ASYMPTOTIC NORMALITY OF A HURST PARAMETER ESTIMATOR BASED ON THE MODIFIED ALLAN VARIANCE

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ABSTRACT. It has been observed that different kinds of real data are characterized by self-similarity and long-range correlations on various time-scales. The memory parameter of a related time series is thus a key quantity in order to predict and control many phenomena. In the present paper we analyze the performance of a memory parameter estimator, $\widehat{\alpha}$, defined by the log-regression on the so-called modified Allan variance. Under the assumption that the signal process is a fractional Brownian motion, with Hurst parameter H, we study the rate of convergence of the empirical modified Allan variance, and then prove that the log-regression estimator $\widehat{\alpha}$ converges to the memory parameter $\alpha = 2H - 2$ of the process. In particular, we show that the deviation $\widehat{\alpha} - \alpha$, when suitably normalized, converges in distribution to a normal random variable, and we compute explicitly its asymptotic variance.

1. INTRODUCTION

There is ample evidence that different kinds of real data (Hydrology, Telecommunication networks, Economics, Biology) exhibit self-similarity and long-range dependence (LRD) on various time scales. By self-similarity we refer to the property that a dilated portion of a realization has the same statistical characterization as the original realization. This can be well represented by a self-similar random process with given scaling exponent H (Hurst parameter). The long-range dependence, also called long-memory, emphasizes the long-range time-correlation between past and future observations and it is thus commonly equated to an asymptotic power-law decrease of the spectral density or, equivalently, of the autocovariance function, of a given stationary random process. In this situation, the memory parameter of the process is given by the exponent d characterizing the power-law of the spectral density. (For a review of historical and statistical aspects of the self-similarity and the long-memory see [5].)

Though a self-similar process can not be stationary (and thus nor LRD), these two proprieties are often related in the following sense. Under the hypothesis that a self-similar process has stationary (or weakly stationary) increments, the scaling parameter H enters in the description of the spectral density and covariance function of the increments, providing an asymptotic power-law with exponent $d = 2H - 1$. Under this assumption, we can say that the self-similarity of the process reflects on the long-range dependence of its increments. The most paradigmatic example of this connection is provided by the fractional Brownian motion and by its increment process, the fractional Gaussian noise [13].

In this paper we will consider the problem of estimating the Hurst parameter H of a self-similar process with weakly stationary increments. Among the different techniques introduced in the literature in order to estimate this parameter, we will focus on a method based on the log-regression of the so-called Modified Allan Variance (MAVAR). The MAVAR is a well known time-domain quantity generalizing the classic Allan variance [6, 7]. It has been proposed for the first time as a traffic analysis tool in [10], and sequently, in a series of paper [10, 9, 8], its performance has been evaluated by simulation and comparison

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with the real IP traffic. These works have pointed out the high accuracy of the method in estimating the parameter H, and have shown that it achieves a highest confidence if compared with the well-established wavelet log-diagram.

The aim of the present work is to substantiate these results from the theoretical point of view, studying the rate of convergence of the estimator toward the memory parameter. In particular, our goal is to provide the precise asymptotic normality of the MAVAR log-regression estimator in order to compute the related confidence intervals. This will be reached under the assumption that the signal process is a fractional Brownian motion. Although this hypothesis is restrictive (indeed this estimator is successfully used for more general processes), the obtained results are a first step toward the mathematical study of the MAVAR log-regression estimator. To our knowledge, there are no similar results in the literature.

Our theorems can be view as a counterpart of the already established results concerning the asymptotic normality of the wavelet log-regression estimator [15, 16, 17]. While the classical Allan variance estimator easily falls into the wavelet framework, as is shown in [4], the log-regression MAVAR estimator seems to us not trivially related to a wavelets family (see sec. 5), and we adopt a different analysis.

The paper is organized as follows. In section 2 we recall the properties of self-similarity and long-range dependence for stochastic processes, and the definition of the fractional Brownian motion; in section 3 we introduce the MAVAR and its estimator, with their main properties; in section 4 we state and prove the main results concerning the asymptotic normality of the estimator; in section 5 we make some comments on the link between the MAVAR and the wavelet estimators and about the modified Hadamar variance, which is a generalization of the MAVAR; in the appendix we recall some results used along the proof.

2. Self-similarity and Long-range dependence

We consider a real-valued stochastic process $X = \{X(t),\, t \in \mathbb{R}\}$, that can be interpreted as the signal process. Sometimes it is also useful to consider the τ-increment of the process X, which is defined, for every $\tau \in \mathbb{R}^+$ and $t \in \mathbb{R}$, as

$$Y_\tau(t) = \frac{X(t+\tau) - X(t)}{\tau}\,. \tag{1}$$

In order to reproduce the behavior of the real data, it is commonly assumed that X satisfies one of the two following properties: (i) Self-similarity; (ii) Long range dependence.

(i) The *self-similarity* of a process X refers to the existence of a parameter $H \in (0,1)$, called *Hurst index or Hurst parameter* of the process, such that, for all $a > 0$, it holds

$$\{X(t)\,, t \in \mathbb{R}\} \stackrel{d}{=} \{a^{-H} X(at),\, t \in \mathbb{R}\}\,. \tag{2}$$

In this case we say that X is a H-self-similar process.

(ii) We first recall that a stochastic process X is *weakly stationary* if it is square-integrable and its autocovariance function, $C_X(t,s) := \mathrm{Cov}(X(t), X(s))$, is translation invariant, namely if

$$C_X(t,s) = C_X(t+r, s+r) \qquad \forall\, t, s, r \in \mathbb{R}\,.$$

In this case we also set $R_X(t) := C_X(t, 0)$.

If X is a weakly stationary process, we say that it displays *long-range dependence, or long-memory*, if there exists $d \in (0,1)$ such that the spectral density of the process, $f_X(\lambda)$, satisfies the condition

$$f_X(\lambda) \sim c_f |\lambda|^{-d} \quad \text{ as } \lambda \to 0\,, \tag{3}$$

for some finite constant $c_f \neq 0$, where we write $f(x) \sim g(x)$ as $x \to x_0$, if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$. Due to the correspondence between the spectral density and the autocovariance function, given by

$$R_X(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\lambda} f_X(\lambda)\, d\lambda,$$

the long-range condition (3) can be often stated in terms of the autocovariance of the process as

$$R_X(t) \sim c_R t^{-\beta} \quad \text{as } |t| \to +\infty\,, \tag{4}$$

for some finite constant $c_R \neq 0$ and $\beta = (1-d) \in (0,1)$.

