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On Casaro Sequence Space of Fuzzy Numbers Defined by a Modulus Function

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Abstract

The main purpose of this paper is to introduce the sequence space $ces^F(f, p)$ of sequence of fuzzy numbers defined by a modulus function. Furthermore, some inclusion theorems have been discussed.

Keywords

Fuzzy Numbers, Modulus Function

Subject Areas: **Functional Analysis, Fuzzy Mathematics**

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1]. And subsequently several authors have discussed the properties of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming and so on. Matloka [2] introduced the bounded and convergent sequences of fuzzy numbers and studied their properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [3], Nanda [4], Esi [5] and many others.

Let $C(R^n)=\{A\subset R^n : A \text{ is compact and convex set}\}$. The space $C(R^n)$ has a linear structure induced by the operations[1]

$$A+B=\{a+b : a\in A, b\in B\} \text{ and } \gamma A=\{\gamma a : a\in A\}$$

for $A, B \in C(R^n)$ and $\gamma \in R$. The Hausdorff distance between A and B in $C(R^n)$ is defined by

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$$\delta_\infty(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{a\in A}\inf_{b\in B}\|a-b\|\right\}.$$

It is well-known that $\left(C\left(R^n\right),\delta_\infty\right)$ is a complete matric space.

A fuzzy number is a function X: R^n to [0, 1] which is normal, fuzzy convex, upper semicontinuous and the closure of $\{X \in R^n: X(x) > 0\}$ is compact. The above properties imply that for each $0 < \alpha \le 1$, the α-level set $X^\alpha = \{X \in R^n: X(x) \ge \alpha\}$ is a non-empty compact, convex subset of R^n, with support X^0. If R^n is replaced by R, then obviously the set $C(R^n)$ is reduced to the set of all closed bounded intervals $A = \left[\overline{A}, \underline{A}\right]$ on R, and also

$$\delta_\infty(A,B) = \max\left\{\left|A-B\right|, \left|\overline{A}-\overline{B}\right|\right\}.$$

Let $L(R)$ denote the set of all fuzzy numbers. The linear structure of $L(R)$ induces the addition $X + Y$ and the scalar multiplication λX in terms of level α-sets, by

$$[X+y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda[X]^\alpha$$

for each $0 \le \alpha \le 1$.

R, the set of real numbers can be embedded in $L(R)$ if we define $\overline{r} \in L(R)$ by

$$\overline{r}(t) = \begin{cases} 1, & \text{if } t = r \\ 0, & \text{if } t \ne r \end{cases}.$$

The additive identity and multiplicative identity of $L(R)$ are denoted by $\overline{0}$ and $\overline{1}$, respectively. Then the arithmetic operations on $L(R)$ are defined as follows:

$$(X \oplus Y)(t) = \sup\{X(s) \wedge Y(t-s)\},\ t \in R,$$

$$(X - Y)(t) = \sup\{X(s) \wedge Y(s-t)\},\ t \in R,$$

$$(X \otimes Y)(t) = \sup\{X(s) \wedge Y(t/s)\},\ t \in R,$$

$$(X/Y)(t) = \sup\{X(st) \wedge Y(s)\},\ t \in R.$$

These operations can be defined in terms of α-level sets as follows:

$$[X \oplus Y]^\alpha = \left[a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha\right],$$

$$[X - Y]^\alpha = \left[a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha\right],$$

$$[X \otimes Y]^\alpha = \left[\min_{i\in\{1,2\}} a_i^\alpha b_i^\alpha, \max_{i\in\{1,2\}} a_i^\alpha b_i^\alpha\right],$$

$$\left[X^{-1}\right]^\alpha = \left[\left(a_1^\alpha\right)^{-1}, \left(a_1^\alpha\right)^{-1}\right],\ a_i^\alpha > 0,\ \in\{1,2\},$$

for each $0 < \alpha \le 1$.

