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AN INITIAL VALUE TECHNIQUE FOR SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS WITH A NEGATIVE SHIFT

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Abstract. In this paper, a numerical method named as Initial Value Technique (IVT) is suggested to solve singularly perturbed boundary value problems for second order ordinary differential equations of reaction-diffusion type with a delay (negative shift). In this technique, the original problem of solving the second order differential equation is reduced to solving four first order singularly perturbed differential equations without delay and one algebraic equation with a delay. The singularly perturbed problems are solved by a second order hybrid finite difference scheme. An error estimate is derived by using supremum norm and it is of order $O(\varepsilon + N^{-2}\ln^2 N)$, where N is a discretization parameter and ε is the perturbation parameter. Numerical results are provided to illustrate the theoretical results.

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1. Introduction

Delay differential equations play an important role in the mathematical modeling of various practical phenomena in the subjects of bioscience and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. A subclass of these equations consist of singularly perturbed ordinary differential equations with a delay, that is an ordinary differential equation in which the highest derivative is multiplied by a small parameter, and involving at least one delay term. Such type of equations arise frequently in the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex [16], the study of bistable devices [2] and variational problems in control theory [10], etc.

It is well known that standard discretization methods for solving singular perturbation problems for differential equations are sometimes unstable and fail

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to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to solve this type of equations, whose accuracy does not depend on the parameter ε, that is the methods which are uniformly convergent with respect to the parameter ε.

In the past, only very few people had worked in the area of numerical methods on Singularly Perturbed Delay Differential Equations(SPDDEs). But in the recent years, there has been a growing interest in this area. In fact, Fevzi Erdogan [6] proposed an exponentially fitted operator method for singularly perturbed first order delay differential equation, Kadalbajoo and Sharma [12, 13, 14] and Mohapatra and Natesan [11] proposed some numerical methods for SPDDEs with a small delay. It may be noted that Lange and Miura [15] gave an asymptotic approximation to solve singularly perturbed second order delay differential equations. In the present paper a numerical method named as IVT is suggested to solve the following boundary value problems for second order ordinary differential equations of reaction-diffusion type with a negative shift (2.1). The initial value method was introduced by the authors Gasparo and Macconi [7, 8, 9]. In fact they applied this method to solve singularly perturbed boundary value problems for differential equations without negative shift(delay).

The present paper is organized as follows. In Section 2, the problem under study with continuous source term is stated. A maximum principle for the DDE is established in Section 3. Further a stability result is derived. An asymptotic expansion for the current problem is derived in Section 4. The present numerical method namely the Initial Value Technique (IVT) is described in Section 5, and an error estimate is derived in Section 6. Section 7 considers the problem with a discontinuous source term. Section 8 presents numerical results. The paper is concluded with a discussion (Section 9).

2. Statement of the problem

Throughout the paper C and C_1 denote generic positive constants. Further, the supremum norm is used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem:

$$\|u\|_D = \sup_{x\in D} \mid u(x) \mid .$$

Motivated by the work of [15], we consider the following Boundary Value Problem (BVP) for SPDDE.

Find $u \in Y = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x),\ x \in \Omega^- \cup \Omega^+, \\ u(x) = \phi(x),\ x \in [-1,0],\ u(2) = l, \end{cases} \tag{2.1}$$

where $a(x) \geq \alpha_1 > \alpha > 0$, $\beta_0 \leq b(x) \leq \beta < 0$, $\alpha_1 + \beta_0 > \eta > 0$, the functions $a,\ b,\ f$ are sufficiently smooth functions on $\overline{\Omega}$, [for example $a \in C^2(\overline{\Omega})$], $\Omega = (0,2)$, $\overline{\Omega} = [0,2]$, $\Omega^- = (0,1)$, $\Omega^+ = (1,2)$ and ϕ is smooth on $[-1,0]$. The above problem is equivalent to

$$(2.2)\qquad Pu(x):\ = \begin{cases} -\varepsilon u''(x) + a(x)u(x) = f(x) - b(x)\phi(x-1),\ x \in \Omega^-, \\ -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x),\ x \in \Omega^+, \end{cases}$$

$$u(0) = \phi(0),\ u(1-) = u(1+),\ u'(1-) = u'(1+),\ u(2) = l,$$

where $u(1-)$ and $u(1+)$ denote the left and right limits of u at $x = 1$, respectively. This BVP (2.1) exhibits boundary layers at $x = 0,\ x = 2$ and interior layers (left and right) at $x = 1$ [15]. Further, the above problem (2.1) has a solution [4].

3. Stability Result

The differential-difference operator P defined in (2.2) satisfies the following maximum principle.

Theorem 3.1. *(Maximum principle) Let* $w \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ *be any function satisfying* $w(0) \geq 0,\ w(2) \geq 0,\ Pw(x) \geq 0,\ \forall x \in \Omega^- \cup \Omega^+$ *and* $w'(1+) - w'(1-) = [w'](1) \leq 0$. *Then,* $w(x) \geq 0,\ \forall x \in \overline{\Omega}$.

