A polynomial bound on the number of comaximal localizations needed in order to make free a projective module

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January 19, 2011

Abstract

Let $\mathbf{A}$ be a commutative ring and M be a projective module of rank k with n generators. Let $h = n-k$. Standard computations show that M becomes free after localizations in $\binom{n}{k}$ comaximal elements (see Theorem 5). When the base ring $\mathbf{A}$ contains a field with at least $hk+1$ non-zero distinct elements we construct a comaximal family G with at most $(hk+1)(nk+1)$ elements such that for each $g \in G$, the module M_g is free over $\mathbf{A}[1/g]$.

Keywords: Generalized Gram coefficients, finitely generated projective module, local freeness.

Introduction

Let $\mathbf{A}$ be a commutative ring and M be a projective module of rank k with n generators. Standard computations show that M becomes free after $\binom{n}{k}$ localizations in comaximal elements (see Theorem 5). This bound is exponential (e.g. when $n = 2k$) and a polynomial bound is expected to be found. This problem is reminiscent of the one of finding a good bound on the number of affine charts for the grassmannian variety of k-dimensional vector subspaces of an n dimensional vector space. Such a good bound is given in [1, Chistov & al.]. In this paper we use their result in order to find a polynomial bound for the first problem (see Theorem 12). More precisely, letting $h = n-k$ we give explicitely $(hk+1)(nk+1)$ convenient comaximal elements. Nevertheless we have to assume that the base ring $\mathbf{A}$ contains a field with at least $hk+1$ non-zero distinct elements.

1 Preliminaries about finitely generated projective modules

Let M be a projective module over $\mathbf{A}$ isomorphic to the image of a projector $P \in \mathbf{A}^{n\times n}$. In this case, the matrix $Q = \mathrm{I}_n - P$ is a presentation matrix for M, that is, $\operatorname{Ker} P = \operatorname{Im} Q$ which implies that $\operatorname{Im} P \simeq \operatorname{Coker} Q$. In this section, we introduce constructive notions about finitely generated projective modules. We start defining the determinantal ideals of a matrix.

Definition 1 *Let $A \in \mathbf{A}^{n\times m}$ be a matrix with coefficients in $\mathbf{A}$ and let $1 \le k \le \min(m,n)$. The determinantal ideal of order k of A is the ideal $\mathcal{D}_k(A)$ generated by the minors of order k of the matrix A. By convention, we let $\mathcal{D}_k(A) = \langle 1\rangle$ for $k \le 0$ and $\mathcal{D}_k(A) = \langle 0\rangle$ for $k > \min(m,n)$.*

When M is projective, it is well known that the determinantal ideals of both matrices P and Q are generated by idempotent elements. Moreover, they allows us to define full rank of matrices with entries in a ring as follows.

Definition 2 *A matrix $A \in \mathbf{A}^{n\times m}$, with $n \le m$, has full rank if $\mathcal{D}_n(A) = \langle 1\rangle$.*

*This work is partially supported by the MICINN project MTM2008-04699-C03-03

Another notion that will be crucial throughout the paper is the following.

Definition 3 *A family of elements $x_1, \dots, x_\ell$ of $\mathbf{A}$ is said comaximal if $\langle x_1, \dots, x_\ell \rangle = \langle 1 \rangle$.*

Next we introduce a constructive definition of the rank of a projective module, that allows us to work without localizing at prime ideals.

We set

$$\mathrm{P}_M(X) = \det(\mathrm{I}_n + XP) = 1 + \mathrm{d}_1(M)X + \cdots + \mathrm{d}_n(M)X^n$$

and $\mathrm{d}_0(M) = 1$, $\mathrm{d}_p(M) = 0$ for $p > n$. This polynomial does only depend on M (see Theorem 5 below).

Definition 4 *Let $\mathrm{R}_M(X) = \sum_i \mathrm{r}_i(M)X^i$ be defined by $\mathrm{R}_M(1+X) = \det(\mathrm{I}_n + XP)$. Then*

- *The module M is said of rank k if $\mathrm{R}_M(X) = X^k$.*
- *The module $M$is said of rank $\leq k$ if $\mathrm{r}_{k+1}(M) = \ldots = \mathrm{r}_n(M) = 0$.*
- *The module $M$is said of rank $> k$ if $\mathrm{r}_0(M) = \ldots = \mathrm{r}_k(M) = 0$.*

We will denote $\mathrm{r}_i(M)$ by r_i. Note that

$$\mathrm{P}_M(X) = \sum_i \mathrm{r}_i (1+X)^i, \quad \mathrm{R}_M(X) = \mathrm{P}_M(X-1) = \det(\mathrm{I}_n + (X-1)P).$$

In fact, we have the following result.

