 ICCES, vol.2, no.4, pp.101-110, 2007

# Stability Analysis of Beams Rotating on an Elastic Ring Application to Turbo machinery Rotor-Stator Contacts

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## Summary

This paper presents a model of flexible beams rotating on the inner surface of an elastic stationary ring. The beams possesses two degrees of freedom, traction/compression and flexure. The in-plane deformations of the ring are considered and a single mode approximation is used. The model has been developed within the rotating frame by use of an energetic method. To better understand the phenomena occurring, the degrees of freedom of the beams can first be treated separately then together. Stability analysis show that even without rubbing, the radial degree of freedom of a beam rotating on an elastic ring can create divergence instabilities as well as mode couplings of the circular structure. When rubbing is considered, the system is unstable as soon as the rotational speed is non null. Moreover rubbing can couple the beams and the ring giving rise to mode coupling instabilities and locus veering phenomena. Finally, a comparison to a more complicated model of a flexible bladed-rotor in contact with an elastic casing shows a very good accordance with the phenomena occurring.

**keywords:** Stability analysis, rotating beams, circular elastic ring, divergence, mode couplings, rubbing, rotor-stator contacts

## Introduction

Moving loads on an elastic structure is a very rich and interesting problem that has been widely studied since it has many occurrences. For instance, the vibrations of circular saw [1, 2], those of computer memory storage disk [3, 4] or brake system noise [5, 6] result actually from the vibrations of a rotating disk excited by stationary loads on its surface. It is now well known that such system can experience divergence instabilities as well as mode couplings. Moreover, it has been shown that the same phenomena occur in the case of a stationary disk excited by rotating loads [7]. However, very few studies could be found that focuses on a ring excited by loads on its inner surface. Recently, Canchi and Parker [8] investigated the problem of parametric instabilities of a circular ring excited by rotating springs. However, rubbing was neglected is this study. This kind of system, with or without rubbing can have application to better understand the dynamics of rotor-stator contacts in Turbo machinery. Thus, the present paper describes a simple model of two degree of freedom rotating beams rubbing on the inner surface of an elastic ring. Then a stability analysis for the ring excited by beams featuring only

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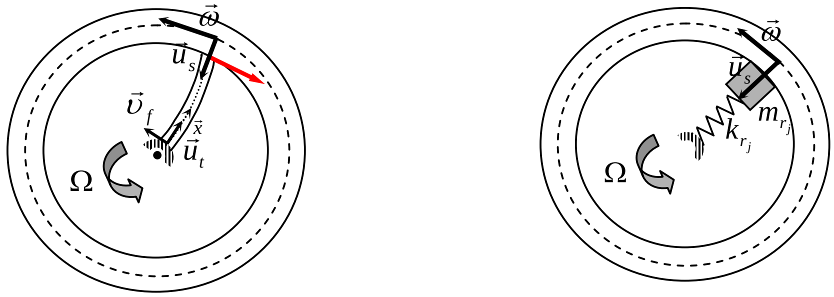
a traction/compression degree of freedom rotating without rubbing is performed. Rubbing is next introduced in the case of beams possessing two degrees of freedom. Lastly, a comparison with the dynamics of a flexible bladed rotor in contact with an elastic casing is done to better understand and to confirm the phenomena observed.

