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THE MIXED MODE CRACK PROBLEM IN A FGM LAYER[+]

X. Long and F. Delale[*]
Department of Mechanical Engineering,
The City College of New York, New York, NY 10031, USA.
and
Graduate School and University Center of CUNY

ABSTRACT

Functionally graded materials (FGMs) are multiphase composites whose composition, microstructure and properties vary gradually. They can be tailored to meet the requirements encountered in practice through the design of their constituents. In this paper, analytical expressions for stress intensity factors of mixed-mode cracks in a FGM strip have been derived for the first time. A parametric study, by varying both the geometric and material parameters, is conducted to determine their effects on the stress intensity factors.

INTRODUCTION

FGMs are multiphase composite materials designed in a way that combines the desirable characteristics of each of the constituents. The distinctive feature of FGMs is that their material properties vary spatially. Though desirable for engineering applications, the variation of mechanical properties of FGMs significantly complicates the mathematical analysis of these materials.

Some crack problems in FGMs have been solved during the past decade. While most of these works are limited to semi-infinite or infinite domains, Wu and Erdogan solved the mode *I* crack problem in a FGM strip in [1]. The geometry of this problem is an internal or edge crack in a strip with finite width. Konda and Erdogan solved the mixed mode crack problem in FGMs [2] for an arbitrarily oriented crack in an infinite nonhomogeneous medium.

In this paper, the more general problem of an arbitrarily oriented crack in a FGM layer is studied. The problem is formulated in terms of a system of singular integral equations, which is solved numerically. The stress intensity factors at the crack tips are computed. A complete parametric study, by varying both the geometric and material parameters is conducted.

1. The Formulation

The crack problem under consideration is a FGM strip of thickness h containing an embedded finite crack on the $y_1 = 0$ plane (Fig. 1). To make the problem mathematically tractable, the Young's modulus of the material is assumed to vary exponentially in the thickness coordinate. The shear modulus is defined by:

$$\mu(x) = \mu_1 e^{\delta x} \quad \text{or} \quad \mu(x_1, y_1) = \mu_1 e^{\beta x_1 + \gamma y_1} \qquad (1)$$

where,

$$\beta = \delta\cos\theta, \qquad \gamma = -\delta\sin\theta \qquad (2)$$

Here, δ is a constant that describes the nonhomogeneity of the material. We assume $\delta \geq 0$ to simplify the discussion. θ is the angle between the crack line and x.

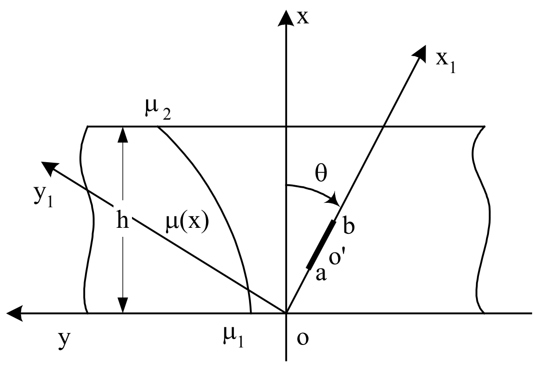

Fig. 1. Crack geometry in the FGM layer

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[*] Author to whom all correspondence should be sent

In a previous study [3], it was shown that the effect of the Poisson's ratio on the stress intensity factors is negligible. Thus in this paper the Poisson's ratio is assumed to be constant.

The mixed boundary value problem will be solved under following boundary and continuity conditions:

$$\begin{aligned} \sigma_{y_1}(x_1,+0) &= \sigma_{y_1}(x_1,-0) \\ \tau_{x_1y_1}(x_1,+0) &= \tau_{x_1y_1}(x_1,-0) \end{aligned} \tag{3}$$

$$\begin{aligned} \sigma_x(0,y) = \sigma_x(h,y) = 0 \\ \tau_{xy}(0,y) = \tau_{xy}(h,y) = 0 \end{aligned} \quad -\infty < y < \infty \tag{4}$$

$$\begin{aligned} v(x_1,+0) = v(x_1,-0) \\ u(x_1,+0) = u(x_1,-0) \end{aligned} \quad x_1 < a \;\; or \;\; x_1 > b \tag{5}$$

$$\begin{aligned} \sigma_{y_1}(x_1,0) = p_1(x_1) \\ \tau_{x_1y_1}(x_1,0) = p_2(x_1) \end{aligned} \quad a < x_1 < b \tag{6}$$

where $p_1(x_1)$ and $p_2(x_1)$ are known crack surface tractions, which can be determined by solving the elasticity problem for the uncracked strip under the given loads.

The solution may be expressed as the sum of two displacement sets in forms of $u_1(x_1,y_1)$, $v_1(x_1,y_1)$ and $u_2(x,y)$, $v_2(x,y)$.