For r in R and X in $L(R)$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X\left(r^{-1}t\right), & \text{if } r \ne 0 \\ 0, & \text{if } r = 0 \end{cases}.$$

Defined a map by d: $L(R) \times L(R) \to R$ by $d(X,Y) = \sup_{0\le\alpha\le1}\delta_\infty\left(X^\alpha, Y^\alpha\right)$. For $X, Y \in L(R)$ defined $X \le Y$ it and only if $X^\alpha \le Y^\alpha$ for any $\alpha \in [0, 1]$. It is known that $(L(R), d)$ is complete metric space [3].

The metric d has the following properties:

$$d(cX, cY) = |c|\, d(X,Y)$$

for $c \in R$ and

$$d(X+Y,Z+W) \le d(X,Z)+d(Y,W).$$

A metric on $L(R)$ is said to be translation invariant if $d(X+Z,Y+Z)=d(X,Y)$ for all $X, Y, Z, \in L(R)$.

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural number into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ [2].

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k p_k$, then for any complex number a_k and b_k.

$$|a_k+b_k|^{pk} \le C\left(|a_k|^{pk}+|b_k|^{pk}\right) \tag{1.1}$$

where $C = \max(1, 2^{H-1})$. Also, for any complex number α,

$$|\alpha|^{pk} \le \max\left(1,|\alpha|^{H}\right), \quad [6]. \tag{1.2}$$

2. Main Result

Let f be a modulus function and $p = (p_k)$ be a bounded sequence of positive real numbers. We now introduce the Cesaro sequence set $ces^F(f, p)$ of sequence of fuzzy numbers using a modulus function f as follows:

$$ces^F(f,p)=\left\{X=(X_k)\in w(F):\sum_{n=1}^{\infty}\left[\frac{1}{n}\sum_{k=1}^{n}d\left(X_k,\bar{0}\right)\right]^{pk}<\infty\right\}.$$

If $f(x)=x$, then $ces(f,p)=cesF(p)$, where

$$ces^F(p)=\left\{X=(X_k)\in w(F):\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k,\bar{0}\right)\right)^{pk}<\infty\right\}.$$

Theorem 2.1. The set $ces^F(f,p)$ of sequence of fuzzy numbers is closed under the coordinate wise addition and scalar multiplication.

Proof. Since it is not hard to see that the set $ces(f,p)$ is closed with respect to the coordinate rise addition and scalar multiplication, so we omit the detail.

Theorem 2.2. The set $ces^F(f,p)$ of sequence of fuzzy numbers is complete metric space with respect to the metric

$$\delta(X,Y)=\left(\sum_{n=1}^{\infty}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d(X_k,Y_k)\right)\right]^{pk}\right)^{\frac{1}{M}}$$

where $M = \max(1, \sup p_k)$ and $X=(X_k), Y=(Y_k)$ are elements of the set $ces^F(f,p)$.

Proof. One can easily establish that δ defines a metric on $ces^F(f,p)$ which is a routine verification, so we leave it to the reader. It remains to prove the completeness of the space $ces^F(f,p)$. Let (X^i) be any Cauchy sequence in $ces(f,p)$, where $X^i=\left(X_0^i,X_1^i,X_3^i,\cdots\right)$. Then, for a give $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$\delta\left(X^i,X^j\right)=\left(\sum_{n=1}^{\infty}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,X_k^j\right)\right)\right]^{pk}\right)^{\frac{1}{M}}<\varepsilon \quad \text{for all } i, j \ge n_0. \tag{2.1}$$

We obtain for each fixed $k \in N$ from (2.1)

$$\sum_{n=1}^{\infty}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,X_k^j\right)\right)\right]^{pk}<\varepsilon^M \quad \text{for every } i, j \ge n_0. \tag{2.2}$$

Since for any fixed natural number T, we have from (2.2)

$$\sum_{n=1}^{T}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,X_k^j\right)\right)\right]^{pk}<\varepsilon^M \quad \text{for every } i,j\geq n_0. \tag{2.3}$$

Since f is continuous, we have from (2.3) that

$$f\left[\lim_{i,j\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,X_k^j\right)\right)\right]=0. \tag{2.4}$$