## Model formulation

The present model consists of an elastic ring rubbed by one or several beams on its inner surface as depicted on Fig. 1a, in the case of one rotating beam. These Euler-Bernoulli beams have two degrees of freedom in the rotating frame, a traction/compression motion $\vec{u}_t$ and a flexural motion $\vec{\upsilon}_f$. An energetic method is used to develop the model thus the degrees of freedom of the $j^{th}$ beam are expressed by the following Ritz functions: $u_{t_j}(x,t) = u_{t_j}(t)\sin\left(\frac{\pi x}{2R_{stat}}\right)$ corresponding to the exact traction/compression mode shape of a beam clamped-free and $\upsilon_{f_j}(x,t) = \upsilon_{f_j}(t)\left(1-\cos\left(\frac{\pi x}{2R_{stat}}\right)\right)$for its flexural degree of freedom, $x$ being the local axis along the beam. Concerning the ring, its in-plane flexural vibrations are considered thus, two degrees of freedom are considered in the rotating frame too: its radial displacement $\vec{u}_s(\phi,t)$ and its tangential displacement $\vec{\omega}(\phi,t)$, $\phi$ being the angular position of the centre of mass of a ring's cross section in the rotating frame. This latter degree of freedom can be expressed by [9]: $\omega(\phi,t) = \sum_{n=2}^{k_{tot}} A_n(t)\cos n\phi + B_n(t)\sin n\phi$ where the rigid body motion has been eliminated. In order to generate as simple a model as possible, only one mode shape, the $n^{th}$ one, is considered for the ring hence: $\omega(\phi,t) = A_n(t)\cos n\phi + B_n(t)\sin n\phi$. Moreover, the considered ring is assumed to be inextensible, implying thus that its radial displacement can be expressed from its tangential one. The free ends of the beams are assumed to remain in steady state contact with the inner surface of the ring thus a link relationship between the pertinent degrees of freedom must be written as follows: $\vec{u}_{t_j}(R_{stat},t) = -\vec{u}_s(\phi_j,t)$. Since an energetic method is used to develop the complete model, the kinetic energy and the potential energy are defined for the beams and for the ring as well. The rubbing strength is introduced by defining its work. The dynamic behaviour of the system is thus described by a mass matrix, a stiffness matrix, a circulatory matrix and a gyroscopic matrix. To better understand the phenomenon appearing within this structure however, this system can be separated into simpler structures. The first such structure consists of beams having only a traction/compression degree of freedom rotating on the ring. Then the effects of the flexural motion of beams rubbing against the ring will be studied.




(a) (b)

Figure 1: a) Model of Euler-Bernoulli beam having two degrees of freedom rubbing on an elastic ring, b) model of radial spring-mass rotating against a ring

## Beams featuring only a traction/compression degree of freedom rotating on the ring

The stability of an elastic ring excited by one or several beams can be investigated by determining the solution $\lambda = a + ib$ of the characteristic equation $\det(\lambda^2\mathbf{M} + \lambda\mathbf{G} + \mathbf{K}) = 0$ where **M**, **G** and **K** are the mass, the gyroscopic and the stiffness matrices of the system. The system becomes unstable if any one or more of the eigenvalues' real parts $a$ is positive.

The beams considered having only a traction/compression degree of freedom can be represented by radial spring-masses (see Fig.1b) in order to easily handle modal parameters. In the case of only one rotating load, the dynamic behaviour of such system can be described by the following matrix equation:

$$\begin{aligned}
&\begin{bmatrix} M_{stat}\left(n^2+1\right) & 0 \\ 0 & M_{stat}\left(n^2+1\right)+m_r n^2 \end{bmatrix} \begin{Bmatrix} \ddot{A}_n \\ \ddot{B}_n \end{Bmatrix} \\
+&\begin{bmatrix} 0 & -2M_{stat} n\Omega\left(n^2+1\right) \\ 2M_{stat} n\Omega\left(n^2+1\right) & 0 \end{bmatrix} \begin{Bmatrix} \dot{A}_n \\ \dot{B}_n \end{Bmatrix} \\
+&\begin{bmatrix} K_{stat} n^2(n^2-1)^2 - M_{stat} n^2\Omega^2\left(n^2+1\right) & 0 \\ 0 & K_{stat} n^2(n^2-1)^2 - M_{stat} n^2\Omega^2\left(n^2+1\right) + (k_r - m_r\Omega^2)n^2 \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \end{Bmatrix} \\
=&\begin{Bmatrix} 0 \\ -m_r R_{stat}\Omega^2 n \end{Bmatrix}
\end{aligned} \tag{1}$$

with $M_{stat} = \rho_{stat} S_{stat} R_{stat} \pi$ and $K_{stat} = \frac{E_{stat} I_{stat} \pi}{R_{stat}^3}$

Figure 2.a) represents a stability analysis of the two nodal diameter mode shape of the ring excited by one rotating spring-mass having $m_r = 100Kg$ and $k_r = 1.10^6 N.m^{-1}$. Two kinds of instabilities appear: divergence (instability at zero frequency) of the forward mode shape and mode coupling (the forward and the backward mode shapes of the ring become two mode shapes having the same eigen-frequency but one of them being stable and the other, unstable). The critical rotational speeds where the system diverges can be expressed analytically

through the Routh-Hurwitz criterion applied to the characteristic polynom $P(s) = \det\left(s^2\mathbf{M}+s\mathbf{G}+\mathbf{K}\right)$. Thus, it turns out that the system can experience divergence between $\Omega_c$ and $\Omega_{c_2}$, with:

$$\Omega_c^2 = \frac{K_{stat}}{M_{stat}}\frac{\left(n^2-1\right)^2}{n^2+1} = \frac{\omega_{stat_n}^2}{n^2} \quad \text{and} \quad \Omega_{c2}^2 = \frac{\Omega_c^2}{1+\frac{m_r}{M_{stat}(n^2+1)}} + \frac{\omega_r^2}{1+\frac{M_{stat}(n^2+1)}{m_r}} \tag{2}$$