The governing equations in (x_1,y_1) system may be expressed as:

$$\begin{aligned} &(\kappa+1)\frac{\partial^2 u_1}{\partial x_1^2} + (\kappa-1)\frac{\partial^2 u_1}{\partial y_1^2} + 2\frac{\partial^2 v_1}{\partial x_1 \partial y_1} + \beta(\kappa+1)\frac{\partial u_1}{\partial x_1} \\ &+ \gamma(\kappa-1)(\frac{\partial u_1}{\partial y_1} + \frac{\partial v_1}{\partial x_1}) + \beta(3-\kappa)\frac{\partial v_1}{\partial y_1} = 0 \\ &(\kappa-1)\frac{\partial^2 v_1}{\partial x_1^2} + (\kappa+1)\frac{\partial^2 v_1}{\partial y_1^2} + 2\frac{\partial^2 u_1}{\partial x_1 \partial y_1} + \gamma(3-\kappa)\frac{\partial u_1}{\partial x_1} \\ &+ \beta(\kappa-1)(\frac{\partial u_1}{\partial y_1} + \frac{\partial v_1}{\partial x_1}) + \gamma(\kappa+1)\frac{\partial v_1}{\partial y_1} = 0 \end{aligned} \tag{7}$$

Here, we allow $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

Assuming $u_1(x_1,y_1)$, $v_1(x_1,y_1)$ as

$$\begin{aligned} u_1(x_1,y_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(y_1,\alpha)e^{-i\alpha x_1}d\alpha \\ v_1(x_1,y_1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(y_1,\alpha)e^{-i\alpha x_1}d\alpha \end{aligned} \tag{8}$$

and substituting in (7), we get:

$$\begin{aligned} U(y_1,\alpha) &= \sum_{j=1}^{4} m_j F_j(\alpha)e^{n_j y_1} \\ V(y_1,\alpha) &= \sum_{j=1}^{4} F_j(\alpha)e^{n_j y_1} \end{aligned} \tag{9}$$

where $F_j(\alpha)$ are unknown functions, $m_j(j=1,...,4)$ are given by

$$m_j = \frac{[2\alpha i + \beta(\kappa-3)]n_j + i\alpha\gamma(\kappa-1)}{(\kappa-1)n_j^2 + (\kappa-1)\gamma n_j - (\kappa+1)(\alpha+i\beta)\alpha} \tag{10}$$

and $n_j(j=1,...,4)$ are the roots of the characteristic equation,

$$\begin{aligned} &n^4 + 2\gamma n^3 + [-2\alpha(\alpha+i\beta) + \gamma^2 \\ &+ \beta^2\frac{\kappa-3}{\kappa+1}]n^2 + \alpha\gamma(-2\alpha - i\beta\frac{8}{\kappa+1})n \\ &+ \alpha^2(\alpha^2 + 2i\alpha\beta - \beta^2 + \gamma^2\frac{3-\kappa}{\kappa+1}) = 0 \end{aligned} \tag{11}$$

From equation (11), the values of n_j are obtained.

To satisfy regularity conditions, u and v must vanish for $x_1^2 + y_1^2 \to \infty$, then the unknown functions $F_j(\alpha)(j=1...4)$ satisfy the relations:

$$\begin{aligned} F_3(\alpha) = F_4(\alpha) = 0, \quad & y > 0 \\ F_1(\alpha) = F_2(\alpha) = 0, \quad & y < 0 \end{aligned} \tag{12}$$

Using generalized Hooke's law, the stresses are found to be:

$$\begin{aligned} &\sigma_{x_1}(x_1,y_1) = \frac{\mu}{2\pi(\kappa-1)} \cdot \\ &\int_{-\infty}^{\infty}\sum_{j=l}^{l+1}[-i\alpha m_j(1+\kappa) + n_j(3-\kappa)]F_j(\alpha)e^{n_j y_1 - i\alpha x_1}d\alpha \\ &\sigma_{y_1}(x_1,y_1) = \frac{\mu}{2\pi(\kappa-1)} \cdot \\ &\int_{-\infty}^{\infty}\sum_{j=l}^{l+1}[-i\alpha m_j(3-\kappa) + n_j(1+\kappa)]F_j(\alpha)e^{n_j y_1 - i\alpha x_1}d\alpha \\ &\tau_{x_1y_1}(x_1,y_1) = \\ &\frac{\mu}{2\pi}\int_{-\infty}^{\infty}\sum_{j=l}^{l+1}[n_j m_j - i\alpha]F_j(\alpha)e^{n_j y_1 - i\alpha x_1}d\alpha \end{aligned} \tag{13}$$

where $l=1$ for $y>0$ and $l=3$ for $y<0$.