Theorem 5 *1. The polynomial $\mathrm{R}_M(X)$ does not depend on the matrix P. The polynomial $\mathrm{R}_M(X)$ verifies $\mathrm{R}_M(XY) = \mathrm{R}_M(X)R_M(Y)$ and $\mathrm{R}_M(1) = 1$. That implies that the set $\{\mathrm{r}_0, \dots, \mathrm{r}_n\}$ defines a basic system of orthogonal idempotents.*

2. *We have $\mathrm{r}_0 = \det(\mathrm{I}_n - P)$ and the ideal $\langle \mathrm{r}_0 \rangle$ is the annihilator of M.*
3. *The localization of $\mathbf{A}$ at r_k, $\mathbf{A}_{\mathrm{r}_k} = \mathbf{A}[1/\mathrm{r}_k]$, is isomorphic to $\mathbf{A} /\langle 1 - \mathrm{r}_k \rangle$. The $\mathbf{A}_{\mathrm{r}_k}$–module M_{r_k} is isomorphic to the submodule $\mathrm{r}_k M$ and its rank is equal to k as $\mathbf{A}_{\mathrm{r}_k}$-module.*
4. *The module M is the direct sum $\mathrm{r}_1 M \oplus \ldots \oplus \mathrm{r}_n M$ (with possible zero summands).*
5. *$\mathcal{D}_k(P) = \langle r_k, r_{k+1}, \dots, r_n \rangle$ for $k \leq n$. If M is of rank k then $\mathrm{d}_k(M) = \mathrm{r}_k = 1$.*
6. *The coefficient $\mathrm{d}_k(M)$ is equal to the sum of all kth principal minors of the matrix P. Furthermore, $\mathrm{r}_k\, \mathrm{d}_k(M) = \mathrm{r}_k$ and if μ is a kth principal minor of P, then $M_{\mathrm{r}_k \mu}$ is a free module of rank k over $\mathbf{A}[1/(\mathrm{r}_k \mu)]$.*
7. *The module M becomes free after localizations at 2^n comaximal elements. This number decreases to $\binom{n}{k}$ when M has constant rank k.*

For proof, see [10], Chapters 5 and 10.

Observe that our definition agrees with the usual definition of projective module of constant rank. More precisely, since M is finitely generated projective, $M_{\mathcal{I}}$ is $\mathbf{A}_{\mathcal{I}}$-free for every $\mathcal{I} \in \mathrm{Spec}(\mathbf{A})$. Following classical definitions, M is said to have constant rank k if $k = \mathrm{rank}_{\mathbf{A}_{\mathcal{I}}} M_{\mathcal{I}}$ for every $\mathcal{I} \in \mathrm{Spec}(\mathbf{A})$, see for example [9] for details.

Thus, given $\mathcal{I} \in \mathrm{Spec}(\mathbf{A})$, if M has constant rank k according to Definition 4, then there exists a nonzero kth principal minor μ of P such that $\mu \notin \mathcal{I}$ and so $k = \mathrm{rank}_{\mathbf{A}_{\mathcal{I}}} M_{\mathcal{I}}$. Conversely, assume now that M has constant rank k according to the usual definition; this implies that $r_k = 1$ and $r_h = 0$ for $h \neq k$, which means that M has rank k according to Definition 4.

Hereafter we will suppose that the module M has constant rank k. It follows from (5) of Theorem 5 that all $(k+1)$th minors of P are equal to zero. Moreover, if $\mathcal{P}_k$ denotes the set of all k–minors of P, the fact of having rank k implies that the set $\{\mu | \mu \in \mathcal{P}_k\}$ is comaximal and a priori it is required $\binom{n}{k}$ localizations to make M free. Remark that this bound is exponential in n; for example, with $n = 2k$ we have $\binom{n}{k} \geq 2^k$. In Section 3 we will discuss how to reduce the number of localizations in an effective way. For convenience for the reader, we first introduce the notion of Gram ideals presented in [5] and [6].

