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Research Article

Positive Solutions for a General Gause-Type Predator-Prey Model with Monotonic Functional Response

Guohong Zhang and Xiaoli Wang

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

Correspondence should be addressed to Guohong Zhang, zgh711@swu.cn

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We study a general Gause-type predator-prey model with monotonic functional response under Dirichlet boundary condition. Necessary and sufficient conditions for the existence and nonexistence of positive solutions for this system are obtained by means of the fixed point index theory. In addition, the local and global bifurcations from a semitrivial state are also investigated on the basis of bifurcation theory. The results indicate diffusion, and functional response does help to create stationary pattern.

1. Introduction

In this paper, we are interested in the following semilinear elliptic system with monotonic functional response under Dirichlet boundary condition:

$$\begin{aligned} -d_1\Delta u &= ug(u) - p(u)v \quad \text{in } \Omega, \\ -d_2\Delta v &= -cv + m(x)p(u)v \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in R^N ($N \geq 1$ is an integer) with a smooth boundary $\partial\Omega$. The two functions u and v represent the densities of the prey and predator, respectively. The positive constants d_1 and d_2 are the diffusion coefficients of the corresponding species, c is the death rate of the predator, and $m(x)$, which is assumed to be space dependent, represents the conversion rate of the prey to predators. The function $g(u)$ denotes the growth rate of the prey species in the absence of predator. Throughout this paper, we impose the following hypotheses on the function $g(u)$.

($\mathbb{H}1$) $g \in C^1([0,\infty))$, $g(0) > 0$, $-\widetilde{g} < g_u(u) < 0$ for all $u \geq 0$ with a positive constant $\widetilde{g}$; there exists a unique positive constant K such that $g(K) = 0$.

Obviously, the classical Logistic growth rate $g(u) = r(1-(u/K))$ satisfies ($\mathbb{H}1$). The function $p(u)$ denotes the functional response of predators to prey. According to different biology backgrounds, the functional response $p(u)$ may have several forms and many important results on the dynamics of predator-prey systems with different functional response have been obtained (see [1–20] and references therein). In many predator-prey interactions, the functional responses satisfies the following hypotheses.

($\mathbb{H}2$) $p \in C^2([0,\infty))$, $p(0) = 0$, $0 < p_u(u) < \widehat{p}$ for all $u > 0$ with a positive constant $\widehat{p}$.

It is easy to see that Holling-type I, Holling-type II, Holling-type III, and Ivelev functional response satisfy hypothesis ($\mathbb{H}2$).

In this work, we aim to understand the influence of diffusion and functional response on pattern formation, that is, the positive solutions of (1.1). Throughout this paper, a solution (u,v) of (1.1) is called a positive solution if $u(x) > 0$, $v(x) > 0$ for all $x \in \Omega$ and $(\partial w/\partial_{\nu_x}), (\partial v/\partial_{\nu_x}) < 0$ for all $x \in \partial\Omega$, where ∂_{ν_x} stand for the outward unit norm to Ω at x. As a consequence, the results indicate the stationary pattern arises when the diffusion coefficient enter into certain regions. In other words, we show that diffusion does help to create stationary pattern and diffusion and functional response can become determining factors in the formation pattern. Furthermore, we also investigate the properties of the nonconstant positive solution by using local bifurcation theory introduced by Crandall and Rabinowitz in [21] and global bifurcation theory introduced by López-Gómez and Molina-Meyer in [22]. We remark that problem (1.1) with Neumann boundary conditions was discussed in [5] recently. We point out that our results about the existence and nonexistence of positive solutions are different from [5] (see Corollary 3.8 and Remark 3.9).

The rest of this paper is organized as follows. In Section 2, some necessary preliminaries are introduced. In Section 3, we will give a priori upper bounds for positive solutions and investigate the existence and nonexistence of positive solutions of (1.1). In Section 4, the local bifurcations about parameter c are investigated. Finally, the results about global bifurcations are obtained in Section 5.

2. Some Preliminaries

In order to give the main results and complete the corresponding proofs, we need to introduce some necessary notations and theorems as the following.

For each $h \in C^\alpha(\Omega)$ $(0 < \alpha < 1)$, let $\lambda_1(h)$ denote the principle eigenvalue of the following eigenvalue problem:

$$\begin{aligned} -d_1\Delta u + h(x)u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Let $\lambda_1^*(h)$ denote the principle eigenvalue of the following eigenvalue problem:

$$-d_2 \Delta u + h(x)u = \lambda u \quad \text{in}\Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

and denote $\lambda_1(0), \lambda_1^*(0)$ by $\lambda_1, \ \lambda_1^*$ for simplicity. It is easy to know that $\lambda_1(h), \lambda_1^*(h)$ is strictly increasing (see [23, 24]).

In order to calculate the indexes at the trivial and semitrivial states by means of the fixed point index theory, we also need to introduce the following theorem.

Theorem 2.1 (see [9, 13]). *Assume $h \in C^\alpha(\Omega)$ $(0 < \alpha < 1)$ and M is a sufficiently large number such that $M > h(x)$ for all $x \in \overline{\Omega}$. Define a positive and compact operator $\mathbb{L} = (-d_1\Delta + M)^{-1}(M - h(x))$. Denote the spectral radius of $\mathbb{L}$ by $r(\mathbb{L})$.*

(i) $\lambda_1(h) > 0$ *if and only if* $r(\mathbb{L}) < 1$;

(ii) $\lambda_1(h) < 0$ *if and only if* $r(\mathbb{L}) > 1$;

(iii) $\lambda_1(h) = 0$ *if and only if* $r(\mathbb{L}) = 1$.

It is easy to see that the corresponding conclusions in Theorem 2.1 are also correct if the positive and compact operator $\mathbb{L} = (-d_1\Delta + M)^{-1}(M - h(x))$ is replaced by $\mathbb{L} = (-d_2\Delta + M)^{-1}(M - h(x))$.

From Theorem 2.1, we see that it is crucial to know the sign of the eigenvalue $\lambda_{1,k}(h)$ to determine the spectral radius of $\mathbb{L}$. The following theorem give some sufficient conditions to determine the sign of the eigenvalue $\lambda_{1,k}(h)$.

