\mathbf{K}$ in (3.21).

Proof. Due to the relations (4.12) the functions G_{m-n} are the linear combinations of U_n^m, therefore if we find the linear ODE for $\mathbf{G}$ having the form (5.17) with the matrix $\mathbf{K}$ having the appropriate structure, we can conclude that this $\mathbf{K}$ should coincide with (3.22).

Differentiate $\mathbf{G}$ with respect to k. Note that the coefficients β_m obey the relations:

$$\beta_m'(k) = \left[\mathrm{i}a_m - \frac{1}{2(k \pm k_0)} \right] \beta_m(k), \tag{5.19}$$

where the sign "+" is chosen for $m = 2, 4$ and "−" is chosen for $m = 1, 3$.

First, it is easy to prove by induction and using Lemma 2(a) that the following recursive relations are valid for each term G_α:

$$G_\alpha'(k) = \mathrm{i}a_j G_\alpha(k) + \sum_{m=0}^{n} \left(\frac{P_{s(\alpha,m)}}{k - k_0} + \frac{M_{s(\alpha,m)}}{k + k_0} \right) G_{t(\alpha,n-m)}(k), \qquad n = N(\alpha) \tag{5.20}$$

where j is the first symbol of α; P_α and M_α are the coefficients obeying the recursive relations:

$$P_m = -\frac{1}{2}\,\mathrm{Odd}[m] \qquad M_m = -\frac{1}{2}\,\mathrm{Even}[m], \tag{5.21}$$

$$P_{\alpha m} = \sum_{j=0}^{n} P_{s(\alpha,j)} \mathcal{F}_\varepsilon[\beta_m^{-1} G_{t(\alpha,n-j)}, k_0] + \frac{\mathrm{Odd}[m]}{2} \mathcal{F}_\varepsilon[(\beta_m)^{-1} G_\alpha, k_0], \qquad n = N(\alpha), \tag{5.22}$$

$$M_{\alpha m} = \sum_{j=0}^{n} M_{s(\alpha,j)} \mathcal{F}_\varepsilon[\beta_m^{-1} G_{t(\alpha,n-j)}, -k_0] + \frac{\mathrm{Even}[m]}{2} \mathcal{F}_\varepsilon[(\beta_m)^{-1} G_\alpha, -k_0], \qquad n = N(\alpha); \tag{5.23}$$

$\mathrm{Odd}[m] = 1$ for odd m, $\mathrm{Odd}[m] = 0$ for even m, $\mathrm{Even}[m] = 1 - \mathrm{Odd}[m]$.

Second, consider the derivatives G_{m-n}'. Using Lemma 3 one can obtain:

$$G_{m-n}' = \mathrm{i}a_m G_{m-n} + \sum_{j=1}^{4} \left(\frac{P_{m-j}}{k - k_0} + \frac{M_{m-j}}{k + k_0} \right) G_{j-n}, \tag{5.24}$$

where P_{m-n} and M_{m-n} are the sums of the terms of P_α and M_α respectively over all indices α starting with m and ending with n.

The equation (5.24) can be rewritten in the matrix form (5.17). Comparing the coefficients conclude that

$$K^+_{m,n} = P_{m-n}, \qquad K^-_{m,n} = M_{m-n}. \tag{5.25}$$

Simplify the formulae for the matrices P_{m-n} and M_{m-n}. Using the notations introduced above and applying Lemma 3, we can find that

$$\begin{aligned} P_{m-n} &= \frac{\mathrm{Odd}[n]}{2}(f_{m-n}(k_0) - \delta_{m,n}) + \sum_{l=1}^{4} P_{m-l} f_{l-n}(k_0), \\ M_{m-n} &= \frac{\mathrm{Even}[n]}{2}(f_{m-n}(-k_0) - \delta_{m,n}) + \sum_{l=1}^{4} M_{m-l} f_{l-n}(-k_0). \end{aligned} \tag{5.26}$$

The systems (5.26) can be solved with respect to the values P_{m-n} and M_{m-n}. Taking into account (5.25) we find the representations (5.18) ■

5.4 *Evolution equations*

Introduce the values

$$B^j_{\alpha m} = \mathrm{i}(\delta_{l,j} - \delta_{j,m})\,\mathcal{C}_\varepsilon[\beta_m^{-1} G_\alpha], \qquad B^j_m = 0, \tag{5.27}$$

l is the first symbol of α; the symbol ε is used according to the rules introduced above, i.e. it denotes "+" if the m is equal to the last symbol of α plus 1, and it denotes "−" if m is equal to the last sign of α minus 1.