Notice that if X is a self-similar process, then it can not be weakly stationary due to the normalization factor a^H. On the other hand, assuming that X is a H-self-similar process with weakly stationary increments, i.e. the quantity

$$\mathbb{E}\left[\big(X(\tau_2+s+t) - X(s+t)\big)\big(X(\tau_1+s) - X(s)\big)\right]$$

does not depend on s, it turns out that the autocovariance function is given by

$$C_X(s,t) = \frac{\sigma_H^2}{2}\left(|t|^{2H} - |t-s|^{2H} + |s|^{2H}\right)\,, \tag{5}$$

with $\sigma_H^2 = \mathbb{E}(X^2(1))$, which is clearly not translation invariant. Consequently, denoting by Y_τ its τ-increment process (see (1)), the autocovariance function of Y_τ is such that

$$R_{Y_\tau}(t) \sim c t^{2H-2} \quad \text{as } |t| \to +\infty\,, \tag{6}$$

for some finite constant $c \neq 0$ depending on H and τ ([3]). In particular, if $H \in (\frac{1}{2}, 1)$, the process Y_τ displays long-range dependence in the sense of (4) with $\beta = 2 - 2H$. Under this assumption, we thus embrace the two main empirical properties of a wide collection of real data.

A basic example of the connection between self-similarity and long-range dependence is provided by the *fractional Brownian motion* $B_H = \{B_H(t),\, t \in \mathbb{R}\}$. This is a centered Gaussian process with autocovariance function given by (5), where

$$\sigma_H^2 = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}. \tag{7}$$

It can be proved that B_H is a self-similar process with Hurst index $H \in (0,1)$, which corresponds, for $H = 1/2$, to the standard Brownian motion. Moreover, its increment process

$$G_{\tau,H}(t) = \frac{B_H(t+\tau) - B_H(t)}{\tau}\,,$$

called fractional Gaussian noise, turns out to be a weakly stationary Gaussian process [13, 20].

In the next sections we will perform the analysis of the modified Allan variance and of the related estimator of the memory parameter.

3. The Modified Allan variance

In this section we introduce and recall the main properties of the Modified Allan variance (MAVAR), and of the log-regression estimator of the memory parameter based on it.

Let $X = \{X(t) : t \in \mathbb{R}\}$ be a stochastic process with weakly stationary increments. Let $\tau_0 > 0$ be the "sampling period" and define the sequence of times $\{t_k\}_{k\geq 1}$ taking $t_1 \in \mathbb{R}$ and setting $t_i - t_{i-1} = \tau_0$, i.e. $t_i = t_1 + \tau_0(i-1)$.

Definition 3.1. For any fixed integer $p \geq 1$, the modified Allan variance (MAVAR) is defined as

$$\begin{aligned}\sigma_p^2 = \sigma^2(\tau_0, p) &:= \frac{1}{2\tau_0^2 p^2}\mathbb{E}\left[\left(\frac{1}{p}\sum_{i=1}^{p}(X_{t_{i+2p}} - 2X_{t_{i+p}} + X_{t_i})\right)^2\right] \\ &= \frac{1}{2\tau^2}\mathbb{E}\left[\left(\frac{1}{p}\sum_{i=1}^{p}(X_{t_i+2\tau} - 2X_{t_i+\tau} + X_{t_i})\right)^2\right],\end{aligned} \tag{8}$$

where we set $\tau := \tau_0 p$. For $p = 1$ we recover the well-known Allan variance.

Let us assume that a finite sample $X_1, \dots, X_n$ of the process X is given, and that the observations are taken at times $t_1, \dots, t_n$. In other words we set $X_i = X_{t_i}$ for $i = 1, \dots, n$.

A standard estimator for the modified Allan variance (MAVAR estimator) is given by

$$\begin{aligned}\widehat{\sigma}_p^2(n) = \widehat{\sigma}^2(\tau_0, p, n) &:= \frac{1}{2\tau_0^2 p^4 n_p}\sum_{h=1}^{n_p}\left(\sum_{i'=h}^{p+h-1}(X_{i'+2p} - 2X_{i'+p} + X_{i'})\right)^2 \\ &= \frac{1}{n_p}\sum_{k=0}^{n_p-1}\left(\frac{1}{\sqrt{2}\tau_0 p^2}\sum_{i=1}^{p}(X_{k+i+2p} - 2X_{k+i+p} + X_{k+i})\right)^2\end{aligned} \tag{9}$$

for $p = 1, \dots, \lfloor n/3 \rfloor$ and $n_p = n - 3p + 1$.

For $k \in \mathbb{Z}$, let us set

$$d_{p,k} = d(\tau_0, p, k) := \frac{1}{\sqrt{2}\,\tau_0 p^2}\sum_{i=1}^{p}(X_{k+i+2p} - 2X_{k+i+p} + X_{k+i})$$

so that we can write

$$\sigma_p^2 = \mathbb{E}\left[d_{p,0}^2\right]$$

and

$$\widehat{\sigma}_p^2(n) = \frac{1}{n_p}\sum_{k=0}^{n_p-1} d_{p,k}^2.$$

3.1. **Some properties.** Let us further assume that X is a H-self-similar process (see (2)) with $X(0) = 0$ and $\mathbb{E}(X(t)) = 0$ for all t. Under these assumptions on X, the process $\{d_{p,k}\}_k$ turns out to be weakly stationary for each fixed p, with $\mathbb{E}[d_{p,k}] = 0$. More precisely, applying the covariance formula (5), it holds

$$\sigma_p^2 = \mathbb{E}\left[d_{p,k}^2\right] = \mathbb{E}\left[d_{p,0}^2\right] = \sigma_H^2\,\tau^{2H-2}\,K(H,p)\,, \tag{10}$$

with $\sigma_H^2 = \mathbb{E}[X(1)^2]$ and

$$K(H,p) := \frac{2}{p}(1 - 2^{2H-2}) + \frac{1}{2p^2}\sum_{\ell=1}^{p-1}\sum_{h=1}^{\ell} P_{2H}\left(\tfrac{h}{p}\right)\,, \tag{11}$$

where P_{2H} is the polynomial of degree $2H$ given by

$$P_{2H}(x) := \left[-6x^{2H} + 4\,(1+x)^{2H} + 4\,(1-x)^{2H} - (2+x)^{2H} - (2-x)^{2H}\right].$$

Since we are interested in the limit for $p \to \infty$, we consider the approximation of the two finite sums in (11) by the corresponding double integral, namely

$$\frac{1}{p^2}\sum_{\ell=1}^{p-1}\sum_{h=1}^{\ell} P_{2H}\left(\tfrac{h}{p}\right) = \int_0^1\int_0^y P_{2H}(x)dxdy + O_H(p^{-1})\,.$$