2 Gram and Vandermonde ideals

Let t be a formal variable. The ring $\mathbf{A}(t)$ is the localization $U^{-1}\mathbf{A}[t]$ with U equal to the set of primitive polynomials (a polynomial is said to be primitive when its coefficients are comaximal). Next we define the quadratic form $\Phi_{t,m}$ on $E' = \mathbf{A}(t)^m$ with values in $\mathbf{A}(t)$ as

$$\Phi_{t,m}(\xi_1, \ldots, \xi_m) \quad = \quad {\xi_1}^2 + t\,{\xi_2}^2 + \cdots + t^{m-1}\,{\xi_m}^2 \ ,$$

and the quadratic form $\Phi_{t,n}$ on $F' = \mathbf{A}(t)^n$ with values in $\mathbf{A}(t)$ as

$$\Phi_{t,n}(\zeta_1, \ldots, \zeta_n) \quad = \quad {\zeta_1}^2 + t\,{\zeta_2}^2 + \cdots + t^{n-1}\,{\zeta_n}^2 \ .$$

The "associated inner products" with $\Phi_{t,m}$ and $\Phi_{t,n}$ will be denoted by $\langle\cdot,\cdot\rangle_{E'}$ and $\langle\cdot,\cdot\rangle_{F'}$ respectively.

Given a linear transformation $\varphi \in \mathcal{L}(E, F)$, we get by extension of scalars a linear transformation φ' in $\mathcal{L}(E', F')$. The matrix A of φ is the same as the matrix of φ'.

Thus, there exists only one linear transformation, $A^\circ : F' \to E'$, which verifies:

$$\forall x \in E', \quad \forall y \in F', \qquad \langle A\,x, y\rangle_{F'} = \langle x, A^\circ\,y\rangle_{E'} \,. \tag{1}$$

If $Q_m = \mathrm{diag}(1,\, t,\, t^2, \ldots, t^{m-1})$ and $Q_n = \mathrm{diag}(1,\, t,\, t^2, \ldots, t^{n-1})$ are the diagonal matrices associated with $\langle\cdot,\cdot\rangle_{E'}$ and $\langle\cdot,\cdot\rangle_{F'}$ respectively, A° is given by

$$A^\circ \;=\; Q_m^{-1}\, A^{\mathrm{t}}\, Q_n.$$

Hence, for all $x \in \mathbf{A}(t)^{m\times 1}$, $y \in \mathbf{A}(t)^{n\times 1}$, we have $(A\,x)^{\mathrm{t}}\, Q_n\, y \;= x^{\mathrm{t}}\, Q_m\, (A^\circ\, y)$. In practice, if $A = (a_{i,j})$, then $A^\circ = (t^{j-i}\, a_{j,i})$.

Definition 6 *The Generalized Gram's Polynomials, $\mathcal{G}'_k(A)(t) = a_k(t) \in \mathbf{A}[t, 1/t]$, and the Generalized Gram's Coefficients, $\mathcal{G}'_{k,\ell}(A) = a_{k,\ell} \in \mathbf{A}$, are given by the following expression:*

$$\left\{\begin{array}{rcl} \det(\mathrm{I}_m + z\,AA^\circ) & = & 1 \,+\, a_1(t)\,z \,+\, \cdots \,+\, a_n(t)\,z^n \\ a_k(t) & = & t^{-k(m-k)} \left(\displaystyle\sum_{\ell=0}^{k(m+n-2k)} a_{k,\ell}\, t^\ell \right). \end{array}\right. \tag{2}$$

Observe that if the matrix A is real, usual Gram's Coefficients are obtained by substituting 1 for t in the expression $\mathcal{G}'_k(A)(t)$. Furthermore, if $\mu_{\alpha,\beta}$ denotes the k–minor where the rows and columns retained are given by subscripts $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subset \{1, \ldots, n\}$ and $\beta = \{\beta_1, \ldots, \beta_k\} \subset \{1, \ldots, m\}$, then the Generalized Gram's Coefficient $a_{k,\ell} = \mathcal{G}'_{k,\ell}(A)$ is given by

$$\mathcal{G}'_{k,\ell}(A) \;=\; \sum_{(\alpha,\beta)\in S_{n,m,k,\ell}} {\mu_{\alpha,\beta}}^2. \tag{3}$$

with

$$|\alpha| = \sum_{i\le k} \alpha_i, \;\; |\beta| = \sum_{i\le k} \beta_i, \;\; S_{n,m,k,\ell} = \{(\alpha,\beta) \mid |\alpha| - |\beta| = \ell - k(m-k)\}\,.$$