Proof. Using the method of proof of Theorem 3.1 of [17] and the test function Φ given by,

$$\Phi(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & x \in [0,1], \\ \frac{3}{8} + \frac{x}{4}, & x \in [1,2], \end{cases}$$

the above theorem can be proved. □

Theorem 3.2. *(Stability Result) For any* $u \in Y$ *we have*

$$(3.1)\qquad |\ u(x)\ | \leq C\ \max\{|\ u(0)\ |, |\ u(2)\ |, \sup_{\xi \in \Omega^- \cup \Omega^+} |\ Pu(\xi)\ |\},\ \forall\ x \in \overline{\Omega}.$$

Proof. This theorem can be proved by using Theorem 3.1 and the barrier function $\psi^{\pm}(x) = C_1\Phi(x) \pm u(x),\ x \in \overline{\Omega}$. □

An immediate consequence of Theorem 3.2 is that the solution of the BVP (2.1) is unique.

4. An asymptotic expansion

Motivated by the works of [18, 19], an asymptotic expansion approximation for the solution of the problem (2.1) is constructed by using the fundamental idea of WKB method [1].

Let $u_0 \in C^0(\Omega^- \cup \Omega^+ \cup \{0,\ 2\})$ be the solution of the reduced problem of (2.1) given by

$$\begin{cases} a(x)u_0(x) + b(x)u_0(x-1) = f(x),\ \Omega^- \cup \Omega^+ \cup \{0,\ 2\}, \\ u_0(x) = \phi(x),\ x \in [-1,\ 0), \end{cases}$$

that is,

$$(4.1)\qquad u_0(x)=\begin{cases}\frac{f(x)-b(x)\phi(x-1)}{a(x)}, & x\in[0,1],\\ \frac{f(x)-b(x)[\frac{f(x-1)-b(x-1)\phi(x-2)}{a(x-1)}]}{a(x)}, & x\in(1,2],\end{cases}$$

and assume that $\| u_0'' \|_{\Omega^-\cup\Omega^+}\leq C$.

Further, let $s(x)=\exp(-\int_0^x\sqrt{\frac{a(s)}{\varepsilon}}ds)$, $\forall x\in\overline{\Omega}$, $t(x)=\exp(-\int_x^1\sqrt{\frac{a(s)}{\varepsilon}}ds)$, $\forall x\in[0,1]$, $v(x)=\exp(-\int_1^x\sqrt{\frac{a(s)}{\varepsilon}}ds)$, $\forall x\in[1,2]$ and $w(x)=\exp(-\int_x^2\sqrt{\frac{a(s)}{\varepsilon}}ds)$, $\forall x\in\overline{\Omega}$ be the solutions of the following problems, respectively:

$$(4.2)\qquad \begin{cases}\sqrt{\varepsilon}s'(x)+\sqrt{a(x)}s(x)=0, \ x\in(0,2],\\ s(0)=1,\end{cases}$$

$$(4.3)\qquad \begin{cases}\sqrt{\varepsilon}t'(x)-\sqrt{a(x)}t(x)=0, \ x\in[0,1),\\ t(1)=1,\end{cases}$$

$$(4.4)\qquad \begin{cases}\sqrt{\varepsilon}v'(x)+\sqrt{a(x)}v(x)=0, \ x\in(1,2],\\ v(1)=1\end{cases}$$

and

$$(4.5)\qquad \begin{cases}\sqrt{\varepsilon}w'(x)-\sqrt{a(x)}w(x)=0, \ x\in[0,2),\\ w(2)=1.\end{cases}$$

An asymptotic expansion approximation to the solution of the original problem (2.1) is given by

$$(4.6)\qquad u_{as}(x)=\begin{cases}u_0(x)+k_1\hat{s}(x)+k_2\hat{t}(x), & x\in[0,1),\\ u_0(x)+k_3\hat{v}^*(x)+k_4\hat{w}(x), & x\in[1,2],\end{cases}$$

where $\hat{s}(x)=[a(x)]^{-1/4}s(x)$, $\hat{t}(x)=[a(x)]^{-1/4}t(x)$, $\hat{v}(x)=[a(x)]^{-1/4}v(x)$, $\hat{w}(x)=[a(x)]^{-1/4}w(x)$ and

$$\hat{v}^*(x)=\begin{cases}0, \ x\in[0,1),\\ \hat{v}(x), \ x\in[1,2].\end{cases}$$