Computing the integral and inserting the result in (11), we get

$$K(H,p) = K(H) + O_H(p^{-1}) \tag{12}$$

where

$$K(H) := \frac{2^{2H+4} + 2^{2H+3} - 3^{2H+2} - 15}{2(2H+1)(2H+2)}\,. \tag{13}$$

From (10) and (12), we get

$$\left|\sigma_p^2 - \sigma_H^2 \tau^{2H-2} K(H)\right| \le \sigma_H^2 \tau^{2H-2} O_H(p^{-1}) = O_H(p^{-(3-2H)})\,. \tag{14}$$

Under these hypothesis on X, one can also prove that the process $\{d_{p,k}\}_{p,k}$ satisfies the stationary condition

$$\mathrm{Cov}(d_{p,k}, d_{p-u,k'}) = \mathrm{Cov}(d_{p,k-k'}, d_{p-u,0}) \qquad \text{for } 0 \le u < p\,. \tag{15}$$

To verify this condition, we write explicitly the covariance as

$$\begin{aligned}
\mathbb{E}[d_{p,k} d_{p-u,k'}] = \frac{1}{2\tau^2(p-u)^2} \sum_{j=1}^{p} \sum_{j'=1}^{p-u} & \mathbb{E}\left[(X_{k+j+2p} - X_{k+j+p})(X_{k'+j'+2(p-u)} - X_{k'+j'+(p-u)})\right] \\
& - \mathbb{E}\left[(X_{k+j+2p} - X_{k+j+p})(X_{k'+j'+(p-u)} - X_{k'+j'})\right] \\
& - \mathbb{E}\left[(X_{k+j+p} - X_{k+j})(X_{k'+j'+2(p-u)} - X_{k'+j'+(p-u)})\right] \\
& + \mathbb{E}\left[(X_{k+j+p} - X_{k+j})(X_{k'+j'+(p-u)} - X_{k'+j'})\right]\,.
\end{aligned}$$

Using the spectral representation of the correlation function for the increments of the process X (see formula (2.2) in [14]), i.e.

$$\begin{aligned}
R(t;\tau_1,\tau_2) &:= \mathbb{E}\left[\big(X(\tau_2+s+t) - X(s+t)\big)\big(X(\tau_1+s) - X(s)\big)\right] \\
&= \int_{\mathbb{R}} \tfrac{(1+\lambda^2)}{\lambda^2} e^{it\lambda}(1-e^{-i\tau_1\lambda})(1-e^{i\tau_2\lambda})\,d\mu(\lambda),
\end{aligned} \tag{16}$$

we get

$$\begin{aligned}
\mathbb{E}[d_{p,k} d_{p-u,k'}] &= \frac{1}{2\,\tau^2} \frac{1}{(p-u)^2} \sum_{j=1}^{p} \sum_{j'=1}^{p-u} r(k-k', j-j', p, u) \\
&= \frac{1}{2\tau^2(p-u)^2} \left[(p-u) r(k-k',0,p,u) + 2 \sum_{j=2}^{p-u} \sum_{h=1}^{j-1} \widehat{r}(k-k',h,p,u) \right. \\
&\quad \left. + \sum_{j=p-u+1}^{p} \sum_{h=j-(p-u)}^{j-1} r(k-k',h,p,u) \right],
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
r(q,h,p,u) &:= R(\tau_0(q+h+u);\tau_0(p-u),\tau_0 p) - R(\tau_0(q+h+p);\tau_0(p-u),\tau_0 p) \\
&\quad - R(\tau_0(q+h-(p-u));\tau_0(p-u),\tau_0 p) + R(\tau_0(q+h);\tau_0(p-u),\tau_0 p) \\
&= \int e^{i\tau_0 h\lambda} e^{i\tau_0 q\lambda} \left(1 - e^{-i\tau_0(p-u)\lambda}\right)^2 \left(1 - e^{i\tau_0 p\lambda}\right)^2 \tfrac{(1+\lambda^2)}{\lambda^2}\,d\mu(\lambda)
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\widehat{r}(q,h,p,u) &:= r(q,h,p,u) + r(q,-h,p,u) \\
&= \int \cos(\tau_0 h\lambda) e^{i\tau_0 q\lambda} \left(1 - e^{-i\tau_0(p-u)\lambda}\right)^2 \left(1 - e^{i\tau_0 p\lambda}\right)^2 \tfrac{(1+\lambda^2)}{\lambda^2}\,d\mu(\lambda).
\end{aligned} \tag{19}$$

This immediately provides the stationary condition (15).

Notice that the third term of (17) cancels for $u = 0$. Moreover, when $u = 0$ and $k = k'$, Eq. (17) provides an alternative formula for the variance σ_p^2.

Under further hypothesis on the second order properties of X, one can get more precise information about the covariance of the process $d_{p,k}$. In the next sections, where we will assume that the process X is a fractional Brownian motion, we will deduce some stronger results (see Lemma 4.3) that will be used in order to carry out the analysis of the estimator.

3.2. **The log-regression MAVAR estimator.** Let n be the sample size, i.e. the number of the observations.

Definition 3.2. Let $\bar{p},\bar{\ell} \in \mathbb{N}$ such that $1 \leq \bar{p}(1+\bar{\ell}) \leq p_{\max}(n) = \lfloor \frac{n}{3} \rfloor$, and let $\underline{w} = (w_0,\dots,w_{\bar{\ell}})$ be a vector of weights satisfying the conditions

$$\sum_{\ell=0}^{\bar{\ell}} w_\ell = 0 \quad \text{and} \quad \sum_{\ell=0}^{\bar{\ell}} w_\ell \log(1+\ell) = 1\,. \tag{20}$$

The *log-regression MAVAR estimator* associated to the weights $\underline{w}$ is defined as

$$\widehat{\alpha}_n(\bar{p},\underline{w}) = \widehat{\alpha}_n(\tau_0,\bar{p},\underline{w}) := \sum_{\ell=0}^{\bar{\ell}} w_\ell \log\left(\widehat{\sigma}^2_{\bar{p}(1+\ell)}(n)\right)\,. \tag{21}$$

Roughly speaking, the idea behind this definition is to use the approximation

$$\left(\widehat{\sigma}^2_{\bar{p}}(n),\dots,\widehat{\sigma}^2_{\bar{p}(1+\bar{\ell})}(n)\right) \cong \left(\sigma^2_{\bar{p}}(n),\dots,\sigma^2_{\bar{p}(1+\bar{\ell})}(n)\right)$$

in order to get, by (14) and (20),

$$\widehat{\alpha}_n(\bar{p},\underline{w}) \cong \sum_{\ell=0}^{\bar{\ell}} w_\ell \log(\sigma^2_{\bar{p}(1+\ell)}) \cong \sum_{\ell=0}^{\bar{\ell}} w_\ell \left[\alpha\log(1+\ell) + \alpha\log(\tau_0\bar{p}) + \log(\sigma^2_H K(H))\right] = \alpha,$$

where $\alpha := 2H-2$. Thus, given the data $X_1,\dots,X_n$ the following procedure is used to estimate α:

- compute the modified Allan variance by (9), for integer values $\bar{p}(1+\ell)$, with $1 \leq \bar{p}(1+\ell) \leq p_{\max}(n)$;
- compute the weighted log-regression MAVAR estimator by (21) in order to get an estimate $\widehat{\alpha}$ of α;
- estimate H by $\widehat{H} = (\widehat{\alpha}+2)/2$.