Applying the boundary and continuity conditions (3), and the expressions of stresses obtained in (13), we find

$$\begin{aligned} F_3(\alpha) &= R_1(\alpha)F_1(\alpha) + R_2(\alpha)F_2(\alpha) \\ F_4(\alpha) &= R_3(\alpha)F_1(\alpha) + R_4(\alpha)F_2(\alpha) \end{aligned} \tag{14}$$

where $R_j(\alpha)$ are known functions.

To determine the two remaining unknown functions $F_1(\alpha)$ and $F_2(\alpha)$, the following new auxiliary functions in terms of the crack surface derivatives are introduced:

$$\begin{aligned} g_1(x_1) &= \frac{\partial}{\partial x_1}[u_1(x_1,+0) - u_1(x_1,-0)], \quad a < |x_1| < b \\ g_2(x_1) &= \frac{\partial}{\partial x_1}[v_1(x_1,+0) - v_1(x_1,-0)], \quad a < |x_1| < b \end{aligned} \tag{15}$$

From definitions of $g_1(x_1)$ and $g_2(x_1)$, one can easily conclude that $g_1(x_1)=0$ and $g_2(x_1)=0$ when $0<|x_1|<a$ or $|x_1|>b$ and

$$\int_a^b g_j(t)dt = 0, \qquad j=1,2 \tag{16}$$

Equations (16) are referred to as the single-valuedness conditions.

Substituting u_1 and v_1 into the definition of $g_1(x_1)$ and $g_2(x_1)$, we get

$$g_1(x_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty} -i\alpha(1+\kappa)[(f_{11}-i\alpha f_{41})F_1 + (f_{12}-i\alpha f_{42})F_2]\frac{e^{-i\alpha x_1}}{R_0}d\alpha$$
$$g_2(x_1)=\frac{1}{2\pi}\int_{-\infty}^{\infty} -i\alpha\{[-(1+\kappa)f_{31}+i\alpha(3-\kappa)f_{21}]F_1 + [-(1+\kappa)f_{32}+i\alpha(3-\kappa)f_{22}]F_2\}\frac{e^{-i\alpha x_1}}{R_0}d\alpha \tag{17}$$

where f_{ij} $(i=1,...,4;\ j=1,2)$ are known.

Inverting the Fourier integrals (17) and substituting F_1 and F_2 into equation (13), we can express σ_{x_1}, σ_{y_1} and $\tau_{x_1y_1}$ in terms of g_1 and g_2, for $y_1>0$ as,

$$\sigma_{x_1}{}^{(1)}(x_1,y_1)=\frac{\mu(x_1,y_1)}{2\pi(1+\kappa)}\int_a^b\sum_{j=1}^{2}h_{1j}{}^{(1)}(x_1,y_1,t)g_j(t)dt$$
$$\sigma_{y_1}{}^{(1)}(x_1,y_1)=\frac{\mu(x_1,y_1)}{2\pi(1+\kappa)}\int_a^b\sum_{j=1}^{2}h_{2j}{}^{(1)}(x_1,y_1,t)g_j(t)dt \tag{18}$$
$$\tau_{x_1y_1}{}^{(1)}(x_1,y_1)=\frac{\mu(x_1,y_1)}{2\pi(1+\kappa)}\int_a^b\sum_{j=1}^{2}h_{3j}{}^{(1)}(x_1,y_1,t)g_j(t)dt$$

where

$$h_{kj}{}^{(1)}(x_1,y_1,t)=\int_{-\infty}^{\infty}K_{kj}{}^{(1)}(y_1,\alpha)e^{i\alpha(t-x_1)}d\alpha \tag{19}$$
$$k=1,2;\quad j=1,2$$

and $K_{kj}{}^{(1)}$ are known kernels.

Similarly, after some manipulation, we obtain σ_{x_1}, σ_{y_1} and $\tau_{x_1y_1}$ for $y_1<0$, as:

$$\sigma_{x_1}{}^{(2)}(x_1,y_1)=\frac{\mu(x_1,y_1)}{2\pi(1+\kappa)}\int_a^b\sum_{j=1}^{2}h_{1j}{}^{(2)}(x_1,y_1,t)g_j(t)dt$$
$$\sigma_{y_1}{}^{(2)}(x_1,y_1)=\frac{\mu(x_1,y_1)}{2\pi(1+\kappa)}\int_a^b\sum_{j=1}^{2}h_{2j}{}^{(2)}(x_1,y_1,t)g_j(t)dt \tag{20}$$
$$\tau_{x_1y_1}{}^{(2)}(x_1,y_1)=\frac{\mu(x_1,y_1)}{2\pi(1+\kappa)}\int_a^b\sum_{j=1}^{2}h_{3j}{}^{(2)}(x_1,y_1,t)g_j(t)dt$$

where

$$h_{kj}{}^{(2)}(x_1,y_1,t)=\int_{-\infty}^{\infty}K_{kj}{}^{(2)}(y_1,\alpha)e^{i\alpha(t-x_1)}d\alpha \tag{21}$$
$$k=1,2;\quad j=1,2$$

and $K_{kj}{}^{(2)}$ are known kernels.

