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SLOW INCREASING FUNCTIONS AND THEIR APPLICATIONS TO SOME PROBLEMS IN NUMBER THEORY

K. Santosh Reddy, V. Kavitha and V. Lakshmi Narayana
Vardhaman College of Engineering, Shamshabad, Hyderabad, Andhra Pradesh, India
E-Mail: santureddyk@gmail.com

ABSTRACT

This article commences with a definition of slow increasing function and moves on to delineate a few properties of slow increasing functions. Besides, several applications in some problems of number theory using the theory of slow increasing functions are also presented to show how useful these functions prove in solving complex problems.

Keywords: slow increasing functions, asymptotically equivalent, sequence of positive integers.

1. INTRODUCTION

Slow increasing functions are defined as follows:

1.1. Definition

Let $f:[a,\infty)\to(0,\infty)$ be a continuously differentiable function such that $f'>0$ and $\lim_{x\to\infty} f(x)=\infty$. Then f is said to be a slow increasing function (s.i.f. in short) if $\lim_{x\to\infty}\frac{xf'(x)}{f(x)}=0$

Write $F=\{f: f \text{ is a s.i.f.}\}$.

1.2. Examples

(i) $f(x)=\log x,\ x>1$ is a s.i.f.

Note that $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\log x=\infty$ and $f'(x)=\frac{1}{x},\forall x>1$ and f' is continuous

Moreover $\lim_{x\to\infty}\frac{xf'(x)}{f(x)}=\lim_{x\to\infty}\frac{1}{x}\times\frac{x}{\log x}=0$

(ii) $f(x)=\log\log x, x>e$ is also a s.i.f.

2. SOME PROPERTIES

2.1. Theorem

Let $f,g\in F$ and let $\alpha>0, c>0$ be two constants then we have

(i) $f+c$ (ii) $f-c$ (iii) cf (iv) fg (v) f^{α} (vi) fog (vii) $\log f$ (viii) $f+g$ all lie in F.

Proof

Given that $f,g\in F$ and $\alpha>0, c>0$ be constants.

Proof of (i), (ii), (iii), and (iv) follows the definition 1.1

(v) Let $h=f^{\alpha}$

Note that $\lim_{x\to\infty}h(x)=\lim_{x\to\infty}f(x)^{\alpha}=\infty$, and $h'(x)=\alpha f(x)^{\alpha-1}f'(x)>0$, and h' is continuous

Moreover

$$\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{x\alpha f(x)^{\alpha-1}f'(x)}{f(x)^{\alpha}}=\alpha\lim_{x\to\infty}\frac{xf'(x)}{f(x)}=0.$$

Hence $h=f^{\alpha}\in F$

(vi) Let $h=fog$ i.e $h(x)=f(g(x))$

Note that $\lim_{x\to\infty}h(x)=\lim_{x\to\infty}f(g(x))=\infty$, and $h'(x)=f'(g(x))g'(x)>0$, and h' is continuous

Moreover

$$\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{xf'(g(x))g'(x)}{f(g(x))}=\lim_{x\to\infty}\frac{g(x)f'(g(x))}{f(g(x))}\times\frac{xg'(x)}{g(x)}=0.$$

Hence $h=fog\in F$

(vii) Let $h=\log f$

Note that $\lim_{x\to\infty}h(x)=\lim_{x\to\infty}\log f(x)=\infty$, and $h'(x)=\frac{f'(x)}{f(x)}>0$, and h' is continuous

Moreover

$$\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{x\frac{f'(x)}{f(x)}}{\log f(x)}=\lim_{x\to\infty}x\frac{f'(x)}{f(x)}\times\frac{1}{\log f(x)}=0.$$

Hence $h=\log f\in F$

(viii) Let $h=f+g$

For sufficiently large x, we have $0\le\frac{xf'}{f+g}\le\frac{xf'}{f}$ and $0\le\frac{xg'}{f+g}\le\frac{xg'}{g}$

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By adding the above, we get $0 \le \lim_{x\to\infty} \frac{xh'(x)}{h(x)} \le \lim_{x\to\infty} \frac{xf'}{f} + \lim_{x\to\infty} \frac{xg'}{g} = 0$

$\therefore \lim_{x\to\infty} \frac{xh'(x)}{h(x)} = 0$ Hence $h = f + g \in F$

2.2. Theorem

Let $f, g \in F$. Define $h(x) = f(x^{\alpha})$ and $k(x) = f(x^{\alpha} g(x))$ for each x, then $h, k \in F$.