Example: Suppose that the matrix A is equal to the generic 2×3 matrix. Then

$$A \;=\; \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad A^\circ \;=\; \begin{bmatrix} a_{11} & t\,a_{21} \\ t^{-1}\,a_{12} & a_{22} \\ t^{-2}\,a_{13} & t^{-1}\,a_{23} \end{bmatrix},$$

and the Generalized Gram's Polynomials are given by $\det(\mathrm{I}_2 + zA\,A^\circ)$:

$$\begin{aligned} 1 \quad &+ \left(a_{2,1}^2 t + \left(a_{1,1}^2 + a_{2,2}^2\right) + \frac{a_{1,2}^2 + a_{2,3}^2}{t} + \frac{a_{1,3}^2}{t^2} \right) z \\ &+ \left(\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}^2 + \frac{\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix}^2}{t} + \frac{\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix}^2}{t^2} \right) z^2. \end{aligned}$$

Definition 7 *Given k, the Gram's ideal $\mathcal{C}_k(A)$ of the matrix A is the ideal generated by the Generalized Gram's Coefficients $\mathcal{G}'_{h,\ell}(A)$ for $h \geq k$.*

$$\mathcal{C}_k(A) \ = \langle \mathcal{G}'_{h,\ell}(A),\, h \geq k \rangle$$

Remark that if « $\mathcal{C}_{k+1}(A) = 0$ and $\mathcal{C}_k(A) = \langle 1 \rangle$ », then degree$(\mathrm{P}_{AA^\circ}(z)) = k$ and $a_k(t)$ is invertible in $\mathbf{A}(t)$.

Proposition 8 *We have $\sqrt{\mathcal{C}_k(A)} = \sqrt{\mathcal{D}_k(A)}$. More precisely, there exists $r \in \mathbb{N}$ which depends only on (m, n, k) such that*

$$\mathcal{D}_k(A)^r \subset \mathcal{C}_k(A) \subset \mathcal{D}_k(A)^2 \subset \mathcal{D}_k(A).$$

Corollary 2.1 *Let M be a projective module over $\mathbf{A}$ isomorphic to the image of a projector $P \in \mathbf{A}^{n\times n}$. Then $\mathcal{D}_k(P) = \mathcal{C}_k(P)$.*

Theorem 9 *Let $A \in \mathbf{A}^{n\times m}$, with $n \leq m$. A has full rank if and only if $\mathcal{C}_n(A) = \mathcal{D}_n(A) = \langle 1 \rangle$.*

For details and proofs see [5] and [6].

3 Localization

Recall that M is a projective module over $\mathbf{A}$ isomorphic to the image of a projector $P \in \mathbf{A}^{n\times n}$, of rank equal to k. Assume that $k \neq 0$ and $k \neq n$. Here we will discuss the number of needed localizations.

Let $n = h + k$. Let us consider a field $\mathbb{K}$ with at least $kh + 2$ distinct elements. Let $Z = (z_i)_{0\leq i\leq kh}$ be a family of $kh + 1$ non-zero distinct elements of $\mathbb{K}$. Under these assumptions, we introduce a result based on Proposition 3 of [1, Chistov & al.].

Proposition 10 *Let $a_0, \ldots, a_{n-1}$ be distinct elements of $\mathbb{K}$. We define $H = H_{n,k}(s) = (s^{i-1}a_{i-1}^{j-1}) \in \mathbb{K}[s]^{n\times k}$. Then, for every matrix $A \in \mathbb{K}^{n\times h}$ of rank equal to h, the polynomial $V_A(s) = \det(A|H)$ is not identically zero and has at most kh roots different from zero.*

Example: For a 4×2 matrix of rank equal to 2, the matrix H is given by

$$\begin{pmatrix} 1 & a_0 \\ s & sa_1 \\ s^2 & s^2a_2 \\ s^3 & s^3a_3 \end{pmatrix}$$

The following corollaries are consequences of Proposition 10.