Here, the constants $k_1,\ k_{2,},\ k_3,$ and k_4 are to be determined so that $u_{as}\in Y$, $u_{as}(0)=\phi(0)$ and $u_{as}(2)=l$. In fact, the constants $k_1,\ k_2,\ k_3$ and k_4 are given by

$$(4.7)\qquad \begin{cases}k_1=\Big[[\phi(0)-u_0(0)]-k_2\hat{t}(0)\Big]a(0)^{1/4},\\ k_2=(B_1+k_3A_{12})/A_{11},\\ k_3=\dfrac{B_2A_{11}-B_1A_{21}}{A_{11}A_{22}+A_{12}A_{21}},\\ k_4=\Big[[l-u_0(2)]-k_3\hat{v}(2)\Big]a(2)^{1/4},\end{cases}$$

where

$$
\begin{aligned}
A_{11} &= a(1)^{-1/4} - a(0)^{1/4}\hat{t}(0)\hat{s}(1),\\
A_{12} &= a(1)^{-1/4} - a(2)^{1/4}\hat{v}(2)\hat{w}(1),\\
A_{21} &= a(1)^{1/4} + \hat{t}(0)\hat{s}(1)a(0)^{1/4}[\sqrt{\varepsilon}a^{-1}(1)a'(1)/4 + \sqrt{a}(1)]\\
&\quad - \sqrt{\varepsilon}a(1)^{-5/4}a'(1)/4,\\
A_{22} &= a(1)^{1/4} + \hat{v}(2)\hat{w}(1)a(2)^{1/4}[-\sqrt{\varepsilon}a^{-1}(1)a'(1)/4 + \sqrt{a}(1)]\\
&\quad + \sqrt{\varepsilon}a(1)^{-5/4}a'(1)/4,\\
B_1 &= (u_0(1+) - u_0(1-)) - (\phi(0) - u_0(0))a(0)^{1/4}\hat{s}(1)\\
&\quad + (l - u_0(2))\hat{w}(1)a(2)^{1/4},\\
B_2 &= \sqrt{\varepsilon}(u_0'(1+) - u_0'(1-)) + \hat{s}(1)a(0)^{1/4}(\phi(0) - u_0(0))\Big[\sqrt{\varepsilon}a^{-1}(1)a'(1)/4\\
&\quad + \sqrt{a}(1)\Big] + \hat{w}(1)a(2)^{1/4}(l - u_0(2))[-\sqrt{\varepsilon}a^{-1}(1)a'(1)/4 + \sqrt{a}(1)].
\end{aligned}
$$

It is easy to see that $\mid k_i \mid\leq C,\ i = 1(1)4.$

Theorem 4.1. *Let u be the solution of the problem (2.1) and u_{as} be its asymptotic expansion approximation defined by (4.6). Then*

$$\| u - u_{as} \|_{\overline{\Omega}} \leq C\varepsilon.$$

Proof. Consider the barrier function

$$
\varphi^{\pm}(x) = \begin{cases} C_1\varepsilon\Big[\Phi(x) + e^{-\sqrt{\frac{\alpha}{\varepsilon}}x} + e^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-x)} + 1 \\ \qquad + e^{-\sqrt{\frac{\alpha}{\varepsilon}}(2-x)}\Big] \pm u(x) - u_{as}(x), x \in [0,1], \\ C_1\varepsilon\Big[\Phi(x) + e^{-\sqrt{\frac{\alpha}{\varepsilon}}x} + 1 + \overline{v}^*(x) \\ \qquad + e^{-\sqrt{\frac{\alpha}{\varepsilon}}(2-x)}\Big] \pm u(x) - u_{as}(x), x \in [1,2], \end{cases}
$$

where

$$
\overline{v}^*(x) = \begin{cases} 0, \ x \in [0,1), \\ e^{-\sqrt{\frac{\alpha}{\varepsilon}}(x-1)}, \ x \in [1,2]. \end{cases}
$$

It is easy to see that $\varphi^{\pm} \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, $\varphi^{\pm}(0) > 0$ and $\varphi^{\pm}(2) > 0$.

Let $x \in \Omega^-$. First we see that

$$
\begin{aligned}
P(u(x) - u_{as}(x)) &= \varepsilon u_0''(x) + k_1\varepsilon\left[\frac{5(a'(x))^2}{(4a(x))^2} - \frac{a''(x)}{4a(x)}\right]\hat{s}(x)\\
&\quad + k_2\varepsilon\left[\frac{5(a'(x))^2}{(4a(x))^2} - \frac{a''(x)}{4a(x)}\right]\hat{t}(x)\\
&\geq -C\varepsilon\left[1 + e^{-\sqrt{\frac{\alpha}{\varepsilon}}x} + e^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-x)}\right],
\end{aligned}
$$

since $\| u_0'' \|_{\Omega^- \cup \Omega^+} \leq C$, $\hat{s}(x) \leq Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}x}$, $\hat{t}(x) \leq Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-x)}$ and a is a smooth function on $[0, 2]$. Then,