Proof

Given that $f, g \in F$. Define $h(x) = f(x^{\alpha})$ and $k(x) = f(x^{\alpha} g(x))$ for each x.

Let $h(x) = f(x^{\alpha})$

Note that $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} f(x^{\alpha}) = \infty$, and $h'(x) = f'(x^{\alpha})\alpha x^{\alpha-1} > 0$, and h' is continuous

Moreover

$$\lim_{x\to\infty} \frac{xh'(x)}{h(x)} = \lim_{x\to\infty} \frac{xf'(x^{\alpha})\alpha x^{\alpha-1}}{f(x^{\alpha})} = \alpha \lim_{x\to\infty} \frac{x^{\alpha} f'(x^{\alpha})}{f(x^{\alpha})} = 0$$

Hence $h(x) = f(x^{\alpha})$ is s.i.f.

Let $k(x) = f(x^{\alpha} g(x))$

Note that $\lim_{x\to\infty} k(x) = \lim_{x\to\infty} f(x^{\alpha} g(x)) = \infty$, and

$k'(x) = f'(x^{\alpha} g(x))\left[\alpha x^{\alpha-1} g(x) + x^{\alpha} g'(x)\right] > 0$ and k' is continuous

Moreover

$$\lim_{x\to\infty} \frac{xh'(x)}{h(x)} = \lim_{x\to\infty} \frac{xf'(x^{\alpha} g(x))\left[\alpha x^{\alpha-1} g(x) + x^{\alpha} g'(x)\right]}{f(x^{\alpha} g(x))}$$

$$= \alpha \lim_{x\to\infty} \frac{x^{\alpha} g(x) f'(x^{\alpha} g(x))}{f(x^{\alpha} g(x))} + \lim_{x\to\infty} \frac{x^{\alpha} g(x) f'(x^{\alpha} g(x))}{f(x^{\alpha} g(x))} \times \frac{xg'(x)}{g(x)} = 0$$

Therefore $k(x) = f(x^{\alpha} g(x))$ is s.i.f. Hence $h, k \in F$

2.3. Theorem

Let $f, g \in F$ be such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] > 0$. Then $\frac{f}{g} \in F$.

Proof

Given that

$f, g \in F, \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] > 0$

Let $h(x) = \frac{f(x)}{g(x)}$ and $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Moreover

$$\lim_{x\to\infty} \frac{xh'(x)}{h(x)} = \lim_{x\to\infty} \frac{x\left(\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\right)}{\left(\frac{f(x)}{g(x)}\right)} = \lim_{x\to\infty} \frac{f'(x)}{f(x)} - \lim_{x\to\infty} \frac{xg'(x)}{g(x)} = 0$$

Hence $\frac{f}{g} \in F$

2.4. Theorem

Let $h : [a, \infty) \to (0, \infty)$ be a continuously differentiable function such that $h'(x) > 0$ and $\lim_{x\to\infty} h(x) = \infty$

(i) Define $g(x) = h(\log x)$. Then $g \in F \Leftrightarrow \lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0$

(ii) Define $k(x) = e^{h(x)}$. Then $k \in F \Leftrightarrow \lim_{x\to\infty} xh'(x) = 0$

Proof

Given that $h'(x) > 0$ and $\lim_{x\to\infty} h(x) = \infty$

(i) Define $g(x) = h(\log x)$ then $g'(x) = \frac{h'(\log x)}{x}$

Suppose $g \in F$ then g satisfies $\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = 0$

i.e. $\lim_{x\to\infty} \frac{x\frac{h'(\log x)}{x}}{h(\log x)} = 0 \Rightarrow \lim_{x\to\infty} \frac{h'(\log x)}{h(\log x)} = 0$

Put $t = \log x$ so that $x \to \infty \Rightarrow t \to \infty$ $\therefore \lim_{t\to\infty} \frac{h'(t)}{h(t)} = 0$

i.e. $\lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0$.