Theorem 2.2 (see [7, 9, 10, 23, 24]). *Let $h(x) \in L^\infty(\Omega)$ and $\varphi \geq 0$, $\varphi \not\equiv 0$ in Ω with $\varphi = 0$ on $\partial\Omega$. Then one has*

(i) *if* $0 \not\equiv -\Delta\varphi + h(x)\varphi \leq 0$, *then* $\lambda_1(h(x)) < 0$;

(ii) *if* $0 \not\equiv -\Delta\varphi + h(x)\varphi \geq 0$, *then* $\lambda_1(h(x)) > 0$;

(iii) *if* $-\Delta\varphi + h(x)\varphi \equiv 0$, *then* $\lambda_1(h(x)) = 0$.

Consider the following equation:

$$-d_1\Delta\varphi = \varphi g(\varphi) \quad \text{in } \Omega,$$
$$\varphi = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where Ω is a bounded domain in R^N ($N \geq 1$ is an integer) with a smooth boundary $\partial\Omega$.

Theorem 2.3 (see [7, 23, 24]). *Assume that the function $g(\varphi) : \overline{\Omega} \to R$ satisfies the following hypotheses:*

(i) $g(\varphi) \in C^1(\Omega)$ *and* $g_\varphi(\varphi) < 0$ *for all* $\varphi \geq 0$;

(ii) $g(\varphi) \leq 0$ *for* $\varphi \geq C$, *where C is a positive constant.*

Then, (2.3) *has a unique positive solution if* $\lambda_1(-g(0)) < 0$.

Let $\Theta(g(\varphi))$ be the unique positive solution of (2.3) when the unique positive solution exists. Denote $\Theta(g(0))$ by Θ for simplicity.

Remark 2.4. It is easy to see that if the function $g(\varphi)$ satisfies the hypothesis $(\mathbb{H}1)$, then it must satisfies the conditions (i) and (ii) in Theorem 2.3. We also point out that the condition $\lambda_1(-g(0)) < 0$ holds if and only if $g(0) > \lambda_1$. Therefore, if the function $g(\varphi)$ satisfies the hypothesis $(\mathbb{H}1)$ and $g(0) > \lambda_1$, then (2.3) has a unique positive solution.

Now, we introduce the fixed point index theory which plays an important role in finding the sufficient conditions for the existence of positive solutions of model (1.1).

Let $\mathbb{E}$ be a real Banach space and let $\mathbb{W} \subset \mathbb{E}$ be the natural positive cone of $\mathbb{E}$. $\mathbb{W} \subset \mathbb{E}$ is a closed convex set. $\mathbb{W}$ is called a total wedge if $\tau\mathbb{W} \subset \mathbb{W}$ and $\mathbb{W} - \mathbb{W} = \mathbb{E}$. For $y \in \mathbb{W}$, define $\mathbb{W}_y = \{x \in \mathbb{E} : y + \gamma \in \mathbb{W} \text{ for some } \gamma > 0\}$ and $S_y = \{x \in \overline{\mathbb{W}}_y : -x \in \overline{\mathbb{W}}_y\}$. Then, $\overline{\mathbb{W}}_y$ is a wedge containing $\mathbb{W}$, y, $-y$, while S_y is a closed subset of $\mathbb{E}$ containing y. Let T be a compact linear operator on $\mathbb{E}$ which satisfies $T(\overline{\mathbb{W}}_y) \subset \overline{\mathbb{W}}_y$. We say that T has property α on $\overline{\mathbb{W}}_y$ if there is a $t \in (0, 1)$ and an $\omega \in \overline{\mathbb{W}}_y \setminus S_y$ such that $(I - tT)\omega \in S_y$. Let $\mathcal{A} : \mathbb{W} \to \mathbb{W}$ be a compact operator with a fixed point $y \in \mathbb{W}$ and $\mathcal{A}$, a Fréchet differentiable at y. Let $\mathbb{L} = \mathcal{A}'(y)$ be the Fréchet derivative of $\mathcal{A}$ at y. Then, $\mathbb{L}$ maps $\overline{\mathbb{W}}_y$ into itself. We denote by $\deg_{\mathbb{W}}(I - \mathcal{A}, \mathbb{D})$ the degree of $I - \mathcal{A}$ in $\mathbb{D}$ relative to $\mathbb{W}$, $\text{index}_{\mathbb{W}}(\mathcal{A}, y)$ the fixed point index of $\mathcal{A}$ at y relative to $\mathbb{W}$. Then, the following theorem can be obtained.

Theorem 2.5 (see [5, 11, 13]). *Assume that $I - \mathbb{L}$ is invertible on $\mathbb{W}_y$.*

(i) *If $\mathbb{L}$ have property α on $\mathbb{W}_y$, then $\text{index}_{\mathbb{W}}(\mathcal{A}, y) = 0$;*

(ii) *If $\mathbb{L}$ does not have property α on $\mathbb{W}_y$, then $\text{index}_{\mathbb{W}}(\mathcal{A}, y) = (-1)^{\sigma}$, where σ is the sum of algebraic multiplicities of the eigenvalues of $\mathbb{L}$ which are greater than 1.*

Finally, we introduce a result about global bifurcation, which was introduced by López-Gómez and Molina-Meyer in [22] and we state here for convenience.

Let U be an ordered Banach space whose positive cone P is normal and has nonempty interior, and consider the nonlinear abstract equation:

$$\mathfrak{F}(\lambda, u) = \mathfrak{L}(\lambda)u + \mathfrak{R}(\lambda, u), \tag{2.4}$$

where

$(\mathbb{H}\mathfrak{L})$ $\mathfrak{L}(\lambda) := I_U - \mathfrak{N}(\lambda) \in \mathcal{L}(U)$, $\lambda \in \mathbb{R}$, is a compact and continuous operator pencil with a discrete set of singular values, denoted by $\mathfrak{G}$.

$(\mathbb{H}\mathfrak{R})$ $\mathfrak{R} \in C(\mathbb{R} \times U; U)$ is compact on bounded sets and

$$\lim_{u \to 0} \frac{\mathfrak{R}(c, u)}{\|u\|_{C(\overline{\Omega})}} = 0 \tag{2.5}$$

uniformly on compact intervals of $\mathbb{R}$.