$$\begin{aligned} P\varphi^{\pm}(x) &= C_1\varepsilon\Big[[a(x)-\alpha]e^{-\sqrt{\frac{\alpha}{\varepsilon}}x} + [a(x)-\alpha]e^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-x)} \\ &\quad + [a(x)-\alpha]e^{-\sqrt{\frac{\alpha}{\varepsilon}}(2-x)}\Big] + C_1\varepsilon\,[a(x)[\Phi(x)+1]] \pm P(u(x)-u_{as}(x)) \\ &\geq C_1\varepsilon\left[[\alpha_1-\alpha]e^{-\sqrt{\frac{\alpha}{\varepsilon}}x} + [\alpha_1-\alpha]e^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-x)} + [\alpha_1-\alpha]e^{-\sqrt{\frac{\alpha}{\varepsilon}}(2-x)}\right] \\ &\quad + C_1\varepsilon\frac{9\alpha_1}{8} \mp C\varepsilon[1+e^{-\sqrt{\frac{\alpha}{\varepsilon}}x} + e^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-x)}]. \end{aligned}$$

Hence $P\varphi^{\pm}(x) \geq 0$ in Ω^- for a suitable choice of C_1.

Similarly, we can show that $P\varphi^{\pm}(x) \geq 0$ in Ω^+ and $[\varphi^{\pm'}](1) < 0$. Then, by Theorem 3.1 we have $\varphi^{\pm}(x) \geq 0,\ x \in \overline{\Omega}$, that is

$$| u(x) - u_{as}(x) | \leq C\varepsilon,\ x \in \overline{\Omega}.$$

Hence the proof of the theorem. □

5. Numerical methods

In this section, an hybrid finite difference scheme for the singularly perturbed problems (4.2), (4.3), (4.4) and (4.5) is described.

5.1. Mesh selection strategy

Since the BVP (2.1) exhibits strong boundary layers at $x = 0$, $x = 2$ and strong interior layers (left and right) at $x = 1$, we choose a piece-wise uniform Shishkin mesh on $[0, 2]$. For this we divide the interval $[0, 2]$ into six subintervals, namely $\Omega_1 = [0, \tau]$, $\Omega_2 = [\tau, 1-\tau]$, $\Omega_3 = [1-\tau, 1]$, $\Omega_4 = [1, 1+\tau]$, $\Omega_5 = [1+\tau, 2-\tau]$ and $\Omega_6 = [2-\tau, 2]$, where $\tau = \min\{0.25, \frac{2\sqrt{\varepsilon}\ln N}{\sqrt{\alpha}}\}$. Let $h = 4N^{-1}\tau$ and $H = 2N^{-1}(1-2\tau)$.

The mesh $\overline{\Omega}^{2N} = \{x_0,\ x_1,\ \ldots,\ x_{2N}\}$ is defined by

$$\begin{cases} x_0 = 0.0,\ x_i = x_0 + ih,\ i = 1(1)\frac{N}{4},\ x_{i+\frac{N}{4}} = x_{\frac{N}{4}} + iH,\ i = 1(1)\frac{N}{2}, \\ x_{i+\frac{3N}{4}} = x_{\frac{3N}{4}} + ih,\ i = 1(1)\frac{N}{4},\ x_{i+N} = x_N + ih,\ i = 1(1)\frac{N}{4}, \\ x_{i+\frac{5N}{4}} = x_{\frac{5N}{4}} + iH,\ i = 1(1)\frac{N}{2},\ x_{i+\frac{7N}{4}} = x_{\frac{7N}{4}} + ih,\ i = 1(1)\frac{N}{4}. \end{cases}$$

5.2. An hybrid finite difference scheme for the problems (4.2), (4.3), (4.4) and (4.5)

Applying the hybrid finite difference scheme given in [20] to the problems (4.2), (4.3), (4.4) and (4.5), we get

$$\text{(5.1)}\quad \begin{cases} L_1 S_i = \begin{cases} \sqrt{\varepsilon}\frac{S_i - S_{i-1}}{h} + \sqrt{a(\frac{x_i + x_{i-1}}{2})}\frac{S_i + S_{i-1}}{2} = 0,\ i = 1(1)\frac{N}{4}, \\ \sqrt{\varepsilon}\frac{S_i - S_{i-1}}{H} + \sqrt{a(x_i)}S_i, = 0\ i = \frac{N}{4} + 1(1)\frac{3N}{4}, \\ \sqrt{\varepsilon}\frac{S_i - S_{i-1}}{h} + \sqrt{a(x_i)}S_i = 0,\ i = \frac{3N}{4} + 1(1)N, \\ \sqrt{\varepsilon}\frac{S_i - S_{i-1}}{h} + \sqrt{a(x_i)}S_i = 0,\ i = N + 1(1)\frac{5N}{4}, \\ \sqrt{\varepsilon}\frac{S_i - S_{i-1}}{H} + \sqrt{a(x_i)}S_i = 0,\ i = \frac{5N}{4} + 1(1)\frac{7N}{4}, \\ \sqrt{\varepsilon}\frac{S_i - S_{i-1}}{h} + \sqrt{a(x_i)}S_i = 0,\ i = \frac{7N}{4} + 1(1)2N, \end{cases} \\ S_0 = 1, \end{cases}$$