Conversely suppose $\lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0$

Put $t = e^x$ so that $x = \log t$ and $x \to \infty \Rightarrow t \to \infty$

$$\Rightarrow \lim_{x\to\infty} \frac{h'(x)}{h(x)} = \lim_{t\to\infty} \frac{h'(\log t)}{h(\log t)} = 0$$

Now $\lim_{t\to\infty} \frac{tg'(t)}{g(t)} = \lim_{t\to\infty} \frac{h'(\log t)}{h(\log t)} = \lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0.$

Hence $g \in F$

(ii) Like proof of (i)

2.5. Theorem

If $f \in F$ then $\lim_{x\to\infty} \frac{\log f(x)}{\log x} = 0.$

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Proof

Given that $f \in F$,

$$\lim_{x\to\infty}\frac{\log f(x)}{\log x} = \lim_{x\to\infty}\frac{\left(\frac{f'(x)}{f(x)}\right)}{\left(\frac{1}{x}\right)} \quad \text{(byL'Hospital's rule)}$$

$$= \lim_{x\to\infty}\frac{xf'(x)}{f(x)} = 0 \text{ i.e. } \lim_{x\to\infty}\frac{\log f(x)}{\log x} = 0.$$

2.6. Theorem

$f \in F$ if and only if to each $\alpha > 0$ there exists x_α such that $\frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] < 0, \forall x > x_\alpha$

Proof

We have

$$\frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] = \frac{f'(x)x^\alpha - f(x)\alpha x^{\alpha-1}}{x^{2\alpha}} = \frac{f(x)}{x^{\alpha+1}}\left[\frac{xf'(x)}{f(x)} - \alpha\right]$$

Suppose $f \in F$ then $\Rightarrow \lim_{x\to\infty}\frac{xf'(x)}{f(x)} = 0$

i.e. For each $\alpha > 0$ there exists x_α such that $\forall x > x_\alpha$

And $\left|\frac{xf'(x)}{f(x)} - 0\right| < \alpha, \ \forall x > x_\alpha \Rightarrow \frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] < 0, \ \forall x > x_\alpha$

To prove the converse assumes that the condition holds.

Let $\alpha > 0$ be given. Then there exist x_α such that $\forall x > x_\alpha$

We have, by hypothesis $\frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] < 0$ this implies that

$$\left|\frac{xf'(x)}{f(x)} - 0\right| < \alpha, \ \forall x > x_\alpha$$

$$i.e. \frac{xf'(x)}{f(x)} \to 0 \text{ as } x \to \infty \Rightarrow \lim_{x\to\infty}\frac{xf'(x)}{f(x)} = 0.$$

Therefore $f \in F$.

2.7. Theorem

If $f \in F$ then $\lim_{x\to\infty}\frac{f(x)}{x^\beta} = 0$, for all $\beta > 0$

Proof

For any α with $0 < \alpha < \beta$, we get by Theorem 2.6, $\frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] < 0$, for all $\forall x > x_\alpha$ for some x_α

This implies that $\frac{f(x)}{x^\alpha}$ is decreasing for $x > x_\alpha$

Hence $\frac{f(x)}{x^\alpha}$ bounded above, say, by M

That is, there exists $M >$ 0 such that $0 < \frac{f(x)}{x^\alpha} < M, \forall x > x_\alpha$

$$\lim_{x\to\infty}\frac{f(x)}{x^\beta} = \lim_{x\to\infty}\frac{f(x)}{x^\alpha}\frac{1}{x^{\beta-\alpha}} = 0$$

2.8. Note

We know that each $f \in F$ is an increasing function. Moreover, by the above theorem, it is clear that $\lim_{x\to\infty}\frac{f(x)}{x^\beta} = 0, \forall \beta > 0$. This shows that the increasing nature of f is slow**.** In other words, f does not increase rapidly. This justifies the name given to the members of F.

From the above theorem, we have the following results:

2.9. Theorem

If $f \in F$ then $\lim_{x\to\infty}\frac{f(x)}{x} = 0$ and $\lim_{x\to\infty} f'(x) = 0$.

Proof

In Theorem 2.7 put $\beta = 1$, toget $\lim_{x\to\infty}\frac{f(x)}{x} = 0$.

If $f \in F$, then $\lim_{x\to\infty}\frac{xf'(x)}{f(x)} = 0$

Since $\lim_{x\to\infty}\frac{f(x)}{x} = 0$ we must have $\lim_{x\to\infty} f'(x) = 0$.