Since f is a modulus function, one can derive by (2.4) that

$$\lim_{i,j\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,X_k^j\right)\right)=0$$

which implies that for each fixed k,

$$\lim_{i,j\to\infty}d\left(X_k^i,X_k^i\right)=0.$$

It means that (X^i) is a Cauchy sequence in $L(R)$ for each fixed $k\in N$. Since $L(R)$ is complete, (X^i) converges in $LC(R)$ that is $X^i\to X_k$ as $i\to\infty$. Using these infinitely many limits, we define the sequence $X=\left(X_0,X_1,X_3,\cdots\right)$. Let as $j\to\infty$ in (2.1), the we obtain $\delta\left(X^i,X\right)\leq\varepsilon$. To show that $X\in ces^F\left(f,p\right)$, consider $n_o\in N$ and $j>k$. Since $\frac{pk}{M}\leq 1$ for all k, then by using Minkowski's inequality and the definition of modulus function f, we have

$$\leq\left(\sum_{n=1}^{n_o}\left[f\left(\frac{1}{n}\left(d\left(X_k^i,X_k\right)+d\left(X_k^i,\bar{0}\right)\right)\right)\right]^{pk}\right)^{\frac{1}{M}}$$

$$\left(\sum_{n=1}^{n_o}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,\bar{0}\right)\right)\right]^{pk}\right)^{\frac{1}{M}}$$

$$\leq\left(\sum_{n=1}^{n_o}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,X_k\right)+f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,\bar{0}\right)\right)\right)\right]^{pk}\right)^{\frac{1}{M}}$$

$$\leq\left(\sum_{n=1}^{n_o}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,X_k\right)\right)\right]^{pk}\right)^{\frac{1}{M}}+\left(\sum_{n=1}^{n_o}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,\bar{0}\right)\right)\right]^{pk}\right)^{\frac{1}{M}}$$

If follows that $\sum_{n=1}^{\infty}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k^i,\bar{0}\right)\right)\right]^{pk}$ is convergent, so that $X\in ces^F\left(f,p\right)$ and the space is complete.

We now investigate some inclusion relations between $ces^F\left(f,p\right)$ spaces.

Theorem 2.3. If $p,q\in \mathrm{R}^+$ and $0\leq p<q<\infty$, then for any modulus function f, $ces^F\left(f,p\right)\subset ces^F\left(f,p\right)$.

Proof. Let $X\in ces^F\left(f,p\right)$. Then

$$\sum_{n=1}^{\infty}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k,\bar{0}\right)\right)\right]^{pk}<\infty.$$

This implies that $\left[f\left(\frac{1}{n}\sum_{k}^{n}<d\left(X_k,\bar{0}\right)\right)\right]^{pk}\leq 1$ for sufficiently large values of n, say $n\geq n_o$ for some fixes $n_o\in N$. Since f is increasing and $p<q$, we have

$$\sum_{n=1}^{n_o}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k,\bar{0}\right)\right)\right]^{qk}\le\sum_{n=1}^{n_o}\left[f\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k,\bar{0}\right)\right)\right]^{pk}<\infty.$$

So, $X\in ces^F\left(f,q\right)$.

Theorem 2.4. Let r, $t\in R^+$ and $p=\min\left(r,t\right)$ then for any modulus function f,

$$ces^F\left(f,p\right)=ces^F\left(f,r\right)\cap ces^F\left(f,t\right).$$

Proof. It follows from Theorem 2.3 that

$$ces^F\left(f,p\right)\subset ces^F\left(f,r\right)\cap ces^F\left(f,t\right).$$

For any complex number $\alpha,\left|\alpha\right|^{pk}\le\left(\left|\alpha\right|^r,\left|\alpha\right|^t\right)$, thus

$$ces^F\left(f,r\right)\cap ces^F\left(f,t\right)\subset ces^F\left(f,p\right)$$

and then the proof is complete.

Theorem 2.5. Let f and g be two modulus functions. Then the following relations hold:

1) $ces^F\left(f,p\right)\cap ces^F\left(g,p\right)\subset ces^F\left(f+g,p\right)$,

2) $ces^F\left(f,p\right)\subset ces^F\left(gof,p\right)$,

3) If $f\left(t\right)\le g\left(t\right)$ for all $t\in\left[0,\infty\right)$, then $ces^F\left(g,p\right)\subset ces^F\left(f,p\right)$.