In the sequel we will give, under suitable assumptions, two convergence results in order to justify these approximations and to get the rate of convergence of $\widehat{\alpha}_n(\bar{p},\underline{w})$ toward $\alpha = 2H-2$. Obviously, we need to take $\bar{p} = \bar{p}(n) \to +\infty$ as $n \to +\infty$ in order to reach jointly the above two approximations.

4. The asymptotic normality of the estimator

Since now on we will always assume that X is a fractional Brownian motion with Hurst index H so that the process $\{d_{p,k}\}_{p,k}$ is also *Gaussian*. Under this assumption and with the notation introduced before, we can state the following results.

Theorem 4.1. *Let $\bar{p} = \bar{p}(n)$ be a sequence of integers such that $\bar{p}(n) \to +\infty$ and $\bar{p}(n)n^{-1} \to 0$, and $\bar{\ell}$ a given integer. Let $\underline{\widehat{\sigma}}^2_n(\bar{p},\bar{\ell})$ be the vector $(\widehat{\sigma}^2_{\bar{p}}(n),\widehat{\sigma}^2_{2\bar{p}}(n),\dots,\widehat{\sigma}^2_{\bar{p}(1+\bar{\ell})}(n))$ and, analogously, set $\underline{\sigma}^2(\bar{p},\bar{\ell}) = (\sigma^2_{\bar{p}},\sigma^2_{2\bar{p}},\dots,\sigma^2_{\bar{p}(1+\bar{\ell})})$. Then it holds*

$$\sqrt{n_{\bar{p}}}(\tau_0\bar{p})^{2-2H}\left(\underline{\widehat{\sigma}}^2_n(\bar{p},\bar{\ell}) - \underline{\sigma}^2(\bar{p},\bar{\ell})\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,W(H))\,, \tag{22}$$

where the covariance matrix $W(H)$ has finite entries given by

$$W_{\ell',\ell}(H) = 2\sigma_H^4(1+\ell)^{4H-4}\left(\eta^2_{\ell',\ell}(H) + \sum_{q\in\mathbb{Z},\,q\neq 0} G^2_{\ell',\ell}(H,q)\right)\,, \quad \textit{for all } 0 \leq \ell' \leq \ell \leq \bar{\ell}\,, \tag{23}$$

with σ_H^2 given in (7) and the functions $\eta_{\ell',\ell}(H)$ and $G_{\ell',\ell}(H,q)$ defined in Lemma 4.4 by (32) and (33).

From this Theorem, as an application of the δ-method, we can state the following result.

Theorem 4.2. *Let $\widehat{\alpha}_n(\bar{p},\underline{w})$ be defined as in (21), for some finite integer $\bar{\ell}$ and a weight-vector $\underline{w}$ satisfying (20). If $\bar{p}=\bar{p}(n)$ is a sequence of integers such that $\bar{p}(n)n^{-1}\to 0$ and $\bar{p}(n)^2n^{-1}\to+\infty$, then*

$$\sqrt{n_{\bar{p}}}\left(\widehat{\alpha}_n(\bar{p},\underline{w})-\alpha\right)\xrightarrow[n\to\infty]{d}\mathcal{N}\left(0,\underline{w}_*^TV(H)\underline{w}_*\right)\,, \tag{24}$$

where $\alpha=2H-2$, the vector $\underline{w}_*$ is such that $[\underline{w}_*]_\ell:=w_\ell(1+\ell)^{2-2H}$, and $V(H)=(\sigma_H^2K(H))^{-2}W(H)$, with σ_H^2, $K(H)$ and $W(H)$ given in (7), (13) and (23) respectively.

Before starting the proof of the above theorems, we need the following results.

Lemma 4.3. *The process $d_{p,k}$ is stationary in k in the sense of (15). Moreover, for all $p,u\in\mathbb{R}^+$, $0\le u<p$, and $k,k'\in\mathbb{Z}$, the covariance is given by*

$$\mathrm{Cov}(d_{p,k},d_{p-u,k'})=\sigma_H^2\tau^{2H-2}G_H(k-k',p,u)\,, \tag{25}$$

where $\tau=\tau_0p$,

$$G_H(q,p,u)=\int L(p,u,\nu)g_H(q,p,u,\nu)\,d\nu \tag{26}$$

and $L(p,u,\nu)$ and $g_H(q,p,u,\nu)$ are some explicitly defined functions such that $G_H(q,p,0)=G_H(-q,p,0)$ and

$$G_H(q,p,u)=O(|q|^{2H-4})\qquad \textit{as } |q|\to+\infty. \tag{27}$$

Proof. As a special case of self-similar process with weakly stationary increments, the process X satisfies the stationary condition (15) and (17). Moreover, recalling (see formula (13) in [1]) that the spectral measure μ of a fractional Brownian motion with Hurst index H takes the explicit form $d\mu(\lambda)=f(\lambda)d\lambda$, with

$$f(\lambda)=\frac{\sigma_H^2\gamma_H}{|\lambda|^{2H-1}(1+\lambda^2)}\quad\text{and}\quad \gamma_H=\frac{\sin(\pi H)}{2\pi}\Gamma(1+2H)\,,$$

from Eq. (18), and after the change of variable $\nu=p\tau_0\lambda=\tau\lambda$, we get

$$\begin{aligned}r(q,h,p,u)&=\sigma_H^2\tau^{2H}\int e^{ih\nu/p}e^{iq\nu/p}\left(1-e^{-i(p-u)\nu/p}\right)^2\left(1-e^{i\nu}\right)^2\frac{\gamma_H}{|\nu|^{2H+1}}\,d\nu\\&=\sigma_H^2\tau^{2H}\int e^{ih\nu/p}g_H(q,p,u,\nu)d\nu\,,\end{aligned}$$

where

$$g_H(q,p,u,\nu):=e^{iq\nu/p}\left(1-e^{-i(p-u)\nu/p}\right)^2\left(1-e^{i\nu}\right)^2\frac{\gamma_H}{|\nu|^{2H+1}}.$$

Analogously, from (19), we get

$$\widehat{r}(q,h,p,u)=\sigma_H^2\tau^{2H}\int\cos(h\nu/p)g_H(q,p,u,\nu)d\nu\,.$$

With this notation, the covariance formula (17) provides the required Eq. (26) with

$$L(p,u,\nu):=\tfrac{1}{2(p-u)}+\tfrac{1}{(p-u)^2}\sum_{j=1}^{p-u-1}\sum_{h=1}^{j}\cos\left(\nu h/p\right)+\tfrac{1}{2(p-u)^2}\sum_{j=1}^{u}\sum_{h=j}^{j+p-u-1}e^{i\nu h/p}\,,$$

where the last term cancels for $u=0$.