$\Omega_c$ being the wave propagation speed in the ring for its $n^{th}$ mode shape and so, its $n^{th}$ critical rotational speed in the rotating frame and $\omega_r^2 = \frac{k_r}{m_r}$. Thus, it appears that both the modal mass and the modal stiffness of the traction/compression degree of freedom of the spring-mass can make the system diverge. In a general manner, the bigger the stiffness is, the larger the speed range where the system can experience divergence is. Concerning the mass, the heavier it is, the earlier the speed range where the system can experience divergence is. The critical speed where the system experiences mode coupling can be determined by knowing that a sufficient condition for the apparition of flutter, in this particular case of an undamped structure, is to have two eigenvalues with real parts null and opposite imaginary parts. Thus, in the case of only one rotating spring-mass, the rotational speeds satisfying the latter condition can be expressed by:

$$\Omega_{mc} = \pm\frac{\left\{m_r\chi\left[\gamma\pm4\sqrt{\xi}\right]\right\}^{1/2}}{m_r n\chi} \tag{3}$$

providing that the square roots can be defined, with:

$$\chi = \left[8M_{stat}\left(n^2+1\right)^2 - m_r\left(n^2-1\right)^2\right] \tag{4}$$

$$\gamma = 4M_{stat}\left(n^2+1\right)\left(2K_{stat}+n^2k_r\right)+m_rn^2\left(4K_{stat}+\left(n^2-1\right)\left[k_r-n^2m_r\Omega_c^2\right]\right) \tag{5}$$

$$\begin{aligned}\xi &= \left(M_{stat}\left(n^2+1\right)+n^2m_r\right)\\ &\quad\cdot\left(M_{stat}\left(n^2+1\right)\left[n^2k_r+2K_{stat}\right]^2+K_{stat}m_rn^2\left[m_rn^2\Omega_c^2-k_r\left(n^2+1\right)\right]\right)\end{aligned} \tag{6}$$

$$K_{stat} = \frac{E_{stat}I_{stat}\pi}{R_{stat}^3}\left(n^3-n\right)^2 \tag{7}$$

Thus the influence of the mass is preponderant in comparison to the one of the stiffness and the heavier the mass, the earlier the system experiences mode couplings. These results and the influence of each parameter on the instable speed range of the system is in general accordance with other studies by Iwan and Moeller [3], Iwan and Stahl [7], Canchi and Parker [8]. Moreover, Fig.2b representing the stability analysis of the two nodal diameter mode shape of the ring excited by three radial spring-masses separated from 60° from each other, shows that under certain conditions, the divergence of the forward mode shape of the ring can be avoided. As a matter of fact, by using the Routh-Hurwitz criterion, a sufficient condition for this is to have a kind of symmetry of the loads in comparison to the ring mode shape excited. This can be written as follows:

$$\left\{\begin{array}{l} \sum_j k_{r_j}\sin^2(n\phi_j) = \sum_j k_{r_j}\cos^2(n\phi_j) \quad and \quad \sum_j k_{r_j}\sin(n\phi_j)\cos(n\phi_j) = 0 \\ \sum_j m_{r_j}\sin^2(n\phi_j) = \sum_j m_{r_j}\cos^2(n\phi_j) \quad and \quad \sum_j m_{r_j}\sin(n\phi_j)\cos(n\phi_j) = 0 \end{array}\right. \tag{8}$$