$(\mathbb{H}\mathbb{P})$ The solutions of (2.4) satisfy the strong maximum principle in the sense that

$$(c, u) \in \mathbb{R} \times (P \setminus \{0\}), \qquad \mathfrak{F}(c, u) = 0 \Longrightarrow u \in \operatorname{Int} P, \tag{2.6}$$

where $\operatorname{Int} P$ stands for the interior of the cone P.

Define the parity mapping $C : \mathfrak{G} \mapsto \{-1, 0, 1\}$ by

$$C(\sigma) := \frac{1}{2}\lim_{\varepsilon \mapsto 0}[\operatorname{Ind}(0, \mathfrak{N}(\sigma+\varepsilon)) - \operatorname{Ind}(0, \mathfrak{N}(\sigma-\varepsilon))], \quad \sigma \in \mathfrak{G}. \tag{2.7}$$

Then, thanks to [25, Theorem 6.2.1], 2.4 possesses a component emanating from $(\lambda, 0)$ at λ_0 if $C(\lambda_0) \in \{-1, 1\}$. Such a component will be subsequently denoted by $\mathfrak{C}_{\lambda_0}$. Then, the following abstract result hold.

Theorem 2.6. *Suppose that $\lambda_0 \in \mathfrak{G}$ satisfies $C(\lambda_0) \neq 0$,*

$$N[\mathfrak{L}(\lambda_0)] = \operatorname{span}[\varphi_0], \quad \varphi_0 \in P \setminus \{0\}, \tag{2.8}$$

and $\mathfrak{N}(\lambda_0)$ is strongly positive in the sense that

$$\mathfrak{N}(\lambda_0)(P \setminus \{0\}) \subset \mathit{Int} P. \tag{2.9}$$

Then, there exists a subcomponent $\mathfrak{C}^P_{\lambda_0}$ of $\mathfrak{C}_{\lambda_0}$ in $\mathbb{R} \times \mathit{Int}\, P$ such that $(\lambda_0, 0) \in \overline{\mathfrak{C}}^P_{\lambda_0}$.

Moreover, if λ_0 is the unique singular value for which 1 is an eigenvalue of $\mathfrak{N}(\lambda)$ to a positive eigenvector, then $\mathfrak{C}^P_{\lambda_0}$ must be unbounded in $\mathbb{R} \times U$.

Remark 2.7. When we are working in a product-ordered Banach space, the conditions (2.6) and (2.9) can be modified as

$$\begin{gathered}(c, u, v) \in [\mathbb{R} \times (P \setminus \{0\}) \times (P \setminus \{0\})], \qquad \mathfrak{F}(c, u, v) = 0 \Longrightarrow (u, v) \in \operatorname{Int} P \times \operatorname{Int} P, \\ \mathfrak{N}(c)([P \setminus \{0\}] \times [P \setminus \{0\}]) \subset \operatorname{Int} P \times \operatorname{Int} P.\end{gathered} \tag{2.10}$$

For the technical details, one can refer to [25, Theorem 7.2.2] and [26, Proposition 2.2]. To avoid a repetition, we omitted it herein.

3. Existence and Nonexistence of Stationary Pattern

At first, we introduce the following lemma which gives the necessary condition for (1.1) to have positive solutions.

Lemma 3.1. *If problem* (1.1) *has a positive solution, then $g(0) > \lambda_1$ and $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$.*

Proof. Assume (u, v) is a positive solution of (1.1). Then, it is obvious that $g(0) > \lambda_1$ and $u < \Theta$ by maximum principle. Because (u, v) satisfies

$$\begin{gathered}-d_2\Delta v = -cv + mp(u)v \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{3.1}$$

we have

$$0 = \lambda_1^*\left(c - mp(u)\right) > \lambda_1^*\left(c - mp(\Theta)\right) = c + \lambda_1^*\left(-mp(\Theta)\right),$$
$$0 = \lambda_1^*\left(c - mp(u)\right) < \lambda_1^*(c) = c + \lambda_1^*. \tag{3.2}$$

So, $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$. □

In the rest of this section, we shall prove that the necessary conditions in Lemma 3.1 are also sufficient conditions by means of the fixed point index theory. So, we need to obtain a priori bound for the positive solutions of (1.1).

Theorem 3.2. *Assume $c > -\lambda_1^*$ and (u, v) is a positive solution of* (1.1)*. Then, one has*

$$u \le g(0), \qquad v \le g(0)\left(\frac{cd_1 m}{d_2} + mg(0)\right)\left\|\left(-\Delta + \frac{c}{d_2}\right)^{-1}\right\|_{C(\overline{\Omega})}. \tag{3.3}$$

Proof. It is obvious that $u(x) \le g(0)$ by the maximum principle. From (1.1), we can find that

$$\begin{aligned} -\Delta(d_1 mu + d_2 v) &= -cv + mp(u)v + mug(u) - mp(u)v \\ &= -cv + mug(u) \\ &= -\frac{c}{d_2}(d_1 mu + d_2 v) + u\left(\frac{cd_1}{d_2}m + mg(u)\right) \end{aligned} \tag{3.4}$$

and hence

$$\left(-\Delta + \frac{c}{d_2}\right)(d_1 mu + d_2 v) = u\left(\frac{cd_1}{d_2}m + mg(u)\right). \tag{3.5}$$

Therefore,

$$\begin{aligned} v &\le \frac{1}{d_2}(d_2 v + d_1 mu) \le \left(-\Delta + \frac{c}{d_2}\right)^{-1} \frac{u}{d_2}\left(\frac{cd_1}{d_2}m + mg(u)\right) \\ &\le \frac{g(0)}{d_2}\left(\frac{cd_1 m}{d_2} + mg(0)\right)\left(-\Delta + \frac{c}{d_2}\right)^{-1}(1) \\ &\le \frac{g(0)}{d_2}\left(\frac{cd_1 m}{d_2} + mg(0)\right)\left\|\left(-\Delta + \frac{c}{d_2}\right)^{-1}\right\|_{C(\overline{\Omega})}. \end{aligned} \tag{3.6}$$

□

Now, we introduce the following notations:

$$
\begin{gathered}
\mathbb{E} = \left\{C_0^1\left(\overline{\Omega}\right) \times C_0^1\left(\overline{\Omega}\right)\right\}, \\
\mathbb{N} = \left\{\varphi \in C_0^1\left(\overline{\Omega}\right) : \varphi \geq 0 \text{ in } \overline{\Omega}\right\}, \\
\mathbb{W} = \mathbb{N} \times \mathbb{N}, \\
\mathbb{D} = \{(u, v) \in \mathbb{W} : u \leq (K+1), v \leq (R+1)\},
\end{gathered}
\tag{3.7}
$$

where $R = (g(0)/d_2)((cd_1 m)/d_2 + mg(0))\|(-\Delta + (c/d_2))^{-1}\|_{C(\overline{\Omega})}$. Take q sufficiently large with $q > \max\{g(0) + p_u(0)R, -c + p(0)\}$ such that $ug(u) - p(u)v + qu$ and $-cv + p(u)v + qv$ are, respectively, monotone increasing with respect to u and v for all $(u, v) \in [0, K] \times [0, R]$.