(5.2)

$$\begin{cases} L_2 T_i = \begin{cases} \sqrt{\varepsilon}\frac{T_{i+1} - T_i}{h} - \sqrt{a(x_i)}T_i = 0,\ i = 0(1)\frac{N}{4}, \\ \sqrt{\varepsilon}\frac{T_{i+1} - T_i}{H} - \sqrt{a(x_i)}T_i = 0,\ i = \frac{N}{4} + 1(1)\frac{3N}{4}, \\ \sqrt{\varepsilon}\frac{T_{i+1} - T_i}{h} - \sqrt{a(\frac{x_{i+1} + x_i}{2})}\frac{T_{i+1} + T_i}{2} = 0,\ i = \frac{3N}{4} + 1(1)N - 1, \end{cases} \\ T_N = 1, \end{cases}$$

(5.3)

$$\begin{cases} L_3 V_i = \begin{cases} \sqrt{\varepsilon}\frac{V_i - V_{i-1}}{h} + \sqrt{a(\frac{x_i + x_{i-1}}{2})}\frac{V_{i-1} + V_i}{2} = 0,\ i = N + 1(1)\frac{5N}{4}, \\ \sqrt{\varepsilon}\frac{V_i - V_{i-1}}{H} + \sqrt{a(x_i)}V_i = 0,\ i = \frac{5N}{4} + 1(1)\frac{7N}{4}, \\ \sqrt{\varepsilon}\frac{V_i - V_{i-1}}{h} + \sqrt{a(x_i)}V_i = 0,\ i = \frac{7N}{4} + 1(1)2N, \end{cases} \\ V_N = 1 \end{cases}$$

and

(5.4)

$$\begin{cases} L_4 W_i = \begin{cases} \sqrt{\varepsilon}\frac{W_{i+1} - W_i}{h} - \sqrt{a(x_i)}W_i = 0,\ i = 0(1)\frac{N}{4}, \\ \sqrt{\varepsilon}\frac{W_{i+1} - W_i}{H} - \sqrt{a(x_i)}W_i = 0,\ i = \frac{N}{4} + 1(1)\frac{3N}{4}, \\ \sqrt{\varepsilon}\frac{W_{i+1} - W_i}{h} - \sqrt{a(x_i)}W_i = 0,\ i = \frac{3N}{4} + 1(1)N, \\ \sqrt{\varepsilon}\frac{W_{i+1} - W_i}{h} - \sqrt{a(x_i)}W_i = 0,\ i = N + 1(1)\frac{5N}{4}, \\ \sqrt{\varepsilon}\frac{W_{i+1} - W_i}{H} - \sqrt{a(x_i)}W_i = 0,\ i = \frac{5N}{4} + 1(1)\frac{7N}{4}, \\ \sqrt{\varepsilon}\frac{W_{i+1} - W_i}{h} - \sqrt{a(\frac{x_{i+1} + x_i}{2})}\frac{W_{i+1} + W_i}{2} = 0,\ i = \frac{7N}{4} + 1(1)2N - 1, \end{cases} \\ W_{2N} = 1. \end{cases}$$

The following theorem gives an error estimate for the above schemes.

Theorem 5.1. *Let $s(x)$, $t(x)$, $v(x)$ and $w(x)$ be the solutions of (4.2), (4.3), (4.4) and (4.5) respectively. Further, let $S = (S_0, S_1, \cdots, S_{2N})$, $T = (T_0, T_1, \cdots, T_N)$, $V = (V_N, \cdots, V_{2N})$ and $W = (W_0, W_1, \cdots, W_{2N})$ be their numerical solutions defined by (5.1), (5.2), (5.3) and (5.4) respectively. Then*

$$\| s - S \|_{\overline{\Omega}^{2N}} \le CN^{-2}\ln^2 N,\ \| t - T \|_{\overline{\Omega}^{2N}\cap[0,1]} \le CN^{-2}\ln^2 N,$$

$$\| v - V \|_{\overline{\Omega}^{2N}\cap[1,2]} \le CN^{-2}\ln^2 N,\ \| w - W \|_{\overline{\Omega}^{2N}} \le CN^{-2}\ln^2 N.$$

Proof. Note that, $\int_a^b \alpha(x)(x-a)^{(p-1)}dx \leq \frac{1}{p}\Big(\int_a^b (\alpha(x))^{1/p}dx\Big)^p$ which holds true for any positive monotonically decreasing function $\alpha(x)$ on $[a,b]$ and for an arbitrary $p \in \mathbb{N}$ [3, 21].