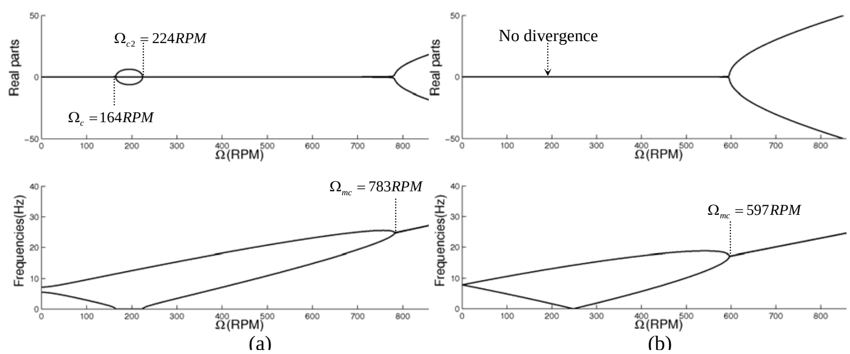



Figure 2: a) Stability analysis of the two nodal diameter mode shape of the ring excited by a) one radial spring-mass, b) three radial spring- masses separated from 60° from each other

## Beams having two degrees of freedom rubbing on the ring

The flexural degree of freedom of the beams as well as rubbing will be now introduced. The system considered is represented on Fig. 1.a. Even in the simple case of one beam featuring just a traction/compression degree of freedom rubbing on the ring, rubbing makes the mass matrix and the stiffness matrix non symmetric, what is known to be characteristic of a potentially-unstable system:

$$\begin{aligned} &\begin{bmatrix} M_{stat}\left(n^2+1\right) & -\mu\left\{1+\frac{h}{2R_{stat}}\left(n^2-1\right)\right\}m_r n \\ 0 & M_{stat}\left(n^2+1\right)+m_r n^2 \end{bmatrix}\left\{\begin{array}{c}\ddot{A}_n \\ \ddot{B}_n\end{array}\right\}+\begin{bmatrix} 0 & -2M_{stat}\,n\Omega\left(n^2+1\right) \\ 2M_{stat}\,n\Omega\left(n^2+1\right) & 0 \end{bmatrix}\left\{\begin{array}{c}\dot{A}_n \\ \dot{B}_n\end{array}\right\} \\ &+\begin{bmatrix} K_{stat}\,n^2(n^2-1)^2-M_{stat}\,n^2\Omega^2\left(n^2+1\right) & -\mu\left\{1+\frac{h}{2R_{stat}}\left(n^2-1\right)\right\}\left(k_r-m_r\Omega^2\right)n \\ 0 & K_{stat}\,n^2(n^2-1)^2-M_{stat}\,n^2\Omega^2\left(n^2+1\right)+(k_r-m_r\Omega^2)n^2 \end{bmatrix}\left\{\begin{array}{c}A_n \\ B_n\end{array}\right\} \\ &=\left\{\begin{array}{l}\mu\left\{1+\frac{h}{2R_{stat}}\left(n^2-1\right)\right\}m_r\Omega^2 R_{stat} \\ -m_r R_{stat}\,\Omega^2 n\end{array}\right\} \end{aligned} \tag{9}$$

Moreover, it can be shown by using once again the Routh-Hurwitz criterion that such undamped system is almost always unstable. When considering to degrees of freedom for each beam, the matrix equation of the associated system is $(2+number\ of\ loads) \times (2+number\ of\ loads)$. Figure 3 represents the stability analysis for one beam having two degrees of freedom rubbing on the two nodal diameter mode shape of the ring with $\mu = 0,1$. As expected, it can be seen on this figure that the system is unstable as soon as the rotational speed is greater than 0 RPM. As the rubbing coefficient increases, instability rises even faster. The flexural degree of freedom of the beam can couple the system and some locus veering can be seen between the backward mode shape of the ring and the flexural motion of the beam as well as mode coupling between this flexural motion and the ring forward mode shape. When $\Omega^2 > \omega_t^2$, $\omega_t$ being the angular frequency of the beam flexural degree of freedom, the beam experiences divergence instability. All simulations have been conducted for a ring's two-nodal diameter mode shape, yet the same phenomena are present for other mode shapes as well. When considering more than one beam, the system is more complex and more locus veering can occur as well as more mode couplings.