2.10. Theorem

Let $f \in F$, then for any $\alpha > -1$ and $\beta \in$ □, the series $\sum_{n=1}^{\infty} n^\alpha f(n)^\beta$ diverges to ∞.

Proof

We write $\sum_{n=1}^{\infty} n^\alpha f(n)^\beta = \sum_{n=1}^{\infty}\left(n^{\alpha+1} f(n)^\beta\right)\frac{1}{n}$

we know that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges to ∞

Given $\alpha > -1 \Rightarrow \alpha + 1 > 0$

If $\beta \geq 0$ then $\lim_{n\to\infty} n^{\alpha+1} f(n)^{\beta} = \infty$

If $\beta > 0$ then

$$\lim_{n\to\infty} \frac{n}{\left(\frac{f(n)^{-\beta}}{n^{\alpha}}\right)} = \lim_{n\to\infty} \frac{n^{\alpha+1}}{f(n)^{-\beta}} = \infty \quad \text{(from Theorem 2.7)}$$

i.e. $\sum_{n=1}^{\infty} n^{\alpha} f(n)^{\beta}$ diverges to ∞

An important byproduct of the above theorem is the following result.

2.11. Theorem

Let $f \in F$. Then for any $\alpha > -1$ and $\beta \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{\int_a^x t^{\alpha} f(t)^{\beta} dt}{\left(\frac{x^{\alpha+1} f(x)^{\beta}}{\alpha+1}\right)} = 1.$$

Proof

From Theorem 2.10, we have $\lim_{n\to\infty} x^{\alpha+1} f(x)^{\beta} = \infty$

$$\Rightarrow \lim_{x\to\infty} \frac{x^{\alpha+1}}{\alpha+1} f(x)^{\beta} = \infty, \quad \forall\ \alpha > -1,\ \forall \beta$$

From Theorem 2.10, we have $\sum_{t=1}^{\infty} t^{\alpha} f(t)^{\beta} = \infty$

$$\Rightarrow \lim_{x\to\infty} \int_a^x t^{\alpha} f(t)^{\beta} dt = \infty$$

Consider $\lim_{x\to\infty} \frac{\int_a^x t^{\alpha} f(t)^{\beta} dt}{\left(\frac{x^{\alpha+1} f(x)^{\beta}}{\alpha+1}\right)}$

$$= \lim_{x\to\infty} \frac{x^{\alpha} f(x)^{\beta}}{x^{\alpha} f(x)^{\beta} + \frac{x^{\alpha+1}}{\alpha+1} \beta f(x)^{\beta-1} f'(x)} \quad \text{(By L'Hospitals's rule)}$$

$$= \lim_{x\to\infty} \frac{x^{\alpha} f(x)^{\beta}}{x^{\alpha} f(x)^{\beta} \left(1 + \frac{\beta}{\alpha+1} \frac{x f'(x)}{f(x)}\right)} = 1$$

2.12. Definition

Let $f, g : [a, \infty) \to (0, \infty)$

(i) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, then f is said to asymptotically equivalent to g. We describe this by writing $f \sim g$.

(ii) $f = O(g)$ Means $f \leq Ag$ for some $A > 0$. In this case we say that f is of large order g.

(ii) $f = o(g)$ Means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. In this case we say that f is of small order g.

2.13. Examples

(i) Consider $f(x) = x^n$, $g(x) = x^n + x$, for all $x > 0$

and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^n}{x^n + x} = 1$

Therefore $f \sim g$.

(ii) $x = O(10x)$ Because $\frac{x}{10x} = \frac{1}{10} \Rightarrow x = \frac{1}{10}(10x)$.

(iii) $x + 1 = o(x^2)$ Because $\lim_{x\to\infty} \frac{x+1}{x^2} = 0$.

As a result of Theorem 2.11, we get the following results as particular cases.

2.14. Theorem

Let $f \in F$. Then we have the following statements.