Define a positive and compact operator $\mathfrak{R} : \mathbb{E} \to \mathbb{E}$ by

$$
\mathfrak{R}(u, v) = \begin{pmatrix} (-d_1\Delta + q)^{-1}\left[ug(u) - p(u)v + qu\right] \\ (-d_2\Delta + q)^{-1}\left[-cv + p(u)v + qv\right] \end{pmatrix}. \tag{3.8}
$$

Remark 3.3. (i) By the maximum principle, it is easy to see that $v \equiv 0$ if $u \equiv 0$ in Ω in system (1.1). On the other hand, if $v \equiv 0$, then we have $-d_1\Delta u = ug(u)$ in Ω and $u = 0$ on $\partial\Omega$. From the assumption ($\mathbb{H}1$), we see that $(\Theta, 0)$ is the only semitrivial solution of (1.1) if $g(0) > \lambda_1$. Moreover, (1.1) does not have any other constant solution except the trivial solution $(0, 0)$.

(ii) Observe that (1.1) is equivalent to $(u, v) = \mathfrak{R}(u, v)$. Then, it is sufficient to prove that $\mathfrak{R}$ has a nonconstant positive fixed point in $\mathbb{D}$ to show that (1.1) has a positive solution.

(iii) From the Remarks (i) and (ii), we can see that it is necessary to calculate the fixed point index of $\mathfrak{R}$ at $(0, 0)$ and $(\Theta, 0)$. By Kronecker's existence theorem [23], we also need to calculate the topological degree of $\mathfrak{R}$ in $\mathbb{D}$ to prove that the necessary conditions in Lemma 3.1 are also sufficient.

At first, we shall calculate the topological degree of the operator $\mathfrak{R}$ in $\mathbb{D}$ and the fixed point index of the operator at $(0, 0)$, that is, $\deg_{\mathbb{W}}(I - \mathfrak{R}, \mathbb{D})$ and $\operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (0, 0))$. It is easy to see that $\mathfrak{R}$ has no fixed point on $\partial\mathbb{D}$. Then, the $\deg_{\mathbb{W}}(I - \mathfrak{R}, \mathbb{D})$ is well defined.

For $\mu \in [0, 1]$, we define a positive and compact operator $\mathfrak{R}_\mu : \mathbb{E} \to \mathbb{E}$ by

$$
\mathfrak{R}_\mu(u, v) = \begin{pmatrix} (-d_1\Delta + q)^{-1}\left[\mu(ug(u) - p(u)v) + qu\right] \\ (-d_2\Delta + q)^{-1}\left[\mu(-cv + p(u)\ v) + qv\right] \end{pmatrix}. \tag{3.9}
$$

Observe that

$$
\mathfrak{R}'(0, 0) = \begin{pmatrix} (-d_1\Delta + q)^{-1}(g(0) + q) & 0 \\ 0 & (-d_2\Delta + q)^{-1}(-c + q) \end{pmatrix} \tag{3.10}
$$

and $S_{(0,0)} = (0, 0)$, $\overline{\mathbb{W}}_{(0,0)} = \mathbb{N} \times \mathbb{N}$; we can obtain the following lemma and we omit the proofs because the calculations are standard.

Lemma 3.4. *Assume that $g(0) > \lambda_1$ and $c > -\lambda_1^*$. Then, one has*

(i) $\deg_{\mathbb{W}}(I - \mathfrak{R}, \mathbb{D}) = 1$,

(ii) $\mathrm{index}_{\mathbb{W}}(\mathfrak{R}, (0,0)) = 0$.

Now, we need to calculate the fixed point index of the operator $\mathfrak{R}$ at $(\Theta, 0)$, that is, $\mathrm{index}_{\mathbb{W}}(\mathfrak{R}, (\Theta, 0))$.

Lemma 3.5. *Assume that $g(0) > \lambda_1$ and $c > -\lambda_1^*$, Then, one has*

(i) *if $-c > \lambda_1^*(-mp(\Theta))$, then $\mathrm{index}_{\mathbb{W}}(\mathfrak{R}, (\Theta, 0)) = 0$;*

(ii) *if $-c < \lambda_1^*(-mp(\Theta))$, then $\mathrm{index}_{\mathbb{W}}(\mathfrak{R}, (\Theta, 0)) = 1$.*

Proof. (i) Observe $\mathfrak{R}(\Theta, 0) = (\Theta, 0)$. Let $\mathbb{L} = \mathfrak{R}'(\Theta, 0)$. Then,

$$\mathbb{L} = \mathfrak{R}'(\Theta, 0) = \begin{pmatrix} (-d_1\Delta + q)^{-1}(g(\Theta) + \Theta g_u(\Theta) + q) & (-d_1\Delta + q)^{-1}(-p(\Theta)) \\ 0 & (-d_2\Delta + q)^{-1}(-c + mp(\Theta) + q) \end{pmatrix}. \tag{3.11}$$

Assume $\mathbb{L}(\xi_1, \xi_2) = (\xi_1, \xi_2)$ for some $(\xi_1, \xi_2) \in \overline{\mathbb{W}}_{(\Theta,0)} = C_0^1(\overline{\Omega}) \times \mathbb{N}$. Then,

$$\begin{gathered} -d_1\Delta\xi_1 - \left(g(\Theta) + \Theta g_u(\Theta)\right)\xi_1 = -p(\Theta)\xi_2 \quad \text{in } \Omega, \\ -d_2\Delta\xi_2 - mp(\Theta)\xi_2 = -c\xi_2 \quad \text{in } \Omega, \\ \xi_1 = \xi_2 = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.12}$$