Corollary 3.1 *Let $A \in \mathbb{K}^{n\times h}$ be a matrix of rank equal to h. Then, there exists $z \in Z$ such that the matrix $(A|H_{n,k}(z))$ is invertible.*

That means that the subspace generated by columns of A is a direct complement to the subspace generated by the columns of $H_{n,k}(z)$ (columns of A and $H_{n,k}(z)$ span $\mathbb{K}^n$).

Corollary 3.2 *Let $\mathbb{L}$ be a field such that $\mathbb{K} \subseteq \mathbb{L}$. Let $Q \in \mathbb{L}^{n\times n}$ be a matrix of rank h. Then there exists $z \in Z$ such that the matrix $[Q|H_{n,k}(z)]$ has full rank, which implies that the Generalized Gram's Polynomial $\mathcal{G}'_n([Q|H_{n,k}(z)])(t)$ is not identically zero.*

Corollary 3.3 *Let Q be a generic $n\times n$ matrix over $\mathbb{K}$ (i.e., the matrix $(q_{i,j})$ in n^2 independent indeterminates over the polynomial ring $\mathbb{K}[q_{i,j}]$). Then, the following system of equations has no solution in any field containing $\mathbb{K}$*

$$Q^2 = Q,\ \mathrm{r}_h(Q) = 1,\ \bigwedge_{\ell,j} \mathcal{G}'_{n,l}\big([Q|H_{n,k}(z_j)]\big) = 0, \tag{4}$$

which implies by the Weak Nullstellensatz that there exists a linear combination of such equations equal to 1, with coefficients in $\mathbb{K}[q_{ij}]$.

The following proposition introduces a sufficient condition for a module to be free.

Proposition 11 *Let E be a projective module of rank h and let F be a module generated by k elements, $f_1, \ldots, f_k$, such that $\mathbf{A}^n = E + F$, with $n = k + h$. Then $\mathbf{A}^n = E \oplus F$ and F is free of rank k, with basis $f_1, \ldots, f_k$.*

Proof.
Suppose $x \in E \cap F$. We will prove $x = 0$. Consider a localization $\mathbf{A}_\mu$ of $\mathbf{A}$ at one comaximal element μ, where E_μ is free of rank h (i.e., $E_\mu \simeq \mathbf{A}_\mu^h$). Let $e_1, \ldots, e_h$ be a basis for E_μ. Hence, $\mathbf{A}_\mu^n = E_\mu + F_\mu = \langle e_1, \ldots, e_h, f_1, \ldots, f_k \rangle$. Since $n = h + k$, we have $\mathbf{A}_\mu^n = E_\mu \oplus F_\mu$ and so $x = 0$ in $\mathbf{A}_\mu^n$. Since that happens for every localization which makes E free, it follows that $x = 0$ in $\mathbf{A}$ and thus $\mathbf{A}^n = E \oplus F$.
This implies that F is a projective module or rank k. Since a projective module of rank k with k given generators is free, we can conclude that F is free, which completes the proof. $\square$

We can now state our result.

Theorem 12 *Let M be a projective $\mathbf{A}$-module of rank k with n generators. Assume that $\mathbf{A}$ contains a field $\mathbb{K}$ with at least $hk + 1$ non-zero distinct elements with $n = h + k$. Then, there exists a comaximal family G, with $|G| \leq (hk + 1)(nk + 1)$, such that for every $g \in G$, the module M_g is a free module of rank k over $\mathbf{A}[1/g]$.*

Proof.
Consider the matrix $Q = \mathrm{I}_n - P$ which is the projector of the complement of M. Observe that Q is of rank $h = n - k$, $Q^2 = Q$ and $\mathrm{r}_h(Q) = 1$. Then, Corollary 3.3 tells us that the family $G = \{\mathcal{G}'_{n,i}([Q|H_{n,k}(z_j)]),\ z_j \in Z, 0 \leq i \leq nk\}$ is comaximal in $\mathbf{A}$. That is, there is a linear combination of $\mathcal{G}'_{n,i}([Q|H_{n,k}(z_j)])$ with coefficients in $\mathbf{A}$ equal to one

$$1 = a_{0,0}\, \mathcal{G}'_{n,0}([Q|H_{n,k}(z_0)]) + a_{1,0}\, \mathcal{G}'_{n,1}([Q|H_{n,k}(z_0)]) + \ldots + a_{nk+1,n-1}\, \mathcal{G}'_{n,nk}([Q|H_{n,k}(z_{hk})]).$$