To handle the function $L(p,u,\nu)$, that enters in the definition of the integral function $G_H(q,p,u)$, we can rewrite the last formula using the identities

$$e^{ih\nu/p}=\sum_{m=0}^{\infty}(ih\nu/p)^m\frac{1}{m!}\quad\text{and}\quad\cos(h\nu/p)=\sum_{m=0}^{\infty}(h\nu/p)^{2m}\frac{(-1)^m}{2m!}\,.$$

Rearranging the summations and using that, for each integer $q\geq 1$,

$$\sum_{h=1}^{\ell}h^q=\frac{(\ell+1)^{q+1}}{q+1}+\sum_{j=1}^{q}c_j(q)(\ell+1)^j\,, \tag{28}$$

where $c_j(q):=\frac{q!\,B_{q+1-j}}{(q+1-j)!j!}$ and the B_k's are the Bernoulli numbers, we get

$$\begin{aligned}
L(p,u,\nu)&=\tfrac{p}{2(p-u)}-\left(\tfrac{p}{p-u}\right)^2\tfrac{1}{\nu^2}\left(\cos\left(\tfrac{\nu(p-u+1)}{p}\right)-\sum_{m=0}^{1}\tfrac{(-1)^m}{2m!}\left(\tfrac{\nu(p-u+1)}{p}\right)^{2m}\right)\\
&+\tfrac12 I_{\{u\neq0\}}\left(\tfrac{p}{p-u}\right)^2\frac{1}{\nu^2}\left(e^{i(u+1)\nu/p}+e^{i(p-u)\nu/p}-e^{i(p+1)\nu/p}\right)\\
&-\tfrac12 I_{\{u\neq0\}}\left(\tfrac{p}{p-u}\right)^2\frac{1}{\nu^2}\sum_{m=0}^{2}\tfrac{i\nu^m}{m!}\left(\left(\tfrac{u+1}{p}\right)^m+\left(\tfrac{p-u}{p}\right)^m-\left(\tfrac{p+1}{p}\right)^m\right)\\
&+\tfrac1p\sum_{m=1}^{\infty}\tfrac{(i\nu)^m}{m!}f_m(p,u)\,,
\end{aligned}\tag{29}$$

where $f_m(p,u)$ is a uniformly bounded function in p and u, such that $f_m(p,0)=0$ if m is odd.

It is easy to check that $G_H(q,p,0)=G_H(-q,p,0)$. In order to study the asymptotic behavior of $G_H(q,p,u)$ as $|q|\to+\infty$, we make the change of variable $\xi=\nu q$, and get

$$G_H(q,p,u)=|q|^{-1}\int L(p,u,\xi/q)g_H(q,p,u,\xi/q)\,d\xi\,.$$

From the definition of $L(p,u,\nu)$ and $g_H(q,p,u,\nu)$ it turns out, as can be easily verified, that $G_H(q,p,u)=O(|q|^{2H-4})$ as $|q|\to+\infty$. □

Lemma 4.4. *Let $\bar p=\bar p(n)$ be a sequence of integers such that $\bar p(n)\to+\infty$ and $\bar p(n)n^{-1}\to0$. For two given integers ℓ,ℓ', with $0\leq\ell'\leq\ell$, set $p_\ell=\bar p(n)(\ell+1)$ and $p_{\ell'}=\bar p(n)(\ell'+1)$. Then*

$$n_{\bar p}(\tau_0\bar p)^{4-4H}\mathrm{Cov}\big(\widehat\sigma^2_{p_\ell}(n),\widehat\sigma^2_{p_{\ell'}}(n)\big)\underset{n\to+\infty}{\longrightarrow}W_{\ell',\ell}(H)\,, \tag{30}$$

where $W_{\ell',\ell}(H)$ is a finite quantity given by

$$W_{\ell',\ell}(H):=2\sigma_H^4(1+\ell)^{4H-4}\left(\eta^2_{\ell',\ell}(H)+\sum_{q\in\mathbb{Z},\,q\neq0}G^2_{\ell',\ell}(H,q)\right)\,, \tag{31}$$

with

$$\eta_{\ell',\ell}(H):=\int L_{\ell',\ell}(\nu)\left(1-e^{-i\frac{1+\ell'}{1+\ell}\nu}\right)^2\left(1-e^{i\nu}\right)^2\tfrac{\gamma_H}{|\nu|^{2H+1}}\,d\nu\,, \tag{32}$$

$$G_{\ell',\ell}(H,q):=q^{2H}\int L_{\ell',\ell}(\xi/q)\left(1-e^{-i\frac{1+\ell'}{1+\ell}\frac{\xi}{q}}\right)^2\left(1-e^{i\xi/q}\right)^2\tfrac{\gamma_H}{|\xi|^{2H+1}}\,d\xi\quad\textit{for } q\in\mathbb{Z}\setminus\{0\} \tag{33}$$

and

$$
\begin{aligned}
L_{\ell',\ell}(x) &:= \lim_{n\to+\infty} L(p_\ell, p_\ell - p_{\ell'}, x)\\
&= \tfrac{1+\ell}{2(1+\ell')} - \left(\tfrac{1+\ell}{1+\ell'}\right)^2 \tfrac{1}{x^2}\left(\cos\left(\tfrac{(1+\ell')x}{1+\ell}\right) - \sum_{m=0}^{1} \tfrac{(-1)^m}{2m!}\left(\tfrac{(1+\ell')x}{1+\ell}\right)^{2m}\right)\\
&+ \tfrac{1}{2} I_{\{\ell\neq\ell'\}}\left(\tfrac{1+\ell}{1+\ell'}\right)^2 \frac{1}{x^2}\left(e^{\frac{\ell-\ell'}{1+\ell}ix} + e^{\frac{1+\ell'}{1+\ell}ix} - e^{ix}\right)\\
&- \tfrac{1}{2} I_{\{\ell\neq\ell'\}}\left(\tfrac{1+\ell}{1+\ell'}\right)^2 \frac{1}{x^2}\sum_{m=0}^{2} \tfrac{ix^m}{m!}\left(\left(\tfrac{\ell-\ell'}{1+\ell}\right)^m + \left(\tfrac{1+\ell'}{1+\ell}\right)^m - 1\right).
\end{aligned}
\tag{34}
$$

Note that $G_{\ell,\ell}(H,q) = G_{\ell,\ell}(H,-q)$.