Taking account of $\xi_2 \in \mathbb{N}$, if $\xi_2 \neq 0$, then we can see from the second equation of (3.12) that $-c = \lambda_1^*(-mp(\Theta))$. This contradicts $-c \neq \lambda_1^*(-mp(\Theta))$. So, $\xi_2 = 0$. Then, we can get from the first equation of (3.12) that

$$\begin{gathered} -d_1\Delta\xi_1 - \left(g(\Theta) + \Theta g_u(\Theta)\right)\xi_1 = 0 \quad \text{in } \Omega, \\ \xi_1 = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.13}$$

If $\xi_1 \neq 0$, then $\lambda_1(-g(\Theta) - \Theta g_u(\Theta)) = 0$. On the other hand, $\lambda_1(-g(\Theta) - g_u(\Theta)\Theta) > \lambda_1(-g(\Theta)) = 0$, which is a contradiction. Therefore, $(\xi_1, \xi_2) = (0,0)$ and $I - \mathbb{L}$ is invertible on $\overline{\mathbb{W}}_{(\Theta,0)}$.

We claim that $\mathbb{L}$ has property α on $\overline{\mathbb{W}}_{(\Theta,0)}$. In fact, set

$$\Psi = \left(-d_2\Delta + q\right)^{-1}\left(-c + mp(\Theta) + q\right). \tag{3.14}$$

Since $-c > \lambda_1^*(-mp(\Theta))$, we can see that $r(\Psi) > 1$ is an eigenvalue of Ψ with a corresponding eigenfunction $\Phi_c > 0$ by Theorem 2.1. Because $S_{(\Theta,0)} = C_0^1(\overline{\Omega}) \times \{0\}$, we know that $(0, \Phi_c) \in \overline{\mathbb{W}}_{(\Theta,0)} \setminus S_{(\Theta,0)}$. Then, we have

$$\begin{aligned} \left(I - r_c^{-1}\mathbb{L}\right)\begin{pmatrix} 0 \\ \phi_c \end{pmatrix} &= \begin{pmatrix} 0 \\ \phi_c \end{pmatrix} - r_c^{-1}\begin{pmatrix} -(-d_1\Delta + q)^{-1} p(\Theta)\phi_c \\ (-d_2\Delta + q)^{-1}(-c + mp(\Theta) + q)\phi_c \end{pmatrix} \\ &= \begin{pmatrix} -(-d_1\Delta + q)^{-1} r_c^{-1} p(\Theta)\phi_c \\ 0 \end{pmatrix} \in S_{(\Theta,0)}. \end{aligned} \tag{3.15}$$

This establishes our claim. Hence, $\operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (\Theta, 0)) = 1$.

(i) From Remark 3.3, we know that the unique nonnegative solutions of (1.1) are $(0,0)$ and $(\Theta, 0)$ if $-c < \lambda_1^*(-mp(\Theta))$. Thus, we have

$$\deg_{\mathbb{W}}(I - \mathfrak{R}, D) = \operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (0,0)) + \operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (\Theta, 0)). \tag{3.16}$$

From Lemma 3.4, we know that $\deg_{\mathbb{W}}(I - \mathfrak{R}, \mathbb{D}) = 1$ and $\operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (0,0)) = 0$. Therefore, we have $\operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (\Theta, 0)) = 1$. □

Now, we can prove that $g(0) > \lambda_1$ and $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$ are also the sufficient conditions for model (1.1) to have a positive solution.

Lemma 3.6. *If $g(0) > \lambda_1$ and $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$, then model* (1.1) *has at least one positive solution.*

Proof. If $g(0) > \lambda_1$ and $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$, by Lemmas 3.4 and 3.5, then we have

$$\deg_{\mathbb{W}}(I - \mathfrak{R}, \mathbb{D}) - \operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (0,0)) - \operatorname{index}_{\mathbb{W}}(\mathfrak{R}, (\Theta, 0)) = 1. \tag{3.17}$$

Hence, model (1.1) has at least one positive solution by *Kronecker's* existence theorem [8]. □

From Lemmas 3.1 and 3.6, we can get the following theorem.

Theorem 3.7. *Problem* (1.1) *has at least one positive solution if and only if $g(0) > \lambda_1$ and $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$.*

Let λ_0 denote the principle eigenvalue of the following eigenvalue problem:

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.18}$$

Then, it is easy to see that the condition $g(0) > \lambda_1$ is equivalent to $d_1 < g(0)/\lambda_0$ and the condition $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$ is equivalent to $(\lambda_1^*(-mp(\Theta)) + mp(\Theta))/\lambda_0 < d_2 < (\lambda_1^* + mp(\Theta))/\lambda_0$. Therefore, one can get the following corollary from Theorem 3.7.

Corollary 3.8. *Problem* (1.1) *has no positive solution if one of the following conditions hold:*

(i) $d_1 > g(0)/\lambda_0$;

(ii) $d_2 > (\lambda_1^* + mp(\Theta))/\lambda_0$;

(iii) $d_2 < (\lambda_1^*(-mp(\Theta)) + mp(\Theta))/\lambda_0$.

Remark 3.9. From Corollary 3.8, we can see that if the prey diffuses so rapidly that $d_1 > g(0)/\lambda_0$, then no positive solution exists. On the other hand, if the predator diffuses so rapidly that $d_2 > (\lambda_1^* + mp(\Theta))/\lambda_0$ or diffuses so slowly that $d_2 < (\lambda_1^*(-mp(\Theta)) + mp(\Theta))/\lambda_0$, then we can also observe the same phenomena. These results are different from the corresponding results in paper [5]. In paper [5], if the predator diffuse so rapidly that $d_2 > \widetilde{D}(d_1)$, where $\widetilde{D}(d_1)$ is a constant, then the corresponding model has at least one positive solution (see [5], Theorem 3.8). How to explain these differences? The key point, we think, lies in the boundary conditions. Different from the reflecting boundary conditions, that is, Neumann boundary condition in [5], the prey and the predator in our model both face lethal boundary conditions, that is, Dirichlet conditions in our model. Therefore, the more rapidly the prey or the predator diffuses, the more possibly they encounter the lethal boundary and then the more possibly they cannot coexist.