In the coordinate system (x,y), the Navier equations for the elastic medium may be expressed as:

$$(\kappa+1)\frac{\partial^2 u}{\partial x^2}+(\kappa-1)\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 v}{\partial x\partial y} + \delta(\kappa+1)\frac{\partial u}{\partial x}+\delta(3-\kappa)\frac{\partial v}{\partial y}=0$$
$$(\kappa-1)\frac{\partial^2 v}{\partial x^2}+(\kappa+1)\frac{\partial^2 v}{\partial y^2}+2\frac{\partial^2 u}{\partial x\partial y} + \delta(\kappa-1)(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})=0 \tag{22}$$

Assuming the solution of $u_2(x,y)$ and $v_2(x,y)$ as

$$u_2(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}qA(\alpha)e^{px}e^{-i\alpha y}d\alpha$$
$$v_2(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}A(\alpha)e^{px}e^{-i\alpha y}d\alpha \tag{23}$$

the characteristic equations for p and q may be expressed as

$$[(\kappa+1)p^2-(\kappa-1)\alpha^2+\delta(\kappa+1)p]q - i\alpha[2p+\delta(3-\kappa)]=0$$
$$(\kappa-1)p^2+\delta(\kappa-1)p-(\kappa+1)\alpha^2 - i\alpha q[2p+\delta(\kappa-1)]=0 \tag{24}$$

solving equations (24), we have:

$$q_j=\frac{(\kappa-1)(p_j+\delta)p_j-(\kappa+1)\alpha^2}{\alpha i[2p_j+\delta(\kappa-1)]},\ j=1..4 \tag{25}$$

Then we can express u_2 and v_2 in terms of the unknown functions $A_j(\alpha)(j=1,...,4)$

$$u_2(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_j^4 q_jA_j(\alpha)e^{p_jx}e^{-i\alpha y}d\alpha$$
$$v_2(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_j^4 A_j(\alpha)e^{p_jx}e^{-i\alpha y}d\alpha \tag{26}$$

From generalized Hooke's law, the stresses corresponding to u_2 and v_2 are

$$\sigma_x{}^{(3)} = \frac{\mu(x,y)}{2\pi(\kappa-1)} \cdot$$
$$\int_{-\infty}^{\infty} \sum_j^4 [(1+k)p_j q_j + (\kappa-3)i\alpha] A_j(\alpha) e^{p_j x - i\alpha y} d\alpha$$
$$\sigma_y{}^{(3)} = \frac{\mu(x,y)}{2\pi(\kappa-1)} \cdot \tag{27}$$
$$\int_{-\infty}^{\infty} \sum_j^4 [-(1+k)i\alpha + (3-\kappa)p_j q_j] A_j(\alpha) e^{p_j x - i\alpha y} d\alpha$$
$$\tau_{xy}{}^{(3)} = \frac{\mu(x,y)}{2\pi} \cdot \int_{-\infty}^{\infty} \sum_j^4 [p_j - i\alpha q_j] A_j(\alpha) e^{p_j x - i\alpha y} d\alpha$$

2. The Integral Equations

At a given point in the medium, the stress state is the sum of the stresses given by equation (27) and (18) or (20), depending on the sign of y_1. The free boundary conditions in equation (4) then yield the following set of equations for the unknown functions $A_j(\alpha)$ $(j = 1,...,4)$ in terms of the auxiliary functions g_1 and g_2,

$$\sum_j^4 [(1+k)p_j q_j + (\kappa-3)i\alpha] A_j(\alpha)$$
$$= \frac{1}{2\pi(1+\kappa)} \sum_{j=1}^{2} \int_a^b F_{1j}(\alpha,t) g_j(t) dt$$
$$\sum_j^4 [p_j - i\alpha q_j] A_j(\alpha)$$
$$= \frac{1}{2\pi(1+\kappa)} \sum_{j=1}^{2} \int_a^b F_{2j}(\alpha,t) g_j(t) dt$$
$$\sum_j^4 [(1+k)p_j q_j + (\kappa-3)i\alpha] A_j(\alpha) e^{p_j h}$$
$$= \frac{1}{2\pi(1+\kappa)} \sum_{j=1}^{2} \int_a^b F_{3j}(\alpha,t) g_j(t) dt$$
$$\sum_j^4 [p_j - i\alpha q_j] A_j(\alpha) e^{p_j h}$$
$$= \frac{1}{2\pi(1+\kappa)} \sum_{j=1}^{2} \int_a^b F_{4j}(\alpha,t) g_j(t) dt \tag{28}$$

where the known functions $F_{kj}{}^{(2)} (k = 1...4,\, j = 1, 2)$ are known functions.