Proof. Since $n/\bar p \to +\infty$, without loss of generality we can assume that $\bar p(1+\bar\ell) \le p_{\max}(n)$ for each n. Let us set $n_\ell = n_{p_\ell}$ and $n_{\ell'} = n_{p_{\ell'}}$. From the definition of the empirical variance and applying the Wick's rule for jointly Gaussian random variable (see the appendix), we get

$$
\begin{aligned}
\mathrm{Cov}(\widehat\sigma^2_{p_\ell}(n), \widehat\sigma^2_{p_{\ell'}}(n)) &= \frac{2}{n_\ell n_{\ell'}} \sum_{k=0}^{n_\ell-1}\sum_{h=0}^{n_{\ell'}-1} \mathrm{Cov}(d_{p_\ell,k}, d_{p_{\ell'},h})^2\\
&= \frac{2}{n_\ell}\mathrm{Cov}(d_{p_\ell,k}, d_{p_{\ell'},k})^2 + \frac{2}{n_\ell n_{\ell'}}\sum_{k=0}^{n_\ell-1}\sum_{h=0,\,h\neq k}^{n_\ell-1} \mathrm{Cov}(d_{p_\ell,k}, d_{p_{\ell'},h})^2\\
&+ \frac{2}{n_\ell n_{\ell'}}\sum_{k=0}^{n_\ell-1}\sum_{h=n_\ell}^{n_{\ell'}-1} \mathrm{Cov}(d_{p_\ell,k}, d_{p_{\ell'},h})^2 .
\end{aligned}
$$

We consider the three terms on the last line separately.

By Lemma 4.3 the first term, multiplied by $n_{\bar p}(\tau_0 \bar p)^{4-4H}$, is equal to

$$
\frac{n_{\bar p}}{n_\ell}(1+\ell)^{4H-4}\sigma_H^4 G_H^2(0, p_\ell, u_{\ell'\ell}),
$$

where we set $u_{\ell'\ell} = p_\ell - p_{\ell'}$. Taking the limit for $n\to\infty$, we get

$$
\lim_n n_{\bar p}(\tau_0\bar p)^{4-4H}\frac{2}{n_\ell}\mathrm{Cov}(d_{p_\ell,k}, d_{p_{\ell'},k})^2 = 2\sigma_H^4(1+\ell)^{4H-4}\eta^2_{\ell',\ell},
$$

with $\eta_{\ell',\ell}$ given in (32).

The second term, multiplied by $n_{\bar p}(\tau_0\bar p)^{4-4H}$, can be rewritten using the function $G_{\ell',\ell}(H,q)$ defined in (33) as

$$
\begin{aligned}
&2\sigma_H^4\left(\frac{\bar p}{p_\ell}\right)^{4-4H}\frac{n_{\bar p}}{n_{\ell'}}\frac{1}{n_\ell}\sum_{k=1}^{n_\ell-1}\sum_{q=1}^{k} G^2_{\ell',\ell}(H,q) + G^2_{\ell',\ell}(H,-q)\\
&+2\sigma_H^4\left(\frac{\bar p}{p_\ell}\right)^{4-4H}\frac{n_{\bar p}}{n_{\ell'}}\sum_{k=1}^{n_\ell-1}\sum_{q=1}^{k} G_H^2(q,p_\ell,u_{\ell'\ell}) - G^2_{\ell',\ell}(H,q)\\
&+2\sigma_H^4\left(\frac{\bar p}{p_\ell}\right)^{4-4H}\frac{n_{\bar p}}{n_{\ell'}}\sum_{k=1}^{n_\ell-1}\sum_{q=1}^{k} G_H^2(-q,p_\ell,u_{\ell'\ell}) - G^2_{\ell',\ell}(H,-q).
\end{aligned}
$$

By Cesaro's lemma, the first term converges to $4\sigma_H^4(1+\ell)^{4H-4}\sum_{q\in\mathbb{Z},\,q\neq0} G^2_{\ell',\ell}(H,q)$, which is finite since $G^2_{\ell',\ell}(H,q) = O(q^{4H-8})$. The second term converges to zero applying once more Cesaro's lemma and noting that

$$\begin{aligned}|G_H^2(q,p_\ell,u_{\ell'\ell})-G_{\ell,\ell'}^2(H,q)| &\leq |G_H(q,p_\ell,u_{\ell'\ell})-G_{\ell,\ell'}(H,q)||G_H(q,p_\ell,u_{\ell'\ell})+G_{\ell,\ell'}(H,q)|\\ &\leq (\bar p)^{-1}O(q^{4H-8}).\end{aligned}$$

Indeed, $G_H(q,p_\ell,u_{\ell'\ell})+G_{\ell,\ell'}(H,q)=O(q^{2H-4})$, while the difference $G_H(q,p_\ell,u_{\ell'\ell})-G_{\ell,\ell'}(H,q)$ is equal to

$$q^{2H}\int\Big(L(p_\ell,u_{\ell'\ell},\xi/q)e^{i\xi/p_\ell}-L_{\ell',\ell}(\xi/q)\Big)\Big(1-e^{-i\frac{1+\ell'}{1+\ell}\frac{\xi}{q}}\Big)^2\Big(1-e^{i\frac{\xi}{q}}\Big)^2\tfrac{\gamma_H}{|\xi|^{2H+1}}\,d\xi$$

where the difference $L(p_\ell,u_{\ell'\ell},\xi/q)e^{i\xi/p_\ell}-L_{\ell',\ell}(\xi/q)$ is equal to

$$\Big(e^{i\xi/p_\ell}-1\Big)L(p_\ell,u_{\ell'\ell},\xi/q)+L(p_\ell,u_{\ell'\ell},\xi/q)-L_{\ell',\ell}(\xi/q)=O(\xi/p_\ell).$$