4. Local Bifurcation

In this subsection, we will employ the local bifurcation theory [21] to investigate the positive solution branches of (1.1) which bifurcate from the semitrivial solution $(\Theta, 0)$ if $g(0) > \lambda_1$. We choose c as the bifurcation parameter and denote by $\Gamma_u = \{(c, \Theta, 0) : c \in R\}$ the semitrivial solution set with the parameter c. The next proposition gives the local bifurcation branch of positive solution of (1.1).

Theorem 4.1. *Assume that $g(0) > \lambda_1$. A branch of positive solutions of* (1.1) *bifurcates from Γ_u if and only if $c = -\lambda_1^*(-mp(\Theta))$. More precisely, there exists a positive number δ such that when $0 < s < \delta$, the local bifurcation positive solutions $(c(s), u(s), v(s))$ from $(-\lambda_1^*(-mp(\Theta)), \Theta, 0)$ have the following form:*

$$\begin{aligned} c(s) &= -\lambda_1^*(-mp(\Theta)) + c_1 s + O\left(s^2\right), \\ u(s) &= \Theta + s\phi^* + O\left(s^2\right), \\ v(s) &= s\psi^* + O\left(s^2\right), \end{aligned} \tag{4.1}$$

where $\psi^* = (-d_1\Delta - g(\Theta) - g_u(\xi)\Theta)^{-1}\phi^*$ with ξ between Θ and u and ϕ^* is the positive eigenfunction corresponding to $c = -\lambda_1^*(-mp(\Theta))$ of the following eigenvalue problem with $\int_\Omega \phi^2 = 1$:

$$\begin{aligned} &-d_2\Delta\phi - mp(\Theta)\phi = -c\phi \quad \text{in } \Omega, \\ &\phi = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

Furthermore, the bifurcation is subcritical, that is, $c'(0) < 0$.

Proof. Let us introduce the change of variable $w = \Theta - u$, which shifts the semitrivial solution $(\Theta, 0)$ to $(0, 0)$.

Introduce an operator $\Phi : \mathbb{R} \times C_0^{2+\alpha} \times C_0^{2+\alpha} \mapsto \mathbb{R} \times C_0^{\alpha} \times C_0^{\alpha}$ as the following:

$$\Phi(c, w, v) = \begin{pmatrix} d_1\Delta + w\big(g(\Theta) + g_u(\xi)\Theta - g_u(\xi)w\big) + p(\Theta - w)v \\ d_2\Delta + \big(-cv + mp(\Theta - w)v\big) \end{pmatrix}, \tag{4.3}$$

where ξ is between Θ and u. We will seek for the degenerate point of the linearized operator $\Phi_{(w,v)}\ (c, 0, 0)$. By a simple calculation, we have

$$\Phi_{(w,v)}(c, 0, 0)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} d_1\Delta\phi + \big(g(\Theta) + g_u(\xi)\Theta\big)\phi + p(\Theta)\psi \\ d_2\Delta\psi + \big(-c + mp(\Theta)\big)\psi \end{pmatrix}. \tag{4.4}$$

When $c = c^* = -\lambda_1^*(-mp(\Theta))$, it is easy to show that $\operatorname{Ker}\Phi_{(w,v)}(c^*, 0, 0) = \operatorname{Span}\{(\phi^*, \psi^*)\}$, where $\psi^* = (-d_1\Delta - g(\Theta) - g_u(\xi)\Theta)^{-1}\phi^* > 0$ in Ω.

If $(\tilde{\phi}, \tilde{\psi}) \in \operatorname{Range}\Phi_{(u,v)}(c^*, 0, 0)$, then there exist $(\phi, \psi) \in C_0^{2+\alpha} \times C_0^{2+\alpha}$ such that

$$\begin{gathered} d_1\Delta\phi + \big(g(\Theta) + g_u(\xi)\Theta\big)\phi + p(\Theta)\psi = \tilde{\phi} \quad \text{in } \Omega, \\ d_2\Delta\psi + \big(-c^* + mp(\Theta)\big)\psi = \tilde{\psi} \quad \text{in } \Omega, \\ \phi = \psi = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.5}$$

By the Fredholm alternative theorem, it is easy to see that the first equation of (4.5) is solvable if and only if $\int_\Omega \tilde{\psi}\psi^* dx = 0$.

For such a solution, the first equation enables us to obtain $\phi = (-d_1\Delta - g(\Theta) - g_u(\xi)\ \Theta)^{-1}(p(\Theta)\psi - \tilde{\phi})$. Therefore, we know that codim Range $\Phi_{(u,v)}(c^*, 0, 0) = 1$. In order to use the local bifurcation theorem [21] at the degenerate point, we need to verify that $\operatorname{Range}\Phi_{(w,v)c}(c^*, 0, 0)(\phi^*, \psi^*) \notin \operatorname{Range}\Phi_{(w,v)}(c^*, 0, 0)$. Here, it can be calculated that

$$\Phi_{(w,v)c}(c^*, 0, 0)\begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix} = \begin{pmatrix} 0 \\ -\psi^* \end{pmatrix}. \tag{4.6}$$

Suppose for contradiction that $\Phi_{(w,v)c}(c^*, 0, 0)(\phi^*, \psi^*) \in \operatorname{Range}\Phi_{w,v}(c^*, 0, 0)$. By (4.4) and (4.6), there exist $(\phi, \psi) \in \mathbb{E}$ such that

$$\begin{gathered} d_1\Delta\phi + \big(g(\Theta) + g_u(\xi)\Theta\big)\phi + p(\Theta)\psi = 0 \quad \text{in } \Omega, \\ d_2\Delta\psi + \big(-c^* + mp(\Theta)\big)\psi = -\psi^* \quad \text{in } \Omega, \\ \phi = \psi = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.7}$$

Then, multiplying the second equation of (4.7) by ψ^* and integrating the resulting expression, we obtain that $\int_\Omega (\psi^*)^2 dx = 0$, which obviously yields a contradiction. Consequently, we can apply the local bifurcation theorem to Φ at $(c, 0, 0)$. Furthermore, by virtue of the Krein-Rutman theorem, we know that the possibility of other bifurcation points except $c = c^*$ is excluded.