Let B^j_{m-n} be the sums of values B^j_α over all indices α starting with m and ending with n.

THEOREM 3'. The following equations are valid

$$\frac{\partial}{\partial a_j} G_{m-n} = \mathrm{i}k\delta_{j,m} G_{m-n} + \sum_{l=1}^{4} B^j_{m-l} G_{l-n}. \tag{5.28}$$

The coefficients A^j_{m-n} of the evolution equations (3.25) are determined by the relation

$$A^j_{m-n} = \mathrm{i}k\delta_{j,m}\delta_{m,n} + B^j_{m-n}. \tag{5.29}$$

Proof. Differentiate the diffraction terms with respect to a_j and apply Lemma 2(b):

$$\frac{\partial}{\partial a_j} G_m = \mathrm{i}k\delta_{jm} G_m, \tag{5.30}$$

$$\frac{\partial}{\partial a_j} G_{\alpha m} = \beta_m F_\varepsilon\left[\beta_m^{-1}\frac{\partial}{\partial a_j} G_\alpha\right] - \mathrm{i}\delta_{m,j}\mathcal{C}_\varepsilon[\beta_m^{-1} G_\alpha] G_m \tag{5.31}$$

The formula (5.31) has a recursive form. Using (5.30) and (5.31) one can prove by induction that non-recursive form for the derivatives of the diffraction terms is:

$$\frac{\partial}{\partial a_j} G_\alpha = \mathrm{i} k \delta_{lj} G_\alpha + \sum_{v=0}^{n} B^j_{s(\alpha,v)} G_{t(\alpha,n-v)}, \qquad n = N(\alpha). \tag{5.32}$$

Applying Lemma 3 one can construct the derivative of the series G_{m-n} in the form (5.28).

Due to the linearity and the relations (4.12), we conclude that the equation

$$\frac{\partial}{\partial a_j} U^m_n = \mathrm{i} k \delta_{j,m} U^m_n + \sum_{l=1}^{4} B^j_{m-l} U^l_n$$

is valid. Comparing this with (3.25), we conclude that the identity (5.29) is valid.

Note also that

$$B^j_{m-n} = (-1)^{n-1} C^m_n (\delta_{j,m} - \delta_{j,n}) \blacksquare$$

6. Numerical results

Let us discuss the numerical aspects of the proposed theory. The computation procedure for such problems stronly depend on the relation between the wavelength and the geometrical sizes of the scatterer (the sizes are $\sim a$). The situation becomes simpler if waves are short comparatively to all (or even some of) sizes. In this case the methods based on the ray thechniques work very well. As the example we should mention the work (**10**), where the combined method utilizing the ray ideas and the approximate Wiener-Hopf method has been developed and very accurate results up to $k_0 a \sim 5$ have been obtained. However, these methods cannot be used when the wavelength has the same order or smaller than the sizes of the scatterer.

The traditional way of solving the diffraction problems with arbitrary wavenumber is the technique of boundary integral equations. However, specific numerical difficulties increase rapidly as $k_0 a$ increases, i.e. the situation is inverse to ray methods.

The current paper is devoted only to establishing the exact theorems concerning the diffraction problems. Applying these results to practical calculations can be an alternative to the traditional integral equations or ray method. Our theorems are valid for arbitrary $k_0 a$. We expect that the following procedure for applying our results can give good results:

- one should formulate the eigenvalue problem for the spectral equation (3.21) with respect to the elements of the matrices $\mathbf{K}^\pm$, taking into account the analytical restrictions on U^m_n;
- one should solve the eigenvalue problem numerically, say, using gradient methods; One can solve this eigenvalue problem for one particular combination of $a_1 \dots a_4$, and then obtain the values of $\mathbf{K}^\pm$ by solving numerically the evolution equations (3.27);
- using the values obtaining on the previous step, one can find the coefficiens and the initial values for the spectral equation (3.21); this equation should be solved numerically;
- using the embedding formula (3.8) or (3.7) one can find the spectral functions related to the plane wave incidence problem. The field then can be reconstructed using (2.23).

The implementation of the procedure described above is beyond the scope of the current paper. However, we would like to uillustrate to validity of at least some elements of our theory by a numerical example based on the technique developed in Sections 4 and 5. Namely, we are going to obtain the approximate (asymptotic) results by truncating the series standing in the right of (5.18).

Our purpose here is to demonstrate the validity of the embedding formula (3.8) and the spectral equation (3.21). We don't touch here the evolution equations, since the equations (3.27) are nonlinear, and the detailed analysis of the stability of their solutions should be performed. It can be the material for another paper.