Proof. 1) Let $X\in ces^F\left(f,p\right)\cap ces^F\left(g,p\right)$. Since

$$\left(f+g\right)\left[d\left(X_k,\bar{0}\right)=f\left[d\left(X_k,\bar{0}\right)\right]+g\left[d\left(X_k,\bar{0}\right)\right]\right],$$

one can easily see that $X\in ces^F\left(f+g,p\right)$ and hence the result follows:

2) Let $X\in ces^F\left(f,p\right)$. Since g is continuous on $[0, \infty)$, there exists an $\beta > 0$ such that $g(\beta) = \varepsilon$ for all $\varepsilon > 0$. Since $X\in ces^F\left(f,p\right)$, there exists a $k_o\in N$ such that $f\left[d\left(X_k,\bar{0}\right)\right]<\beta$ for all $k\ge k_o$. Therefore, we can get

$$g\left(f\left[d\left(X_k,\bar{0}\right)\right]\right)\le g\left(\beta\right)=\varepsilon.$$

From this, we can easily see that the $X\in ces^F\left(gof,p\right)$.

3) Since $f\left(t\right)\le g\left(t\right)$ for all $t\in\left[0,\infty\right)$, we have $f\left[d\left(X_k,\bar{0}\right)\right]\le g\left[d\left(X_k,\bar{0}\right)\right]$. This leads us to the consequence that $X\in ces^F\left(g,p\right)$ which implies that $X\in ces^F\left(f,p\right)$.

Taking modulus function f^u instead of f in the space $ces^F\left(f,p\right)$, we can define the composite space $ces^F\left(f,p\right)$ as follows: For a fixed $u\in N$, we may define

$$ces^F\left(f^u,p\right)=\left\{X=\left(X_k\right)\in w\left(F\right):\sum_{n=1}^{\infty}\left[f^u\left(\frac{1}{n}\sum_{k=1}^{n}d\left(X_k,\bar{0}\right)\right)\right]^{pk}<\infty\right\}.$$

Theorem 2.6. Let f be a modulus function and $v\in N$. Then,

1) $ces^F\left(f^u,p\right)\subset ces^F\left(p\right)$ if $\lim_{t\to\infty}\frac{f\left(t\right)}{t}=\beta>0$.

2) $ces^F\left(p\right)\subset ces^F\left(f^u,p\right)$ if there exists a positive constant γ such that $f(t)\le\gamma t$ for all $t\in[0, \infty)$.

Proof. 1) Following the proof of the Proposition 1 of Maddox [7], we have

$$\beta=\lim_{t\to\infty}\frac{f\left(t\right)}{t}=\inf\left\{\frac{f\left(t\right)}{t}:t>0\right\}$$

so, $0\le\beta\le f(1)$. By definition of β we have $\beta t\le f(t)$ for all $t\ge 0$. Since f is increasing, then we have

$\beta^2 t \le f^2(t)$. So by induction, we have $\beta^v t \le f^v(t)$. Let $X \in ces^F(f^v, p)$ and using (1.20), we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d\left(X_k,\overline{0}\right)\right)^{pk} \le \sum_{n=1}^{\infty}\left[\beta^{-v} f^v\left(\frac{1}{n}\sum_{k=1}^{n} d\left(X_k,\overline{0}\right)\right)\right]^{pk} \le \max\left(1,\beta^{-vH}\right)\sum_{n=1}^{\infty}\left[f^v\left(\frac{1}{n}\sum_{k=1}^{n} d\left(X_k,\overline{0}\right)\right)\right]^{pk}$$

and therefore $X \in ces^F(p)$.

2) Since $f(t) \le \gamma t$ for all $t \in [0,\infty)$ and f is an increasing function, we have $f^u(t) \le \gamma^u t$ for each $v \in N$. Let $X \in ces^F(p)$ and using (1.2), we have

$$\sum_{n=1}^{\infty}\left[f^u\left(\frac{1}{2}\sum_{k=1}^{n} d\left(X_k,\overline{0}\right)\right)\right]^{pk} \le \max\left(1,\alpha^{-vH}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} d\left(X_k,\overline{0}\right)\right)^{pk}$$

and hence $X \in ces^F(f^u, p)$.

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