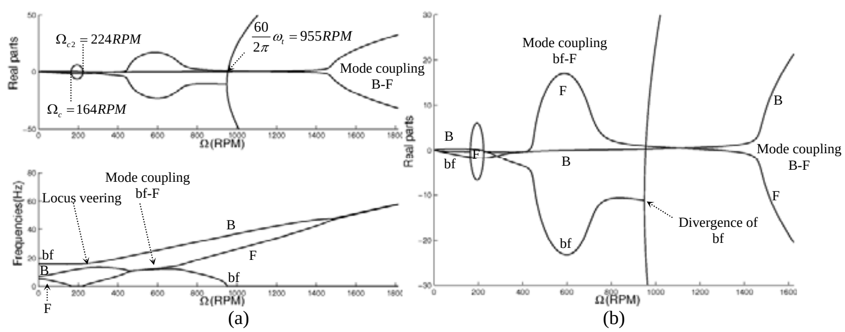
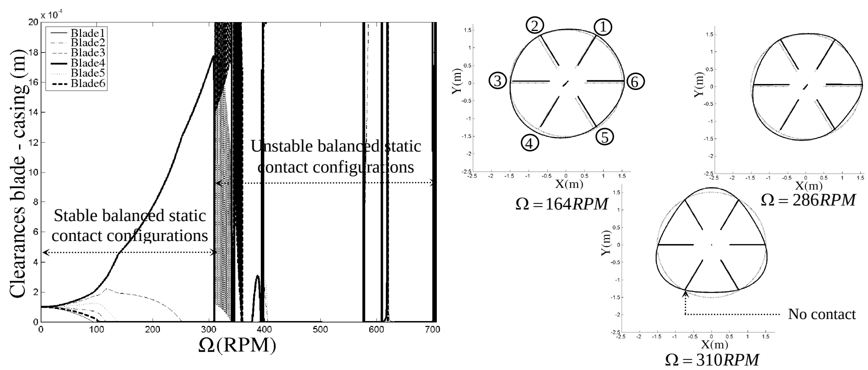



Figure 3: Stability analysis of the two nodal diameter mode shape of the ring rubbed by a beam with $\mu = 0,1$, b) being the associated zoom, bf = beam flexure, F = Forward, B = Backward

## Application to rotor-stator contacts

To end this study, an application of this model improving the understanding of complex dynamics in rotor-stator contacts will be presented. The rotor displayed in this section has been developed in the rotating frame by using an energetic method [10, 11]. It is composed of a shaft modelled by an Euler-Bernoulli beam, connected to a rigid disk modelled by a concentrated mass with rotational inertia. This shaft is set on bearings at multiple locations. On the rigid disk is clamped a full set of flexible blades also modelled by Euler-Bernoulli beams. In the rotating frame, two degrees of freedom are considered for the shaft: two orthogonal translations in the disk's plane, and one degree of freedom for each blade defining its deflection. A Rayleigh-Ritz approximation is used to express the degrees of freedom of these different parts. The casing considered is the same as in the latter section but with internal viscous damping and with more than one mode shape considered. The contact between the flexible blade tips and the flexible casing is introduced by assuming no rubbing between the two structures. In the rotating frame, this non linear problem is a static one. By increasing the rotational speed of the rotor, the evolution of the clearance between each blade tip and the casing can be followed as a function of the its rotational speed as shown in Fig. 4a, in the particular case of a rotor having six blades (see Fig. 4b). It can be seen that the first blade to touch the casing is the blade ①, then successively the blades ⑥, ② and ⑤, when the rotational speed increases. The associated deformed shape, at 164 RPM, is plotted on Fig. 4b. When the rotational speed increases again, the blade ③ touches the stator. Thus, all the blades are in contact with the casing apart from the blade ④ (see Fig. 4b at $\Omega = 286RPM$). The system keeps this configuration while the rotational speed increases, until 310 RPM corresponding actually to the three nodal diameter mode shape critical speed of the casing (see Fig. 4b). It appears that up to this rotational speed, all the balanced static contact configurations found are stable ones but after this speed, the system has only unstable balanced static contact configurations. It could have been thought that from 310 RPM the system might have stable dynamic contact configurations but, all the time integrations performed diverged on the three nodal diameter mode shape of the casing. This phenomenon can be explained by using the simple model of rotating loads on an elastic structure studied previously.