(i) $\int_a^x f(t)^{\beta} dt \sim x f(x)^{\beta}$

(ii) $\int_a^x f(t) dt \sim x f(x)$ (iii) $\int_a^x \frac{1}{f(t)} dt \sim \frac{x}{f(x)}$

Proof

Let $f \in F$

(i) Put α = 0 in Theorem 2.11, we get

$$\lim_{x\to\infty} \frac{\int_a^x f(t)^{\beta} dt}{x f(x)^{\beta}} = 1 \Rightarrow \int_a^x f(t)^{\beta} dt \sim x f(x)^{\beta}$$

(ii) Put α = 0, β = 1 in Theorem 2.11, we get

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$$\lim_{x\to\infty}\frac{\int_a^x f(t)dt}{xf(x)}=1 \Rightarrow \int_a^x f(t)dt \sim xf(x)$$

(iii) Put α = 0, β = -1 in Theorem 2.11, we get

$$\lim_{x\to\infty}\frac{\int_a^x \frac{1}{f(t)}dt}{\frac{x}{f(x)}}=1 \quad \Rightarrow \int_a^x \frac{1}{f(t)}dt \sim \frac{x}{f(x)}$$

2.15. Theorem

Let $f \in F$. Then

(i) $\lim_{x\to\infty}\frac{f(x+c)}{f(x)}=1,$ For any $c\in\mathbb{R}$

(ii) If $f'(x)$ is decreasing then $\lim_{x\to\infty}\frac{f(cx)}{f(x)}=1,$ for any $c\in\mathbb{R}$

Proof

Let $f \in F$

(i) Case (a). Suppose $c>0$

By Lagrange's mean value theorem, There exists a $t\in(x,x+c)$ such that

$$f(x+c)-f(x)=(x+c-x)f'(t)$$

$$\Rightarrow \quad 0\le\frac{f(x+c)-f(x)}{f(x)}=\frac{cf'(t)}{f(x)}$$

$$\Rightarrow \quad 0\le\lim_{x\to\infty}\frac{f(x+c)-f(x)}{f(x)}=\lim_{x\to\infty}\frac{cf'(t)}{f(x)},\ t\in(x,x+c)$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x+c)}{f(x)}-1=0, \text{ since } \lim_{x\to\infty}f'(x)=0 \quad \text{(by Theorem 2.9)}$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{f(x+c)}{f(x)}=1.$$

Case (b). Suppose $c<0$

By Lagrange's mean value theorem there exists $t\in(x+c,x)$ such that

$$f(x)-f(x+c)=(x-x-c)f'(t)$$

$$\Rightarrow \quad 0\le\frac{f(x)-f(x+c)}{f(x)}=-\frac{cf'(t)}{f(x)}$$

$$\Rightarrow \quad 0\le\lim_{x\to\infty}\frac{f(x)-f(x+c)}{f(x)}=-c\lim_{x\to\infty}\frac{f'(t)}{f(x)},\ t\in(x+c,x)$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x+c)}{f(x)}-1=0, \text{ since } \lim_{x\to\infty}f'(x)=0 \quad \text{(by Theorem 2.9)}$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{f(x+c)}{f(x)}=1.$$

(ii) Case (a). Suppose $c>1$

By Lagrange's mean value theorem there exists $t\in(x,cx)$ such that

$$f(cx)-f(x)=(cx-x)f'(t)$$

$$\Rightarrow \quad 0\le\frac{f(cx)-f(x)}{f(x)}=\frac{(c-1)xf'(t)}{f(x)}$$

$$\Rightarrow 0\le\lim_{x\to\infty}\frac{f(cx)-f(x)}{f(x)}=(c-1)\lim_{x\to\infty}\frac{xf'(t)}{f(x)},\ t\in(x,cx)$$

And $f(x)$ is decreasing $\Rightarrow \quad f'(x)>f'(t)$

There fore $\lim_{x\to\infty}\frac{f(cx)}{f(x)}-1=0$, since $\lim_{x\to\infty}f'(x)=0$ (by Theorem 2.9)

$$\Rightarrow \quad \lim_{x\to\infty}\frac{f(cx)}{f(x)}=1.$$

Case (b). Suppose $c<1$

By Lagrange's mean value theorem there exists $t\in(cx,x)$ such that

$$f(x)-f(cx)=(x-cx)f'(t)$$

$$\Rightarrow \quad 0\le\frac{f(x)-f(cx)}{f(x)}=\frac{(1-c)xf'(t)}{f(x)}$$

$$\Rightarrow 0\le\lim_{x\to\infty}\frac{f(x)-f(cx)}{f(x)}=(1-c)\lim_{x\to\infty}\frac{xf'(t)}{f(x)},\ t\in(cx,x)$$