In order to investigate the bifurcation direction from $(-\lambda_1^*(-mp(\Theta)), \Theta, 0)$, substituting $(c(s), u(s), v(s))$ into the second equation of (1.1) and differentiating it with respect to s, setting $s = 0$, we have

$$-d_2 \Delta v_{ss}(0) = 2c_1\phi + \lambda_1^*(-mp(\Theta))v_{ss}(0) - 2mp_w(\Theta)\phi\psi + mp(\Theta)v_{ss}(0). \tag{4.8}$$

Multiplying (4.8) by ϕ and applying divergence theorem, we obtain

$$-\int_\Omega 2c_1\phi^2 dx = \int_\Omega \left(d_2\Delta\phi + \lambda_1^*(-mp(\Theta))\phi + mp(\Theta)\phi\right)v_{ss}(0)dx - \int_\Omega 2mp_u(\Theta)\phi^2\psi\, dx. \tag{4.9}$$

By (4.2), the terms including $v_{ss}(0)$ in (4.10) can be dropped out. Then, we can get

$$c_1 = -\frac{\int_\Omega mp_u(\Theta)\phi^2\psi\, dx}{\int_\Omega \phi^2 dx} = -\int_\Omega mp_u(\Theta)\phi^2\psi\, dx. \tag{4.10}$$

According to hypothesis ($\mathbb{H}2$), we have $p_u(\Theta) > 0$ and $c_1 < 0$. Then, we know that the bifurcation direction from $(-\lambda_1^*(-mp(\Theta)), \Theta, 0)$ is subcritical. $\square$

Remark 4.2. According to the theory of Rabinowitz [27], we can see that there is a continuum C_{c^*} of the set of non-trivial solutions of (1.1) with $(c^*, 0, 0) \in \overline{C}_{c^*}$ under the conditions of Theorem 4.1 and the continuum $\mathcal{C}_{c^*}$ consists of two subcontinua: $\mathcal{C}_{c^*}^+$, filled in by coexistence states, and $\mathcal{C}_{c^*}^-$, filled in by component-wise negative solution pairs in a neighborhood of $(-\lambda_1^*(-mp(\Theta)), \Theta, 0)$. However, this does not necessarily implies that the subcontinuum $\mathcal{C}_{c^*}^+$ satisfies the global alternative of Rabinowitz [27] by the reasons already explained by Dancer [12] and López-Gómez and molina-meyer [22]. Instead, the existence of a global subcontinuum $\mathcal{C}_{c^*}^+$ of the set of positive solutions with $(-\lambda_1^*(-mp(\Theta)), \Theta, 0) \in \mathcal{C}_{c^*}^+$ follows by slightly adapting [22, Theorem 1.1]. Therefore, in the following subsection, we shall study the global bifurcation from $(\Theta, 0)$ by using the global bifurcation theory of [22].

5. Global Bifurcation

In this subsection, basing on the results in Theorem 4.1, we can obtain the following results about global bifurcation from $(\Theta, 0)$ by using the global bifurcation theory introduced by López-Gómez, Molina-Meyer in [22].

Theorem 5.1. *Assume that $g(0) > \lambda_1$. Then, if one chooses c as the main continuation parameter of* (1.1)*, there exists an unbounded component $\mathcal{C}_{c^*}^+ \subset R \times \mathbb{E}$ of the set of positive solutions of* (1.1) *such that*

$$(c, u, v) = \left(-\lambda_1^*(-mp(\Theta)), \Theta, 0\right) \in \mathcal{C}_{c^*}^+, \qquad P_c\mathcal{C}_{c^*}^+ = \left(-\lambda_1^*, -\lambda_1^*(-mp(\Theta))\right), \tag{5.1}$$

where P_c stands for the projection operator into the c-component of the tern. Moreover, $\mathcal{C}_{c^*}^+$ must bifurcate from infinity at $c = -\lambda_1^*$.

Proof. Let $w = \Theta - u$. Then, (1.1) is equivalent to the following problem:

$$\begin{aligned} -d_1 \Delta w &= w\big(g(\Theta) + g_u(\xi)\Theta - g_u(\xi)w\big) + p(\Theta - w)v \quad \text{in } \Omega, \\ -d_2 \Delta v &= -cv + mp(\Theta - w)v \quad \text{in } \Omega, \\ w &= v = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.2}$$

where ξ is between Θ and u. Introduce an operator $\mathfrak{F} : \mathbb{R} \times \mathbb{E} \mapsto \mathbb{E}$ as the following:

$$\mathfrak{F}(c, w, v) = \begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} (-d_1\Delta)^{-1}\big[w\big(g(\Theta) + g_u(\xi)\Theta - g_u(\xi)w\big) + p(\Theta - w)v\big] \\ -(d_2\Delta)^{-1}\big[-cv + mp(\Theta - w)v\big] \end{pmatrix} \tag{5.3}$$

for every $c \in \mathbb{R}$ and $(w, v) \in \mathbb{E}$. Obviously, $\mathfrak{F}(c, 0, 0) = 0$ for all $c \in \mathbb{R}$ and by elliptic regularity $\mathfrak{F}(c, w, v) = 0 \Leftrightarrow (w, v)$ is a classic solution of (5.2).

Subsequently, for every $(w, v) \in \mathbb{E}$, we consider

$$\begin{aligned} \mathfrak{L}(c)\begin{pmatrix} w \\ v \end{pmatrix} &= \begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} (-d_1\Delta)^{-1}\big(g(\Theta) + \Theta g_u(\xi)\big) & (-d_1\Delta)^{-1}p(\Theta) \\ 0 & (-d_2\Delta)^{-1}\big(-c + mp(\Theta)\big) \end{pmatrix}\begin{pmatrix} w \\ v \end{pmatrix} \\ \mathfrak{R}(c, w, v) &= \begin{pmatrix} -(-d_1\Delta)^{-1}\big[-g_u(\xi)w^2 + p(\Theta - w)v - p(\Theta)v\big] \\ -(-d_2\Delta)^{-1}\big[mp(\Theta - w)v - p(\Theta)v\big] \end{pmatrix}. \end{aligned} \tag{5.4}$$

It is easy to see that $\mathfrak{R}(c, 0, 0) = 0$ and $D_{(w,v)}\mathfrak{R}(c, 0, 0) = (0, 0)$. Then, we have

$$\begin{aligned} \mathfrak{L}(c) &= D_{(w,v)}\mathfrak{F}(c, 0, 0), \\ \mathfrak{F}(c, w, v) &= \mathfrak{L}(c)\begin{pmatrix} w \\ v \end{pmatrix} + \mathfrak{R}(c, w, v). \end{aligned} \tag{5.5}$$

Define an operator

$$\mathfrak{N}(c)\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} (-d_1\Delta)^{-1}\big(g(\Theta) + \Theta g_u(\xi)\big) & (-d_1\Delta)^{-1}p(\Theta) \\ 0 & (-d_2\Delta)^{-1}\big(-c + mp(\Theta)\big) \end{pmatrix}\begin{pmatrix} w \\ v \end{pmatrix}. \tag{5.6}$$

By the Ascoli-Arzelá theorem and the classical Schauder estimates, we know that (5.6) is a compact linear operator. Owing to $\mathfrak{L}(c) = I - \mathfrak{N}(c)$, we can see that $\mathfrak{L}(c)$ is Fredholm of index zero.