The third term, multiplied by $n_{\bar p}(\tau_0\bar p)^{4-4H}$, converges to zero. This can be shown using Lemma 4.3 to rewrite the covariance, and then rearranging the summations as follows:

$$\begin{aligned}&\sum_{h=n_\ell}^{n_{\ell'}-1}\sum_{k=0}^{n_\ell-1}G_H^2(-(h-k),p_\ell,u_{\ell'\ell})=\sum_{h'=1}^{3u_{\ell'\ell}}\sum_{k'=0}^{n_\ell-1}G_H^2(-(k'+h'),p_\ell,u_{\ell'\ell})\\ &=\sum_{k'=0}^{n_\ell-1}\sum_{r=k'+1}^{k'+3u_{\ell'\ell}}G_H^2(-r,p_\ell,u_{\ell'\ell})=\sum_{k'=0}^{n_\ell-1}\sum_{r=1}^{k'+3u_{\ell'\ell}}G_H^2(-r,p_\ell,u_{\ell'\ell})-\sum_{k'=0}^{n_\ell-1}\sum_{r=1}^{k'}G_H^2(-r,p_\ell,u_{\ell'\ell})\\ &=\sum_{\widehat k=0}^{n_{\ell'}-1}\sum_{r=1}^{\widehat k}G_H^2(-r,p_\ell,u_{\ell'\ell})-\sum_{\widehat k=0}^{3u_{\ell'\ell}-1}\sum_{r=1}^{\widehat k}G_H^2(-r,p_\ell,u_{\ell'\ell})-\sum_{k'=0}^{n_\ell-1}\sum_{r=1}^{k'}G_H^2(-r,p_\ell,u_{\ell'\ell})\end{aligned}$$

The first and the third term of this equation, divided respectively by $n_{\ell'}$ and n_ℓ, converge to the same limit $\sum_{q=1}^\infty G_{\ell',\ell}^2(H,q)$ and thus cancel. The contribution from the second term is also zero in the limit. Indeed, from the Cesaro's lemma, we get

$$2\lim_n\left(\frac{\bar p}{p_\ell}\right)^{4-4H}\frac{n_{\bar p}u_{\ell'\ell}}{n_{\ell'}n_\ell}\cdot\frac{1}{u_{\ell'\ell}}\sum_{\widehat k=0}^{3u_{\ell'\ell}-1}\sum_{r=1}^{\widehat k}G_H^2(-r,p_\ell,u_{\ell'\ell})=0\,,$$

since $\lim_n\frac{n_{\bar p}u_{\ell'\ell}}{n_{\ell'}n_\ell}=0$ while the remaining part converges to a finite quantity. This concludes the proof of the lemma. □

Proof of Theorem 4.1. As before, without loss of generality, we can assume that $\bar p(1+\bar\ell)\leq p_{\max}(n)$ for each n. Moreover, set again $n_\ell=n_{p_\ell}$ and $n_{\ell'}=n_{p_{\ell'}}$. For a given real vector $\underline{v}^T=(v_0,\dots,v_{\bar\ell})$, let us consider the random variable $T_n=T(\bar p(n),\bar\ell,\underline{v})$ defined as a linear combination of the empirical variances $\widehat\sigma_{\bar p}^2(n),\dots,\widehat\sigma_{\bar p(1+\bar\ell)}^2(n)$ as follows

$$T_n:=\sum_{\ell=0}^{\bar\ell}v_\ell\widehat\sigma_{\bar p(1+\ell)}^2(n)=\sum_{\ell=0}^{\bar\ell}\frac{v_\ell}{n_\ell}\sum_{k=0}^{n_\ell-1}d_{\bar p(1+\ell),k}^2\,.$$

In order to prove the convergence stated in Theorem (4.1), we have to show that the random variable $\sqrt{n_{\bar p}}\,(\tau_0\bar p)^{2-2H}\Big(T_n-\sum_{\ell=0}^{\bar\ell}v_\ell\sigma_{\bar p(1+\ell)}^2\Big)$ converges to the normal distribution with zero mean and variance $\underline{v}^TW(H)\underline{v}$. To this purpose, we note that

$$\sqrt{n_{\bar p}}\,(\tau_0\bar p)^{2-2H}\left(T_n-\sum_{\ell=0}^{\bar\ell}v_\ell\sigma_{\bar p(1+\ell)}^2\right)=V_n^TA_nV_n-\mathbb{E}[V_n^TA_nV_n]$$

where V_n is the random vector with entries $d_{\bar{p}(1+\ell),k}$, $0 \le \ell \le \bar{\ell}$, $0 \le k \le n_\ell - 1$, and A_n is the diagonal matrix with entries

$$[A_n]_{\left(\bar{p}(1+\ell),k\right),\left(\bar{p}(1+\ell),k\right)} = \frac{\sqrt{n_{\bar{p}}}(\tau_0\bar{p})^{2-2H}v_\ell}{n_\ell} = O((\tau_0\bar{p})^{2-2H}n^{-1/2}).$$

Therefore, condition (1) of Lemma A.1 is satisfied since, by Lemma 4.4,

$$\mathrm{Var}[V_n^T A_n V_n] = n_{\bar{p}}\,(\tau_0\bar{p})^{4-4H}\sum_{\ell=0}^{\bar{\ell}}\sum_{\ell'=0}^{\bar{\ell}} v_\ell v_{\ell'}\mathrm{Cov}\big(\widehat{\sigma}^2_{\bar{p}(1+\ell)}(n), \widehat{\sigma}^2_{\bar{p}(1+\ell)}(n)\big) =$$

$$n_{\bar{p}}\,(\tau_0\bar{p})^{4-4H}\left(\sum_{\ell=0}^{\bar{\ell}} v_\ell^2\mathrm{Cov}\big(\widehat{\sigma}^2_{\bar{p}(1+\ell)}(n), \widehat{\sigma}^2_{\bar{p}(1+\ell)}(n)\big) + 2\sum_{\ell=1}^{\bar{\ell}}\sum_{\ell'=0}^{\ell} v_\ell v_{\ell'}\mathrm{Cov}\big(\widehat{\sigma}^2_{\bar{p}(1+\ell)}(n), \widehat{\sigma}^2_{\bar{p}(1+\ell')}(n)\big)\right)$$

$$\underset{n\to+\infty}{\longrightarrow} \sum_{\ell=0}^{\bar{\ell}} v_\ell^2 W_{\ell,\ell}(H) + 2\sum_{\ell=1}^{\bar{\ell}}\sum_{\ell'=0}^{\ell} v_\ell v_{\ell'} W_{\ell',\ell}(H) = \underline{v}^T W(H)\underline{v}.$$