From equation (28), $A_j(\alpha)$ may be obtained as:

$$A_j(\alpha) = \sum_{i=1}^{2} \int_a^b C_{ji}(\alpha,t) g_i(t) dt \tag{29}$$

where

$$C_{ji}(\alpha,t) = \sum_{k=1}^{4} b_{jk}(\alpha) F_{ki}(\alpha,t), \quad (j = 1...4,\, i = 1, 2) \tag{30}$$

Here, the matrix (b_{jk}) is the inverse of (a_{jk}) given by

$$a_{1j}(\alpha) = (1+k)p_j q_j + (\kappa-3)i\alpha$$
$$a_{2j}(\alpha) = p_j - i\alpha q_j$$
$$a_{3j}(\alpha) = [(1+k)p_j q_j + (\kappa-3)i\alpha] e^{p_j h} \tag{31}$$
$$a_{4j}(\alpha) = [p_j - i\alpha q_j] e^{p_j h}$$

Applying the boundary conditions

$$\sigma_{y_1}(x_1, +0) = p_1(x_1)$$
$$\tau_{x_1 y_1}(x_1, +0) = p_2(x_1) \qquad (a < x < b) \tag{32}$$

where $p_1(x_1)$ and $p_2(x_1)$ are crack surface tractions, we obtain:

$$\sigma_{y_1}{}^{(1)}(x_1, 0) + \sin^2\theta \sigma_x{}^{(3)}(x_1\cos\theta, x_1\sin\theta)$$
$$+ \cos^2\theta \sigma_y{}^{(3)}(x_1\cos\theta, x_1\sin\theta) \tag{33}$$
$$- 2\sin\theta\cos\theta \tau_{xy}{}^{(3)}(x_1\cos\theta, x_1\sin\theta) = p_1(x_1)$$
$$\tau_{x_1 y_1}{}^{(1)}(x_1, 0) + \sin\theta\cos\theta[\sigma_y{}^{(3)}(x_1\cos\theta, x_1\sin\theta)$$
$$+ \tau_{xy}{}^{(3)}(x_1\cos\theta, x_1\sin\theta)(\cos^2\theta - \sin^2\theta) \tag{34}$$
$$- \sigma_x{}^{(3)}(x_1\cos\theta, x_1\sin\theta)] = p_2(x_1)$$

Substituting the expressions of stresses and extracting the singular parts of (33), we obtain

$$\frac{1}{\pi} \int_a^b \{\frac{g_2(t)}{t - x_1} + \sum_{j=1}^{2} [k_{1j}{}^{(1)}(x_1,t) + k_{2j}{}^{(1)}(x_1,t)] g_j(t)\} dt$$
$$= \frac{(1+\kappa)}{2\mu(x_1, 0)} p_1(x_1) \tag{35}$$

where the kernels are

$$k_{11}{}^{(1)}(x_1,t) = \frac{1}{4} h_{21}{}^{(1)}(x_1, 0, t)$$
$$k_{12}{}^{(1)}(x_1,t) =$$
$$\frac{1}{4} \int_{-\infty}^{\infty} [K_{22}{}^{(1)}(0,\alpha) - \lim_{\alpha\to\infty} K_{22}{}^{(1)}(0,\alpha)] e^{i\alpha(t - x_1)} d\alpha$$
$$k_{2j}{}^{(1)}(x_1,t) \tag{36}$$
$$= \frac{(1+\kappa)}{4(\kappa-1)} \int_{-\infty}^{\infty} \sum_i^4 \{[(1+k)p_i q_i + (\kappa-3)i\alpha]\sin^2\theta$$
$$+ [-(1+k)i\alpha + (3-\kappa)p_i q_i]\cos^2\theta$$
$$- 2\sin\theta\cos\theta(\kappa-1)[p_i - i\alpha q_i]\}$$
$$C_{ij}(\alpha,t) e^{p_i x_1 \cos\theta - i\alpha x_1 \sin\theta} d\alpha$$

Note that for $\alpha \to \infty$

$$K_{21}{}^{\infty} = 0$$
$$K_{22}{}^{\infty} = -2i\frac{|\alpha|}{\alpha}e^{-|\alpha|y} \tag{37}$$