To solve (3.21) we need to find the constant matrices $\mathbf{K}^{\pm}$ and to calculate U_n^m known at some reference point k_r to use them as the initial conditions. We use for this the technique of the diffraction series. Namely, the formulas (5.18) are used do evaluate the coefficients of the equation, and the formulae (4.12) are used to evaluate $U_n^m(k_r)$.

Numerical calculations were conducted as follows:

- The parameters of the problem were chosen as $k_0 = 1 + 0.2\mathrm{i}$, (this corresponds to the wavelength close to 6), $a_1 = -12$, $a_2 = -4$, $a_3 = 4$, $a_4 = 12$. The ratio of the wavelength to the width is close to 1. The angle of incidence is chosen as $\psi = \pi/2$. This corresponds to $k_* = 0$.
- The coefficients $\mathbf{K}^{\pm}$ of the equation (3.21) were estimated using the formula (5.18). The series f_{m-n} were truncated such that only the term with the indices of order 1 were left. Thus, the approximation for $\mathbf{f}$ was chosen as

$$\mathbf{f} \approx \begin{pmatrix} 0 & f_{12} & 0 & 0 \\ f_{21} & 0 & f_{23} & 0 \\ 0 & f_{32} & 0 & f_{34} \\ 0 & 0 & f_{43} & 0 \end{pmatrix}.$$

- The reference point k_r was chosen equal to 0. The initial data for the equation (3.21), namely, the values $U_n^m(0)$, are found using the formulae (4.12). The matrix $\mathbf{G}$ was estimated as the truncated series. We left only the terms with the order not higher than 1:

$$\mathbf{G} \approx \begin{pmatrix} G_1 & G_{12} & 0 & 0 \\ G_{21} & G_2 & G_{23} & 0 \\ 0 & G_{32} & G_3 & G_{34} \\ 0 & 0 & G_{43} & G_4 \end{pmatrix}.$$

- Using the coefficients and the initial data found on the previous steps, the equation (3.21) was solved numerically on the segment $k = (-80, 80)$.
- The functions $U_n^m(k)$ were substituted into the embedding formula (3.8). As the result, the function $U_1(k) + U_3(k)$ was obtained for $k_* = 0$. The absolute value of this function on the segment $k = (0, 1)$ is shown in Fig. 5.
- The check of the numerical solution obtained above has been performed using the Fourier transform. Namely, the field and its normal derivative was calculated for $y = +0$ numerically by using the formula (2.23). The numerical values of the function found on the previous step were substituted into this formula and the discrete Fourier transform was performed. The results are shown in Fig. 6,7,8. One can see that the field on the strips is approximately equal to $-u^{\mathrm{in}} (\equiv -1)$, and the normal derivative of the field

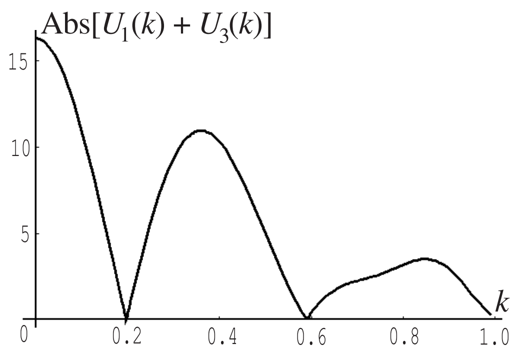

Fig. 5 The spectral function of the field

is approximately equal to zero on the segments $x \in (-\infty, a_1) \cup (a_2, a_3) \cup (a_4, \infty)$. It means that the boundary conditions (2.4), (2.5) are approximately fulfilled.

As the result of the conducted numerical procedure, we can conclude that the approximate method based on the spectral equation gives reasonable results, even being very rough. We should repeat that the proper numerical use of our theorems requires a more sophisticated procedure.