$$\begin{aligned}
\mid L_1(S_i - s(x_i)) \mid &\leq C\sqrt{\varepsilon} \mid \int_{x_{i-1}}^{x_i} (x-x_{i-1})s'''(x)dx \mid \\
&\quad + C \mid \int_{x_{i-1}}^{x_i} (x-x_{i-1})s''(x)dx \mid, \ i = 1(1)N/4 \\
&\leq C\varepsilon^{-1} \int_{x_{i-1}}^{x_i} (x-x_{i-1})\exp(-\sqrt{\alpha}x/\sqrt{\varepsilon})dx \\
&\leq C\varepsilon^{-1}\frac{1}{2}\Big(\int_{x_{i-1}}^{x_i} \exp(-\sqrt{\alpha}x/2\sqrt{\varepsilon})dx\Big)^2 \\
&\leq C\varepsilon^{-1}\Big(\tau N^{-1}\Big)^2 \exp(-\sqrt{\alpha}\eta_i^*/\sqrt{\varepsilon}) \text{ (using mean value theorem)} \\
&\leq CN^{-2}\ln^2 N,
\end{aligned}$$

where $\eta_i^* \in (x_{i-1}, x_i)$. Further,

$$\begin{aligned}
\mid L_1(S_i - s(x_i)) \mid &\leq C\sqrt{\varepsilon} \mid \int_{x_{i-1}}^{x_i} s''(x)dx \mid, \ i = N/4+1(1)2N \\
&\leq C\sqrt{\varepsilon}^{-1} \int_{x_{i-1}}^{x_i} \exp(-\sqrt{\alpha}x/\sqrt{\varepsilon})dx \\
&= -C\sqrt{\alpha}^{-1}\Big(\exp(-\sqrt{\alpha}x_i/\sqrt{\varepsilon}) - \exp(-\sqrt{\alpha}x_{i-1}/\sqrt{\varepsilon})\Big) \\
&= C\sqrt{\alpha}^{-1}\exp(-\sqrt{\alpha}x_{i-1}/\sqrt{\varepsilon})\Big(1 - \exp(-\sqrt{\alpha}H/\sqrt{\varepsilon})\Big) \\
&\leq CN^{-2}.
\end{aligned}$$

Then, by the stability result (Lemma 3. [20]), we have the desired result. That is,

$$\mid s(x_i) - S_i \mid \leq CN^{-2}\ln^2 N, \ i = 0(1)2N.$$

In the same way one can derive the results for the IVP (4.4). The TVPs (4.3) and (4.5) can be tackled by using the transformations $\tilde{x} = 1 - x$ and $\tilde{x} = 2 - x$, respectively. □

Let U_{0_i} be the values of the solution of the problem (4.1) at $x = x_i$ defined by

$$U_{0_i} = \begin{cases} \frac{f(x_i)}{a(x_i)} - \frac{b(x_i)}{a(x_i)}\phi(x_i - 1), \ i = 0(1)N, \\ \frac{f(x_i)}{a(x_i)} - \frac{b(x_i)}{a(x_i)}U_{0_{i-N}}, \ i = N+1(1)2N. \end{cases} \tag{5.5}$$

5.3. A numerical solution to the BVP (2.1)

A numerical solution to the original problem (2.1) is given by

$$U_i = \begin{cases} U_{0_i} + k_1[a(x_i)]^{-1/4}S_i + k_2[a(x_i)]^{-1/4}T_i, \ i = 0(1)N, \\ U_{0_i} + k_3[a(x_i)]^{-1/4}V_i + k_4[a(x_i)]^{-1/4}W_i, \ i = N+1(1)2N, \end{cases} \tag{5.6}$$

where U_{0_i} is defined by (5.5), S, T, V, W are numerical solutions of the problems (4.2),(4.3),(4.4), (4.5) respectively and $k_1,\ k_2,\ k_3,\ k_4$ are defined by (4.7). An error estimate for this numerical solution is derived in the following section.

6. Error estimate

Theorem 6.1. *Let $u(x)$ be the solution of the problem (2.1). Further, let $U = (U_0, \cdots, U_{2N})$ be its numerical solution defined by (5.6). Then*

$$\| u - U \|_{\overline{\Omega}^{2N}} \leq C(\varepsilon + N^{-2} \ln^2 N).$$

Proof. From Theorems 4.1 and 5.1, we have

$$\| u - u_{as} \|_{\overline{\Omega}} \leq C\varepsilon, \| s - S \|_{\overline{\Omega}^{2N}} \leq CN^{-2} \ln^2 N,\ \| t - T \|_{\overline{\Omega}^{2N} \cap [0,1]} \leq CN^{-2} \ln^2 N,$$
$$\| v - V \|_{\overline{\Omega}^{2N} \cap [1,2]} \leq CN^{-2} \ln^2 N,\ \| w - W \|_{\overline{\Omega}^{2N}} \leq CN^{-2} \ln^2 N.$$