Similarly, equation (34) yields:

$$\frac{1}{\pi}\int_a^b\{\frac{g_1(t)}{t-x_1}+\sum_{j=1}^{2}[k_{1j}{}^{(2)}(x_1,t)+k_{2j}{}^{(2)}(x_1,t)]g_j(t)\}dt = \frac{(1+\kappa)}{2\mu(x_1,0)}p_2(x_1) \tag{38}$$

with

$$k_{12}{}^{(2)}(x_1,t)=\frac{1}{4}h_{32}{}^{(1)}(x_1,0,t) \tag{39}$$

$$k_{11}{}^{(2)}(x_1,t)= \frac{1}{4}\int_{-\infty}^{\infty}[K_{31}{}^{(1)}(0,\alpha)-\lim_{\alpha\to\infty}K_{31}{}^{(1)}(0,\alpha)]e^{i\alpha(t-x_1)}d\alpha \tag{40}$$

$$k_{2i}{}^{(2)}(x_1,t)= \frac{(1+\kappa)}{4(\kappa-1)}\int_{-\infty}^{\infty}\sum_{j}^{4}\{(\cos^2\theta-\sin^2\theta)(\kappa-1)[p_j-i\alpha q_j] + 2\sin\theta\cos\theta(1-\kappa)[i\alpha+p_jq_j\} C_{ji}(\alpha,t)e^{p_jx_1\cos\theta-i\alpha x_1\sin\theta}d\alpha \tag{41}$$

The single valuedness conditions (16) complete the formulation of the problem, i.e.

$$\int_a^b g_j(t)dt=0, \qquad j=1,2 \tag{42}$$

3. The Numerical Solution

The system of Cauchy-type singular integral equations obtained above can be solved numerically. Here we employ the collocation method used in [1,4].

First, to normalize the integral interval $a<x,t<b$ to $-1<r,s<1$, we define

$$t=\frac{b-a}{2}r+\frac{b+a}{2}$$
$$x_1=\frac{b-a}{2}s+\frac{b+a}{2}$$
$$g_1(t)=\phi_1(r) \qquad g_2(t)=\phi_2(r)$$
$$p_1(x_1)=f_1(s) \qquad p_2(x_1)=f_2(s) \tag{43}$$
$$\mu(x_1,0)=m(s,0)$$
$$q_{ij}{}^{(n)}(s,r)=\frac{b-a}{2}k_{ij}{}^{(n)}(x_1,t)$$
$$(i=1,2,\ j=1,2,\ n=1,2)$$

In terms of Chebyshev polynomials $U_m(s)$ and $T_n(r)$ the integral equations become:

$$\sum_{n=1}^{\infty}c_n{}^{(1)}U_{n-1}(s)+ \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-1}^{1}\sum_{j=1}^{2}[q_{1j}{}^{(1)}(s,r)+q_{2j}{}^{(1)}(s,r)]c_n{}^{(j)}\frac{T_n(r)}{\sqrt{1-r^2}}dr = \frac{(1+\kappa)}{2m(s,0)}f_1(s) \qquad -1<s<1$$
$$\sum_{n=1}^{\infty}c_n{}^{(2)}U_{n-1}(s)+ \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-1}^{1}\sum_{j=1}^{2}[q_{1j}{}^{(2)}(s,r)+q_{2j}{}^{(2)}(s,r)]c_n{}^{(j)}\frac{T_n(r)}{\sqrt{1-r^2}}dr = \frac{(1+\kappa)}{2m(s,0)}f_2(s) \qquad -1<s<1 \tag{44}$$

Equation (44) can be solved by truncating the series, and choosing the collocation points s_n as

$$T_N(s_n)=0 \qquad s_n=\cos((2n-1)\frac{\pi}{2N} \qquad n=1...N \tag{45}$$

The stress intensity factors are defined by

$$k_1(a)=\lim_{x\to a}\sqrt{2(x-a)}\sigma_{yy}(x,0)$$
$$k_2(a)=\lim_{x\to a}\sqrt{2(x-a)}\tau_{xy}(x,0)$$
$$k_1(b)=\lim_{x\to b}\sqrt{2(b-x)}\sigma_{yy}(x,0) \tag{46}$$
$$k_2(b)=\lim_{x\to b}\sqrt{2(b-x)}\tau_{xy}(x,0)$$

After determining $c_n{}^{(1)}$ and $c_n{}^{(2)}$, we can express the stress intensity factors at the crack tips as

$$k_1(a)=\sqrt{\frac{b-a}{2}}\frac{2\mu(a,0)}{1+\kappa}\sum_{n=1}^{\infty}(-1)^n c_n{}^{(1)}$$
$$k_2(a)=\sqrt{\frac{b-a}{2}}\frac{2\mu(a,0)}{1+\kappa}\sum_{n=1}^{\infty}(-1)^n c_n{}^{(2)}$$
$$k_1(b)=-\sqrt{\frac{b-a}{2}}\frac{2\mu(b,0)}{1+\kappa}\sum_{n=1}^{\infty}c_n{}^{(1)} \tag{47}$$
$$k_2(b)=-\sqrt{\frac{b-a}{2}}\frac{2\mu(b,0)}{1+\kappa}\sum_{n=1}^{\infty}c_n{}^{(2)}$$