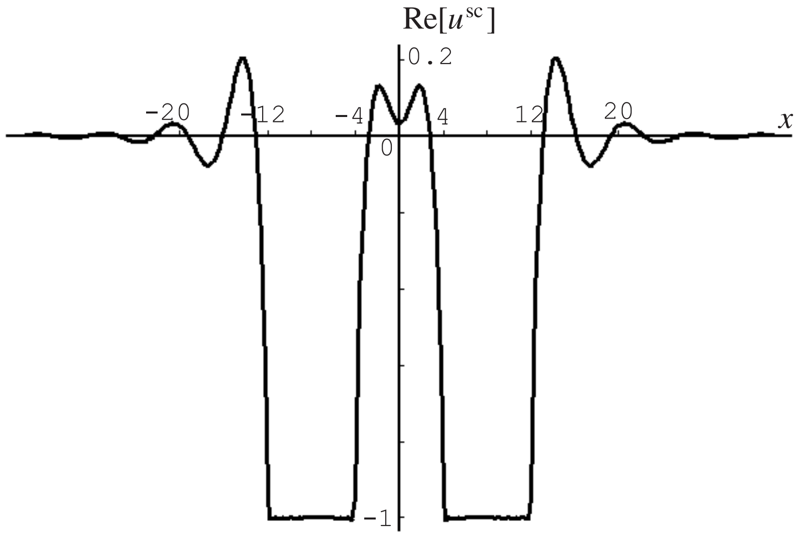

Fig. 6 The real part of the scattered field on the x-axis

7. Concluding remarks

The main result obtained in this paper is the spectral equation (3.21) having the rational coefficients known up to several constant parameters. Although this result does not provide

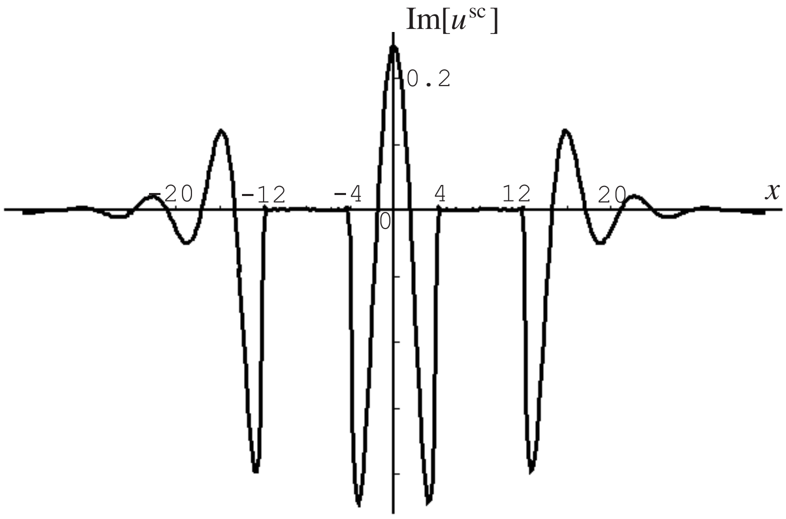

Fig. 7 The imaginary part of the scattered field on the x-axis

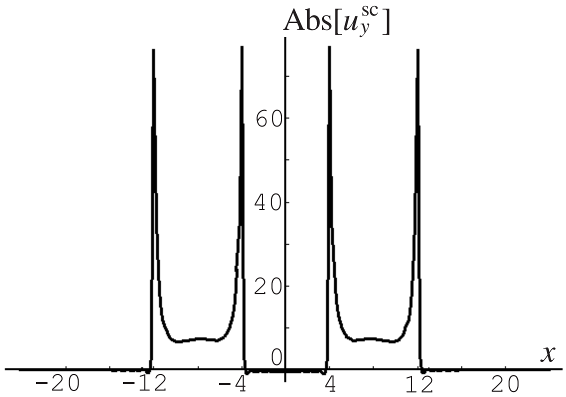

Fig. 8 The absolute value of the normal derivative of the scattered field on the x-axis

the solution of the diffraction problem in the closed form, it indicates the class of functions, in which the solution can be found. Moreover, this result establishes the link between the diffraction theory and the theory of confluent Fuchsian equations. In terms of this theory, the solution of the diffraction problems is found by introducing some *special functions* (namely, $U_n^m(k)$ or G_{m-n}). As it is typical for the special functions, an eigenvalue problem should be solved to determine the parameters of the equation. Some useful properties of new special functions are formulated as evolution equations. An effective method for calculations of the unknown constants and the initial data for the spectral equation is provided by the diffraction series technique.

Any set of the ideal strips located in one plane can be treated using our method.

All equations are generalized in obvious way. In the case of non-ideal strips having the impedance boundary conditions on their sides, the spectral equation cannot be derived, at least in the form (3.21), however, the embedding formula and the evolution equation remain valid with slightly changed coefficients.

Acknowledgements

Author is grateful to Prof. V.M.Babich, Prof. M.H.Williams and Dr. R.V.Craster for helpful discussion. The work was supported by the RFBR grants NN 00-15-96530, 01-02-06119.

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