And $f(x)$ is decreasing $\Rightarrow \quad f'(x)>f'(t)$

There fore $\lim_{x\to\infty}\frac{f(cx)}{f(x)}-1=0$,

since $\lim_{x\to\infty}f'(x)=0$ (by Theorem 2.9)

$$\Rightarrow \lim_{x\to\infty}\frac{f(cx)}{f(x)}=1.$$

2.16. Theorem

Suppose $f\in F$ is such that $f'(x)$ is decreasing. If $0<c_1\le c_2$ and g is a function such that $c_1\le g(x)\le c_2$ then $\lim_{x\to\infty}\frac{f(g(x)x)}{f(x)}=1.$

Proof

Suppose $f \in F$ is such that $f'(x)$ is decreasing

If $0 < c_1 \le g(x) \le c_2 \;\Rightarrow\; f(c_1x) \le f(g(x)x) \le f(c_2x)$ since f is decreasing

$$\Rightarrow \quad \frac{f(c_1x)}{f(x)} \le \frac{f(g(x)x)}{f(x)} \le \frac{f(c_2x)}{f(x)}$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{f(c_1x)}{f(x)} \le \lim_{x\to\infty}\frac{f(g(x)x)}{f(x)} \le \lim_{x\to\infty}\frac{f(c_2x)}{f(x)}$$

$$\Rightarrow \quad 1 \le \lim_{x\to\infty}\frac{f(g(x)x)}{f(x)} \le 1 \qquad \text{(By Theorem 2.15)}$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{f(g(x)x)}{f(x)} = 1.$$

3. APPLICATIONS OF SLOW INCREASING FUNCTIONS TO SOME PROBLEMS OF NUMBER THEORY

This section details some applications in problems pertaining to number theory.

We begin with the following important definition.

3.1. Definition

Let $f \in F$. Through out (a_n) denotes a strictly increasing sequence of positive integers such that

$$a_1 > 1 \text{ and } \lim_{n\to\infty}\frac{a_n}{n^s f(n)} = 1 \text{ for some } s \ge 1.$$

i.e. $a_n \sim n^s f(n)$

There exist several such sequences.

For example $a_n = p_n$, the sequence of prime numbers in increasing order, $f(x) = \log x$ and $s = 1$.

By prime number theorem we have $\lim_{n\to\infty}\frac{p_n}{n\log n} = 1$

3.2. Theorem

Let $f:(a,\infty)\to(1,\infty)$ be a s.i.f. $(a>1)$ and $\lim_{x\to\infty}\int_b^x \frac{tf'(t)^\beta}{f(t)}dt = \infty$ $(a<b)$. Suppose (a_n) be the sequence of positive integers such that $a_n \sim n^s f(n)$ $(s \ge 1)$. (1)

Then

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_1.a_2...a_n}}{a_n} = \frac{1}{e^s}.$$

Proof

Given that $f:(a,\infty)\to(1,\infty)$ be a s.i.f. $(a>1)$ and $\lim_{x\to\infty}\int_b^x \frac{tf'(t)^\beta}{f(t)}dt = \infty$ $(a<b)$

And

$a_n \sim n^s f(n)$ $(s \ge 1)$

$$\Rightarrow \quad \log a_n = s\log n + \log f(n) + o(1)$$

If n' is positive integer in interval $[a,\infty)$

Then

$$\sum_{k=n'}^{n}\log a_k = s\sum_{k=n'}^{n}\log k + \sum_{k=n'}^{n}\log f(k) + \sum_{k=n'}^{n}o(1) \qquad (2)$$

Since $\log x$ is increasing and positive in (a,∞)

Now

$$\sum_{k=n'}^{n}\log k = \int_{n'}^{n}\log x\,dx + \mathrm{O}(\log n)$$

$$= n\log n - n + \mathrm{O}(\log n)$$

$$= n\log n - n + o(n) \qquad (3)$$

On the other hand if $\varepsilon > 0$ we have for all $n \ge n'$ the inequality $|o(1)| < \varepsilon$

Therefore for $n > n'$, we have

$$\frac{\left|\sum_{k=n'}^{n}o(1)\right|}{n} \le \frac{\sum_{k=n'}^{n}|o(1)|}{n} < \frac{\varepsilon(n-n'+1)}{n} \le 2\varepsilon$$

i.e. $\sum_{k=n'}^{n}o(1) = o(n)$ (4)