and the crack surface openings as

$$u(x_1,+0)-u(x_1,-0) = -\sqrt{(a'^2-x'^2)}\sum_{n=1}^{\infty}\frac{1}{n}c_n{}^{(1)}U_{n-1}(x'/a')$$
$$v(x_1,+0)-v(x_1,-0) = -\sqrt{(a'^2-x'^2)}\sum_{n=1}^{\infty}\frac{1}{n}c_n{}^{(2)}U_{n-1}(x'/a') \tag{48}$$

where $a' = (b-a)/2$ is the half crack length and

$$x' = x - \frac{b+a}{2} \tag{49}$$

4. Results and Discussion

The system of integral equations is solved for various values of geometric and material parameters. The loading is uniform strain at infinity, with

$$\varepsilon_{yy}(x, \pm\infty) = \varepsilon_0 \tag{50}$$

where ε_0 is a given constant.

The crack surface traction defined by (6) may be expressed as,

$$p_1(x_1,0) = -\frac{8\mu_1 e^{\delta x\cos\theta}}{1+\kappa}\varepsilon_0\cos^2\theta$$
$$p_2(x_1,0) = -\frac{8\mu_1 e^{\delta x\cos\theta}}{1+\kappa}\varepsilon_0\cos\theta\sin\theta \tag{51}$$

For the nonhomogeneous medium considered, ν is assumed to be a constant and assigned the value of 0.3 in the calculations. The measure of material nonhomogeneity is given in the form of the nondimensional constant δh, with the value of $e^{\delta h} = 10.0$. The crack length is given by another nondimensional variable: a'/h.

The calculations were carried out for various crack lengths combined with the angle θ varying from 0^0 to near 90^0. Figures 2 - 4 show the variation of the stress intensity factors with the angle θ. All stress intensity factors are normalized with $K_0 = \sigma_0\sqrt{a'}$, where σ_0 is the normalizing stress defined as $\sigma_0 = \frac{8\mu_1}{1+\kappa}\varepsilon_0$.

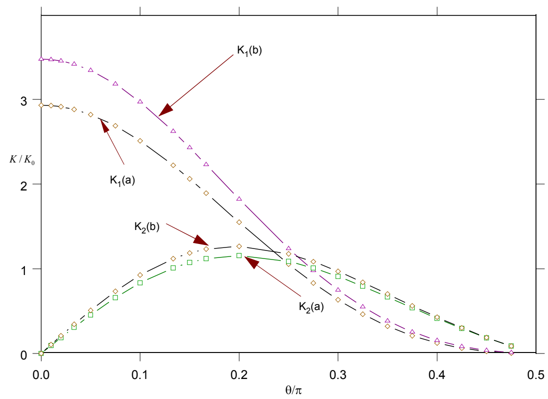

Fig 2. Variation of the normalized stress intensity factors K/K_0 with θ/π for an internal inclined crack in a FGM strip under uniform strain, $a'/h = 0.05$

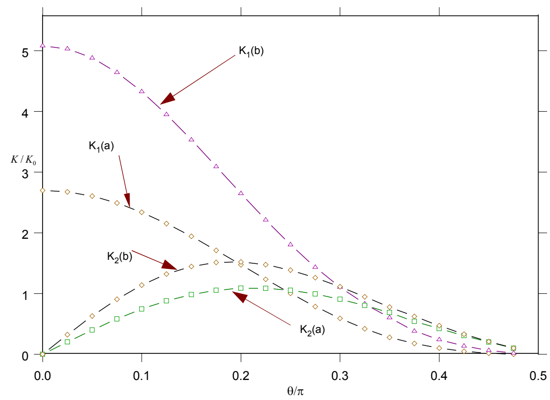

Fig. 3. Variation of the normalized stress intensity factors K/K_0 with θ/π for an internal inclined crack in a FGM strip under uniform strain, $a'/h = 0.20$

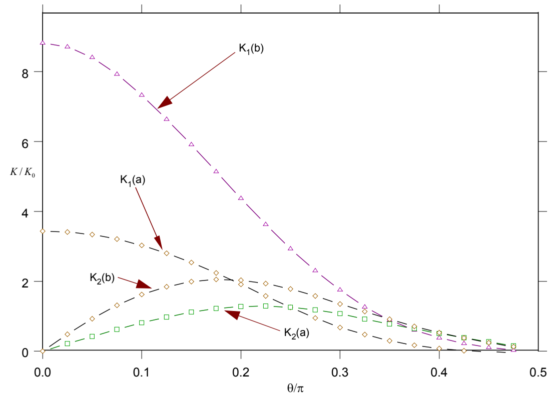

Fig. 4. Variation of the normalized stress intensity factors K/K_0 with θ/π for an internal inclined crack in a FGM strip under uniform strain, $a'/h = 0.35$

It can be observed that the stress intensity factors for mode *I* crack ($K_1(a)$ and $K_1(b)$) decrease as θ increases. At the same time, the stress intensity factors for mode *II* crack increase with the crack angle increasing from 0^o to a certain degree, and then decrease as the crack angle increases. The stress intensity factors $(K_1)_s$ are always greater than $(K_2)_s$ at the beginning, where the problem is mostly under mode *I* deformation. After θ increases to a certain degree, $(K_1)_s$ become smaller than $(K_2)_s$, as mode *II* loading starts to dominate. The length of a crack does not affect this trend, while the values of stress intensity factors change significantly for different length of cracks.