Effectively, it has been shown that an elastic ring excited by rotating loads could experience two kind of instabilities: divergence instabilities and mode couplings. In order to know if the divergence of the rotor coupled with the flexible casing is due to this phenomenon, the latter model of rotating loads on the elastic ring can be used with the modal parameters of the present rotor. The latest stable balanced contact configuration obtained statically and by time integration consists in five contacts: at blades ①,②,③, ⑤ and ⑥. Figures 5a and 5b represent the comparison between respectively the results (real parts and eigen-frequencies) obtained with the three nodal diameter mode shape of the simplified model having five moving loads configured like blades ①,②,③, ⑤ and ⑥ with the rotor parameters and, the Campbell diagram and the associated decay rate plot for the flexible casing in contact at blades ①,②,③, ⑤ and ⑥. The importance as well as the influence of the other mode shapes of the stator on each other appear through all the other instabilities that can be seen on Fig. 5b and not on Fig. 5a. However, these figures confirm that the stator experiences divergence through its three nodal diameter




(a) (b)

Figure 4: a) Evolution of the clearances between the blades and the flexible stator, b) contact configurations

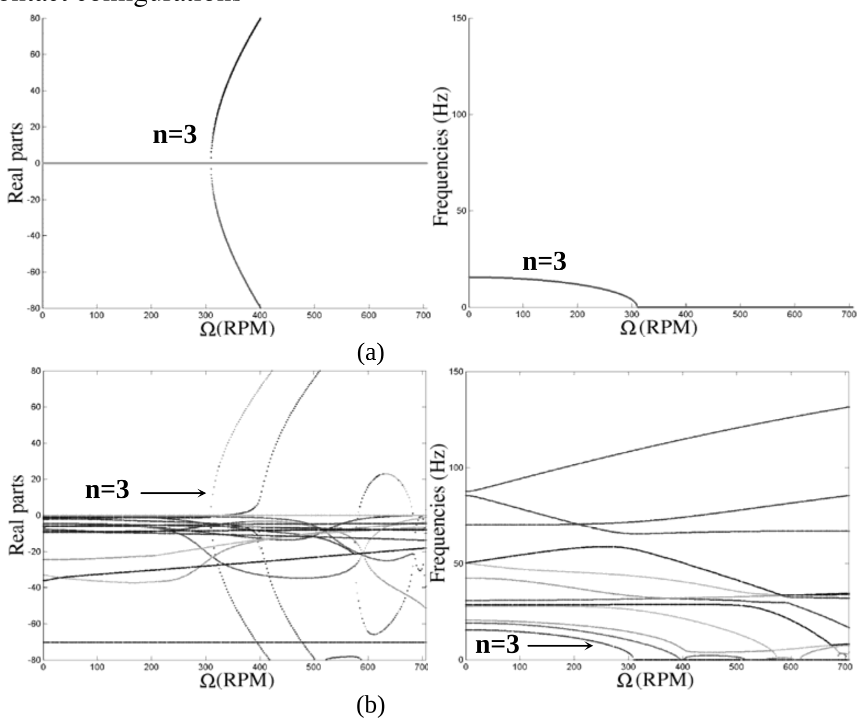



Figure 5: Comparison of real parts and eigen-frequencies between a) the three nodal diameter mode shape of the stator excited by five rotating loads angularly set like blades ①,②,③, ⑤ and ⑥ with the rotor parameters and b) the stator excited by the blades ①,②,③, ⑤ and ⑥

mode shape just after 310 RPM which is the critical speed $\Omega_c$ of this mode shape. Moreover, with this configuration of rotor i.e. with six blades separated by 60° from each other, there is no contact configuration that satisfies the four conditions (see Eq. 8) to avoid the divergence of the three nodal diameter mode shape of the stator. Thus, as shown by Iwan and Moeller [3] and Iwan and Stahl [7], the single mode approximation for the casing provides a good estimate of the overall response of the complete rotating system including critical speeds and instability regions.

## Conclusion

The stability of rotating beams rubbing on an elastic ring has been studied in this article. An energy model of flexible beams possessing two degrees of freedom in steady-state contact with an elastic ring possessing just one in-plane mode shape has been developed within the rotating frame. This model, devoid of time-dependent terms, has been studied from a stability point of view. It appears that even without rubbing, the radial dynamics of a beam can make the system experience divergence and mode couplings. The associated critical speeds have been expressed analytically. Moreover, it has been shown that rubbing always makes the system unstable once the beam's rotational speed is nonzero. As the rubbing coefficient rises, the gradient of the eigenvalue real parts also rises. The beams' flexural degree of freedom yields mode couplings and locus veering with the ring. The influence of several beams rubbing on a ring has been examined and some sufficient conditions have been exhibited to avoid the divergence of the forward mode shape of the ring. Finally, after comparison to a complete model of flexible rotor in contact with an elastic ring, the simple model appeared to give good estimate of the overall dynamics of the latter system and so was very useful to better understand complicated dynamics.

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