Then

$$\begin{aligned}
| u(x_i) - U_i | &\leq | u(x_i) - u_{as}(x_i) | + | u_{as}(x_i) - U_i |,\ i = 0(1)2N \\
&\leq \begin{cases} | u(x_i) - u_{as}(x_i) | + | u_0(x_i) - U_{0_i} | + | k_1 [a(x_i)]^{-1/4} || s(x_i) - S_i | + \\ \qquad k_2 [a(x_i)]^{-1/4} || t(x_i) - T_i |\ \ i = 0(1)N \\ | u(x_i) - u_{as}(x_i) | + | u_0(x_i) - U_{0_i} | + | k_3 [a(x_i)]^{-1/4} || v(x_i) - V_i | + \\ \qquad k_4 [a(x_i)]^{-1/4} || w(x_i) - W_i |,\ i = N+1(1)2N \end{cases} \\
&\leq \begin{cases} C\varepsilon + CN^{-2} \ln^2 N,\ i = 0(1)N \\ C\varepsilon + CN^{-2} \ln^2 N,\ i = N+1(1)2N. \end{cases}
\end{aligned}$$

That is,

$$| u(x_i) - U_i | \leq C(\varepsilon + N^{-2} \ln^2 N),\ i = 0(1)2N. \tag{6.1}$$

□

Note: If we assume that $\sqrt{\varepsilon} \leq CN^{-1}$ in the above theorem, we have almost second order convergence.

7. Discontinuous source term

7.1. Statement of the problem and numerical method

In the previous section it was assumed that f is continuous on $[0, 2]$. Motivated by the works of [5, 19] we suppose $f(x)$ has a simple discontinuity at $x = 1$, that is, $f(1-) \neq f(1+)$.

Consider the following BVP with a discontinuous source term.

Find $u \in Y$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x) = \begin{cases} f_1(x),\ x \in \Omega^-, \\ f_2(x),\ x \in \Omega^+, \end{cases} \\ u(x) = \phi(x),\ x \in [-1, 0],\ u(2) = l, \\ f_1(1-) \neq f_2(1+). \end{cases} \tag{7.1}$$

It is assumed that the conditions stated in Section 2 for the coefficients and the relations among $a,\ b,\ c$ and ϕ hold true and f is smooth in $\Omega^-\cup\Omega^+$. It can be easily verified that the maximum principle and stability result are valid for the above problem (7.1). Now, an asymptotic expansion approximation for the solution of the problem (7.1) is given by (4.6).

We can prove a similar result of Theorem 4.1 for (7.1). With regard to the numerical method, the same mesh selection strategy described in Section 5.1 can be adopted here.

8. Numerical results

In this section, four examples are given to illustrate the numerical method discussed in this paper. The exact solutions of the test problems are not known. Therefore, we use the usual double mesh principle to estimate the error and compute the experiment rate of convergence to our computed solution. For this we put

$$D_\varepsilon^M = \max_{0\le i\le M} \mid U_i^M - U_{2i}^{2M} \mid,$$

where U_i^M and U_{2i}^{2M} are the i^{th} components of the numerical solutions on meshes of M and $2M$ points respectively. We compute the uniform error and rate of convergence as

$$D^M = \max_\varepsilon D_\varepsilon^M \text{ and } \quad p^M = \log_2\left(\frac{D^M}{D^{2M}}\right).$$

Here, $M = 2N$. The set of values $\{2^{-4},\ 2^{-5},\ \cdots, 2^{-23}\}$ is taken as the range of $\sqrt{\varepsilon}$ for the following problems.

Example 8.1. (Constant coefficient)

$$\begin{cases} -\varepsilon u''(x) + 5u(x) - u(x-1) = 1,\ x\in\Omega^-\cup\Omega^+, \\ u(x) = 1,\ x\in[-1,0],\ u(2) = 2. \end{cases} \tag{8.1}$$

Example 8.2. (Variable coefficient)

$$\begin{cases} -\varepsilon u''(x) + (x+5)u(x) - u(x-1) = 1,\ x\in\Omega^-\cup\Omega^+, \\ u(x) = 1,\ x\in[-1,0],\ u(2) = 2. \end{cases} \tag{8.2}$$

Example 8.3. (Discontinuous source term)

$$\begin{cases} -\varepsilon u''(x) + 5u(x) - u(x-1) = \begin{cases} 1,\ x\in\Omega^-, \\ -1,\ x\in\Omega^+, \end{cases} \\ u(x) = 1,\ x\in[-1,0],\ u(2) = 2. \end{cases} \tag{8.3}$$

Example 8.4.

$$\begin{cases} -\varepsilon u''(x) + 2u(x) - 2u(x-1) = 0,\ x\in\Omega^-\cup\Omega^+, \\ u(x) = 1,\ x\in[-1,0],\ u(2) = 2. \end{cases} \tag{8.4}$$

Though this example does not satisfy the condition $\alpha+\beta_0 > 0$, the present method still gives good results (Table 4).

Tables 1-4 present the values of D^M and p^M for the problems stated in Examples 8.1-8.4 respectively. Figures 1 and 2 represent the numerical solutions of the problems stated in Examples 8.1 and 8.3 respectively.