In order to complete the proof of Theorem 5.1, we shall use [22, Theorem 1.1]. So, it is necessary to check the assumptions in Theorem 2.6.

Proof of $(\mathbb{HL})$. Since $\mathfrak{L}(c)$ is Fredholm of index zero, we know that $c^* \in \mathfrak{C}$ if and only if c^* is an eigenvalue of $\mathfrak{L}(c)$, that is, if $\dim N[\mathfrak{L}(c^*)] \geq 1$. Note that $\dim N[\mathfrak{L}(c)] \geq 1$ if and only if there exists $(w,v) \in \mathbb{E} \setminus \{(0,0)\}$ such that

$$\begin{gathered} -d_1 \Delta w = \left(g(\Theta) + \Theta g_u(\xi)\right) w - p(\Theta) v \quad \text{in } \Omega, \\ -d_2 \Delta v = mp(\Theta) v - cv \quad \text{in } \Omega, \\ w = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.7}$$

If $v = 0$, then

$$\begin{gathered} -d_1 \Delta w = \left(g(\Theta) + \Theta g_u(\xi)\right) w \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{5.8}$$

and hence $w = 0$ (if $w \neq 0$, then we have $0 = \lambda_1(-g(\Theta) - g_u(\xi)) > \lambda_1(-g(\Theta)) = 0$, a contradiction). Then, we must have $v \neq 0$. So, $\dim N[\mathfrak{L}(c^*)] \geq 1$ if and only if $-c$ is an eigenvalue of $-d_2\Delta - mp(\Theta)$ in Ω. Consequently, the set of singular values of $\mathfrak{L}(c)$ is indeed discrete and hence the assumption $(\mathbb{HL})$ is fulfilled. □

Proof of $(\mathbb{H}\mathfrak{R})$. From the definition of the operator $\mathfrak{R}$, it is easy to see that the assumption follows directly by a simple calculation. □

Proof of $(\mathbb{HP})$. It is easy to see that $\mathbb{E}$ can be regarded as an ordered Banach Space with respect to the order induced by the product cone $\mathbb{P}$. Using the the strong maximum principle, we can show that $(c,w,v) \in [\mathbb{R} \times (P \setminus \{0\}) \times (P \setminus \{0\})] \cap \Gamma^{-1}(0)$ imply that $w, v > 0$ for all $x \in \Omega$ and $\partial_w/\partial_{\nu_x}, \partial_v/\partial_{\nu_x} < 0$. The assumption $(\mathbb{HP})$ is fulfilled. □

Now, we can prove Theorem 5.1 according to the general framework of [22]. Firstly, note that $C(\sigma) \neq 0$ if and only if $\mathrm{Ind}(0, \mathfrak{N}(\lambda))$ changes as λ crosses σ; we can see that $C(c^*) \neq 0$ from Theorem 4.1. Considering the operator $\mathfrak{N}(c)$ defined by (5.6), it is not difficult to check that $c = -\lambda_1^*(-mp(\Theta))$ is the unique value of c for which 1 is an eigenvalue of $\mathfrak{N}(c)$ to a positive eigenfunction and

$$N\left[\mathfrak{L}\left(-\lambda_1^*\left(mp(\Theta)\right)\right)\right] = N\left[I - \mathfrak{N}\left(-\lambda_1^*\left(mp(\Theta)\right)\right)\right] = \mathrm{span}\left[(\phi,\psi)\right], \tag{5.9}$$

where (ϕ,ψ) are the corresponding eigenfunctions defined in Theorem 4.1. At last, for $g(0) > \lambda_1$ and $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$, we can see that

$$\mathfrak{N}(c)([P \setminus \{0\}] \times [P \setminus \{0\}]) \subset \mathrm{Int}\,\mathbb{P} = \mathrm{Int}\,P \times \mathrm{Int}\,P. \tag{5.10}$$

Following from [22, Theorem 1.1], we know that there exists an unbounded component $\mathcal{C}_{c^*}^+ \subset \mathbb{R} \times \mathbb{E}$ of the set of positive solutions of (1.1) such that $(c,u,v) = (-\lambda_1^*(-mp(\Theta)), \Theta, 0) \in \mathcal{C}_{c^*}^+$ and $P_c\mathcal{C}_{c^*}^+ = (-\lambda_1^*, -\lambda_1^*(-mp(\Theta)))$ due to Theorem 3.7.To complete the proof of Theorem 5.1, we suppose that $-\lambda_1^* < c < -\lambda_1^*(-mp(\Theta))$ and let (u, v) be a positive solution of (1.1). Then, by Theorem 3.2, we have $u(x) \leq g(0)$ for all $x \in \overline{\Omega}$ and

$$\|v\|_{C(\overline{\Omega})} \leq g(0)\left(\frac{cd_1m}{d_2} + mg(0)\right)\left\|\left(-\Delta + \frac{c}{d_2}\right)^{-1}\right\|_{C(\overline{\Omega})}. \tag{5.11}$$

Note that

$$\lim_{c \to -\lambda_1^*} g(0)\left(\frac{cd_1m}{d_2} + mg(0)\right)\left\|\left(-\Delta + \frac{c}{d_2}\right)^{-1}\right\|_{C(\overline{\Omega})} = \infty. \tag{5.12}$$

Therefore, we know that $\mathcal{C}_{c^*}^+$ must bifurcate from infinity at $c = -\lambda_1^*$. The proof of Theorem 5.1 is completed. □

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