Now, consider an element of G, for example $g = \mathcal{G}'_{n,i}([Q|H_{n,k}(z_j)])$. Then $\mathcal{C}_m([Q|H_{n,k}(z_j)]) = \langle 1 \rangle$ and the matrix $[Q|H_{n,k}(z_j)]$ has full rank over the localization $\mathbf{A}[1/g]$. We can claim by Proposition 10 combined with Proposition 11 that the columns of $H_{n,k}(z_j)$ define a basis of a free module, direct complement of $\operatorname{Im} Q$. Such a basis is transformed by P into a basis of $\operatorname{Im} P$ and so we can conclude that M_g is free as $\mathbf{A}[1/g]$-module. We have, thus, shown that G is the comaximal family we searched. $\square$

Observe that for most of the cases, $\binom{n}{k} > (hk+1)(nk+1)$. However, the ring $\mathbf{A}$ is supposed to contain a field $\mathbb{K}$ with at least $hk + 2$ different elements. In [2, Chistov & al.], they improve the bound on the minimal number of elements in the field. For a finite field $\mathbb{K}$ with $|\mathbb{K}| > \min(h, k)$, they build a family of $(hk + 1)^3$ matrices with the property described in Corollary 3.1.

Conclusion

Concerning the general problem of finding few comaximal localizations upon which a given projective module becomes free we can add the following remarks.

1. For rank one modules and an arbitrary $n > 1$ there exist projective modules M generated by n elements but not by $n - 1$ elements (see e.g. [12, Swan, 1962]). Moreover, if there exist k comaximal elements s_i such that every M_{s_i} is free, generated by x_i, then the $x_i's$ generate M and $k \geq n$. In conclusion the general bound n cannot be improved.

2. Following the same reasoning for projective modules of rank 2, the fact of finding a module of rank 2 with $2n$ generators that cannot be generated by $2n-1$ elements would imply that at least n comaximal free localizations are needed. But here n is much less than the general bound $\binom{2n}{2} = n(n-1)$.

3. Using the formal Nusllstellensatz instead of Nullstellensatz, it is possible to slightly weaken hypothesis of Theorem 12: we don't need a field $\mathbb{K}$ with at least $hk + 1$ non-zero distinct elements inside $\mathbf{A}$, it is sufficient to assume we have in $\mathbf{A}$ a family of $hk + 1$ elements v_i such that all v_i and all $v_i - v_j$ for $i \neq j$ are invertible.

4. It remains unknown (at least for us) if the generic case (an idempotent matrix F of size n whose entries are indeterminates over $\mathbb{Z}$ constrained only by $F^2 = F$ and $\mathrm{r}_k(\operatorname{Im} F) = 1$) admits a bound better than $\binom{n}{k}$ on the number of comaximal free localizations.

5. It is known from [11, Serre, 1957/58] and [7, Forster, 1964] that over a Noetherian ring $\mathbf{A}$ of Krull dimension k any projective module of rank $r = k + \ell$ can be written as $\mathbf{A}^\ell \oplus N$ where N is a rank k projective module generated by at most $2k$ elements. It follows that in this case the general bound $\binom{n}{r}$ can be replaced by any general bound for rank k modules generated by $2k$ elements, e.g. $\binom{2k}{k}$.
Moreover the Noetherian hypothesis has been removed in [8, Heitmann, 1984]. Other improvements are due to [13, Swan, 1967] and [4, 3, Coquand, 2004, 2007]. In [4, 3], proofs are constructive and more details can be found in [10, Lombardi & Quitté, chapter 14]. Nevertheless the corresponding algorithms are far from being implemented.

6. The result given in the present paper shows that some link can be established between the minimal number of affine charts for some grassmannian variety and the minimal number of comaximal free localizations for a projective module of constant rank. It should be interesting to understand better this kind of links.

7. Observe that in practice Theorem 12 implies on the one hand that the module M is given by a projector matrix P or a presentation matrix Q, and on the other hand, the computation of the family G. By combining the results of the paper with standard computations in computer linear algebra over an arbitrary computable ring (as in [6]), we can conclude that computations required in such a theorem are of polynomial arithmetic complexity. Moreover if the determinants whose addition defines the coefficients of every $\mathcal{G}'_{n,i}([Q|H_{n,k}(z_j)])$ are of polynomially bounded size, then the bit complexity is also polynomial w.r.t. the size of the data.

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