Table 1: Numerical results for Example 8.1

M	32	64	128	256	512	1024	2048
D^M	1.2175e-1	5.2206e-2	1.8447e-2	6.5158e-3	2.1589e-3	6.5625e-4	2.0517e-4
p^M	1.2217	1.5008	1.5013	1.5937	1.7179	1.6774	-

Table 2: Numerical results for Example 8.2

M	32	64	128	256	512	1024	2048
D^M	1.6717e-1	7.4650e-2	2.7140e-2	9.3384e-3	3.1724e-3	9.6871e-4	2.9972e-4
p^M	1.1630	1.4597	1.5391	1.5576	1.7114	1.6924	-

Table 3: Numerical results for Example 8.3

M	32	64	128	256	512	1024	2048
D^M	1.5007e-1	6.4346e-2	2.2737e-2	8.0311e-3	2.6609e-3	8.0887e-4	2.5288e-4
p^M	1.2217	1.5008	1.5013	1.5937	1.7179	1.6774	-

Table 4: Numerical results for Example 8.4

M	32	64	128	256	512	1024	2048
D^M	7.0787e-2	3.0352e-2	1.0725e-2	3.7882e-3	1.2551e-3	3.8154e-4	1.1928e-4
p^M	1.2217	1.5008	1.5013	1.5937	1.7179	1.6774	-

Table 5: Numerical results

	$\sqrt{\varepsilon} = 2^{-6}$			
N	2^8	2^9	2^{10}	2^{11}
	Example 8.1.			
Maximum error	1.4437e-3	4.3886e-4	1.3720e-4	4.1348e-5
Rate of convergence	1.7179	1.6774	1.7304	-
	Example 8.2.			
Maximum error	1.9500e-3	5.9545e-4	1.8423e-4	5.6093e-5
Rate of convergence	1.7114	1.6925	1.7156	-

9. Discussion

A BVP for one type of SPDDEs of reaction-diffusion type is considered. To obtain an approximate solution for this type of problems, an asymptotic

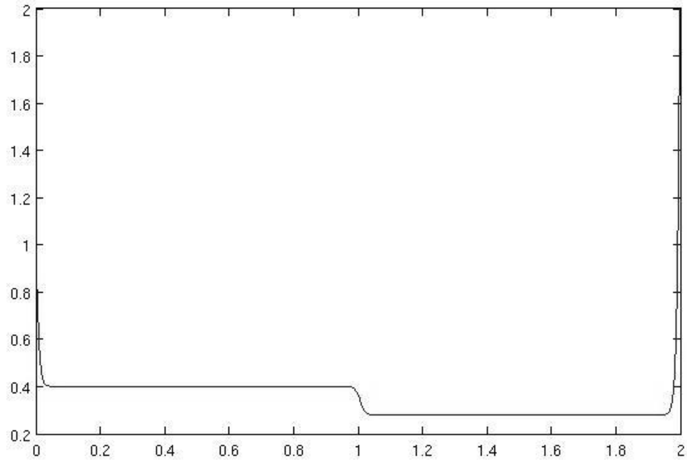

Figure 1: Graph of the numerical solution of Example 8.1

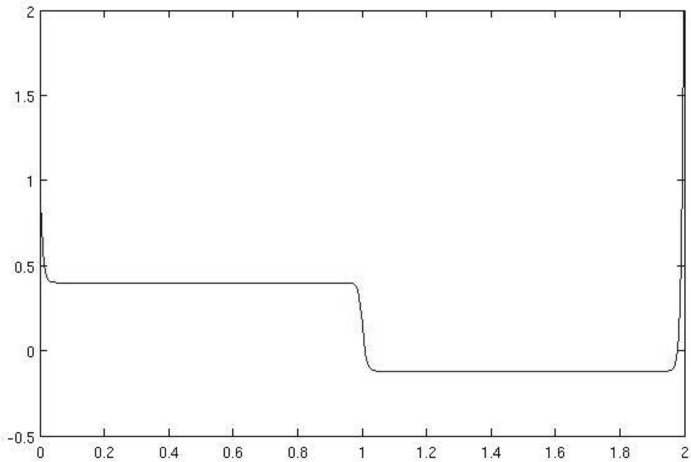

Figure 2: Graph of the numerical solution of Example 8.3

numerical method (IVT) is presented. The method is shown to be of order $O(\varepsilon+N^{-2}\ln^2 N)$ and, that is, the method has almost second order convergence with respect to ε. This is very much reflected on the numerical results. Also a problem with a discontinuous source term is considered. In fact, if the above condition $\sqrt{\varepsilon} \leq CN^{-1}$ is not met, our method yields an almost second order convergence for some problems (Examples 8.1, 8.2), as illustrated in Table 5. To illustrate the nature of boundary layers, graphs are plotted in Figures 1 and 2 for the problems given in Examples 8.1 and 8.3 respectively.

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