We find that

$$\sum_{k=n'}^{n}\log f(k) = \int_{n'}^{n}\log f(x)dx + \mathrm{O}(\log f(n))$$

$$= n\log f(n) - \int_{n'}^{n}\frac{xf'(x)}{f(x)}dx + \mathrm{O}(\log f(n)) \qquad (5)$$

We know that

$$\lim_{n\to\infty}\frac{\log f(n)}{n} = \lim_{x\to\infty}\frac{f'(n)}{f(n)} \qquad \text{(By L'Hospital's rule)}$$

$$= \lim_{n\to\infty}\frac{nf'(n)}{f(n)} \times \frac{1}{n} = 0.$$

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$$\Rightarrow \quad O(\log f(n)) = o(n) \tag{6}$$

Now

$$\lim_{x\to\infty} \frac{\int_{n'}^{x} \frac{tf'(t)}{f(t)} dt}{x} = \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0 \quad \text{(By L'Hospital's rule)}$$

$$\Rightarrow \quad \int_{n'}^{n} \frac{xf'(x)}{f(x)} dx = o(n) \tag{7}$$

From (5), (6) and (7), we get

$$\sum_{k=n'}^{n} \log f(k) = n \log f(n) + o(n) \tag{8}$$

From (2), (3), (4) and (7), we get

$$\sum_{k=1}^{n} \log a_k = s(n\log n - n + o(n)) + n \log f(n) + o(n) + o(n)$$

$$\Rightarrow \quad \sum_{k=1}^{n} \log a_k = sn\log n - sn + n \log f(n) + o(n) \tag{9}$$

But

$$\sum_{k=1}^{n} \log a_k = \log a_1 + \log a_2 + \ldots + \log a_n$$

$$= \log a_1 a_2 \ldots a_n \quad \Rightarrow a_1 a_2 \ldots a_n = \exp\left(\sum_{k=1}^{n} \log a_k\right)$$

$$\Rightarrow \quad \sqrt[n]{a_1 a_2 \ldots a_n} = \exp\left(\frac{\sum_{k=1}^{n} \log a_k}{n}\right)$$

$$\Rightarrow \exp\left(\frac{\sum_{k=1}^{n} \log a_k}{n}\right) = \exp\left(\frac{sn\log n - sn + n\log f(n) + o(n)}{n}\right) \quad \text{(By 9)}$$

$$= \exp\left(\log n^s - s + \log f(n) + o(1)\right)$$

$$= \exp\left(\log n^s f(n) + o(1)\right) e^{-s}$$

$$= \frac{\exp\left(\log n^s f(n) + o(1)\right)}{e^s} \sim \frac{n^s f(n)}{e^s} \sim \frac{a_n}{e^s}$$

Therefore $\sqrt[n]{a_1 a_2 \ldots a_n} \sim \frac{a_n}{e^s} \Rightarrow \frac{\sqrt[n]{a_1 a_2 \ldots a_n}}{a_n} \sim \frac{1}{e^s}$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\sqrt[n]{a_1 a_2 \ldots a_n}}{a_n} = \frac{1}{e^s}.$$

In view of the above theorem and prime number theorem implies the following.

3.3. Theorem

Let p_n be the sequence of prime numbers. Then

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\sqrt[n]{p_1 p_2 \cdots p_n}}{p_n} = \frac{1}{e}.$$

Proof

In theorem 3.2 put $a_n = p_n$, $f(x) = \log x$ and $s = 1$.

Let $c_{n,k}$ be the sequence of integers which have in their prime factorization k prime factors. Rafael Jakimczuk $[4]$ proved that

$$c_{n,k} \sim \frac{(k-1)!n\log n}{(\log\log n)^{(k-1)}}$$

As a result of previous theorem, we have the following result.

3.4. Theorem

$$\lim_{n\to\infty} \frac{\sqrt[n]{c_{1,k}.c_{2,k} \ldots c_{n,k}}}{c_{n,k}} = \frac{1}{e}.$$

Proof

In Theorem 3.2 put $a_n = c_{n,k}$, $f(x) = (k-1)!\log n$ and $s = 1$.

CONCLUSIONS

We apply the results discussed in this article to look into some of the applications in number theory.

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