Fig. 5 shows $K_1(b)$ for different crack under the same loading. It should be pointed out that for $\theta = 0$ the problem reduces to the one studied in [1] with the results obtained here matching those in [1]. Fig. 6 shows the stress intensity factors for different crack lengths when $\theta = 45^0$.

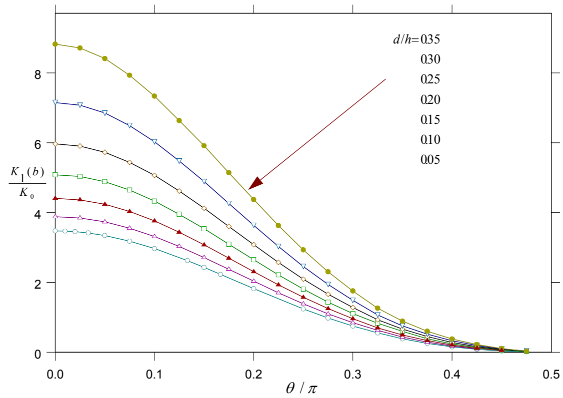

Fig. 5. Variation of the normalized stress intensity factors $K_1(b)/K_0$ with θ/π for an internal inclined crack in a FGM strip under uniform strain, for various a'/h ratios

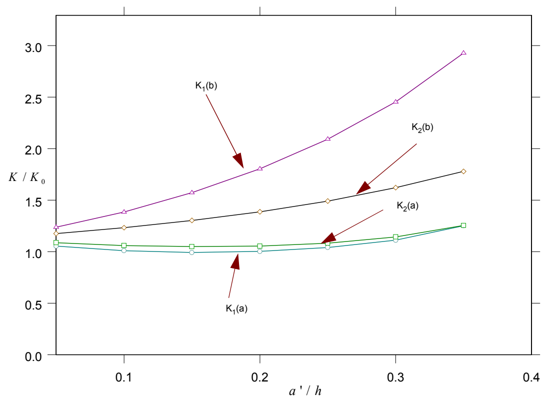

Fig. 6. Variation of the normalized stress intensity factors K/K_0 with a'/h for an internal inclined crack in a FGM strip under uniform strain, $\theta = \pi/4$

The crack surface openings are displayed in Fig. 7 and Fig. 8. Fig. 7 depicts the crack surface opening in y_1 direction with the crack length $a'/h = 0.35$. Fig. 8 shows the corresponding opening in x_1 direction, where o' (the origin of x') is the center of the crack (Fig. 1).

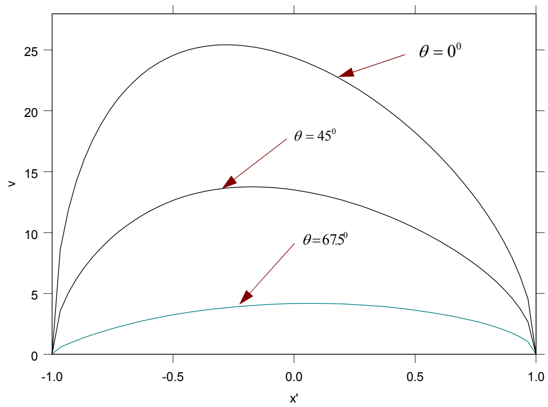

Fig. 7. Crack surface openings in the y_1 direction for $a'/h = 0.35$, $\theta = 0^0$, 45^0 and 67.5^0

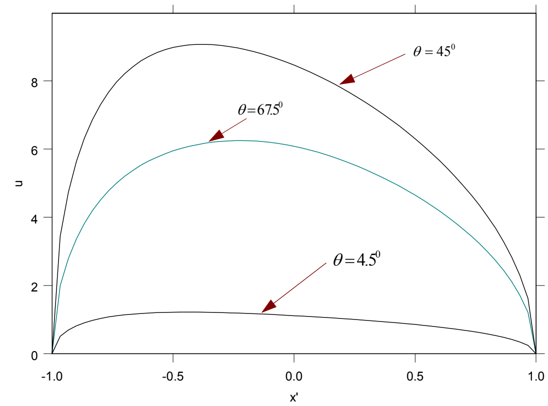

Fig. 8. Crack surface openings in the x_1 direction for $a'/h = 0.35$, $\theta = 0^0$, 45^0 and 67.5^0

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