Moreover, condition (2) of Lemma A.1 is verified. Indeed, if C_n is the covariance matrix of the random vector V_n and $\rho[C_n]$ denotes its spectral radius, then, by Lemma A.2, we have

$$\rho[C_n] \le \sum_{\ell=0}^{\bar{\ell}} \rho\big[C_n\big(\bar{p}(1+\ell)\big)\big],$$

where $C_n\big(\bar{p}(1+\ell)\big)$ is the covariance matrix of the subvector $[d_{\bar{p}(1+\ell),0}, \dots, d_{\bar{p}(1+\ell),n_\ell-1}]^T$. By Lemma 4.3 and (40), we then have

$$\begin{aligned}
\rho\big[C_n\big(\bar{p}(1+\ell)\big)\big] &\le \sigma_H^2[\tau_0(1+\ell)\bar{p}]^{2H-2}\left[G_H(0,\bar{p}(1+\ell),0) + 2\sum_{q=1}^{n_\ell-1} G_H(q,\bar{p}(1+\ell),0)\right]\\
&= \sigma_H^2[\tau_0(1+\ell)\bar{p}]^{2H-2}\left[G_H(0,\bar{p}(1+\ell),0) + 2\sum_{q=1}^{n_\ell-1} G_{\ell,\ell}(H,q)\right]\\
&\quad + 2\sigma_H^2[\tau_0(1+\ell)\bar{p}]^{2H-2}\sum_{q=1}^{n_\ell-1}\left(G_H(q,\bar{p}(1+\ell),0) - G_{\ell,\ell}(H,q)\right)\\
&= O\big((\tau_0\bar{p})^{2H-2}\big)\,,
\end{aligned}$$

where in the last step we used the convergence of the sequence $G_H(0,\bar{p}(1+\ell),0)$ and of the series $\sum_{q=0}^{+\infty} G_{\ell,\ell}(H,q)$, together with the inequality

$$|G_H(q,\bar{p}(1+\ell),0) - G_{\ell,\ell}(H,q)| \le (\bar{p})^{-1}O(q^{2H-4})\,.$$

□

Proof of Theorem 4.2. By the assumptions on the sequence $\bar{p} = \bar{p}(n)$ and inequality (14), it holds

$$\sqrt{n_{\bar{p}}}(\tau_0\bar{p})^{2-2H}\left|\sigma^2_{\bar{p}(1+\ell)} - \sigma_H^2[\tau_0\bar{p}(1+\ell)]^{2H-2}K(H)\right| \le \sigma_H^2(1+\ell)^{2H-2}\sqrt{n_{\bar{p}}}O_H(1/\bar{p}) \underset{n\to+\infty}{\longrightarrow} 0\,.$$

Thus, from Theorem 4.1 we get

$$\sqrt{n_{\bar{p}}}\left[(\tau_0\bar{p})^{2-2H}\underline{\widehat{\sigma}}_n^2(\bar{p},\bar{\ell}) - \underline{\sigma}^2_*\right] \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}(0, W(H))\,, \tag{35}$$

where $\underline{\sigma}^2_*$ is the vector with elements

$$[\underline{\sigma}^2_*]_\ell := \sigma_H^2(1+\ell)^{2H-2}K(H) \quad \text{for } 0 \le \ell \le \bar{\ell}.$$

Now we observe that if $f(\underline{x}) := \sum_{\ell=0}^{\bar{\ell}} w_\ell \log(x_\ell)$, then, by (21) and (20), we have

$$\widehat{\alpha}_n(\bar{p}, \underline{w}) = f(\underline{\widehat{\sigma}}^2(\bar{p}, \bar{\ell})) = f\big((\tau_0 \bar{p})^{2-2H} \underline{\widehat{\sigma}}^2(\bar{p}, \bar{\ell})\big).$$

Moreover, $\alpha = f(\underline{\sigma}_*^2)$ and $\nabla f(\underline{\sigma}_*^2) = (\sigma_H^2 K(H))^{-1} w^*$. Therefore, by the application of the δ-method, the convergence (35) entails the convergence

$$\sqrt{n_{\bar{p}}}\,(\widehat{\alpha}_n(\bar{p}, \underline{w}) - \alpha) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \nabla f(\underline{\sigma}_*^2) W(H) \nabla f(\underline{\sigma}_*^2)\right) ,$$

where

$$\nabla f(\underline{\sigma}_*^2) W(H) \nabla f(\underline{\sigma}_*^2) = \underline{w}_*^T V(H) \underline{w}_* ,$$

and thus concludes the proof. □

5. Some comments

5.1. **The modified Allan variance and the wavelet estimators.** Let $X = \{X(t),\, t \in \mathbb{R}\}$ be a self-similar process with weakly stationary increments and consider the generalized process $Y = \{Y(t),\, t \in \mathbb{R}\}$ defined through the set of identities

$$\int_{t_1}^{t_2} Y(t)dt = X(t_2) - X(t_1)\,, \qquad \forall t_1, t_2 \in \mathbb{R}\,.$$

In short, we write $Y = \dot{X}$. From this definition and with the notation introduced in Sec. 3, we can rewrite the MAVAR as

$$\sigma^2(\tau_0, p) = \frac{1}{2\tau_0^2 p^2} \mathbb{E}\left[\left(\frac{1}{p}\sum_{i=1}^{p}\left(\int_{t_{i+p}}^{t_{i+2p}} Y(t)dt - \int_{t_i}^{t_{i+p}} Y(t)dt\right)\right)^2\right] \tag{36}$$

and its related estimator as

$$\widehat{\sigma}^2(\tau_0, p, n) = \frac{1}{2\tau_0^2 p^2 n_p} \sum_{k=1}^{n_p}\left[\frac{1}{p}\sum_{i=1}^{p}\left(\int_{t_{i+k+p}}^{t_{i+k+2p}} Y(t)dt - \int_{t_{i+k}}^{t_{i+k+p}} Y(t)dt\right)\right]^2 . \tag{37}$$

Now we claim that, for p fixed, the quantity

$$d(\tau_0, p, k) := \frac{1}{\sqrt{2}p^2\tau_0} \sum_{i=1}^{p}\left(\int_{t_{i+k+p}}^{t_{i+k+2p}} Y(t)dt - \int_{t_{i+k}}^{t_{i+k+p}} Y(t)dt\right)$$

can be set in correspondence with a family of discrete wavelet transforms of the process Y, indexed by τ_0 and k. To see that, let us fix $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, and set $\tau_0 = 2^j$ and $t_1 = 2^j$, so that $t_{i+k} = 2^j(i+k)$, for all $i \in \mathbb{N}$. With this choice on the sequence of times, it is not difficult to construct a wavelet $\psi(s)$ such that

$$d_{k,j} := d(2^j, p, k) = \langle Y; \psi_{k,j}\rangle \quad \text{with} \quad \psi_{k,j}(s) = 2^{-j}\psi(2^{-j}s - k)\,. \tag{38}$$

An easy check shows that the function

$$\psi(s) := \sum_{i=1}^{p} \psi^i(s)\,, \qquad \psi^i(s) := \frac{1}{\sqrt{2}p^2}\left(\mathrm{I}_{[i+p, i+2p]}(s) - \mathrm{I}_{[i,i+p]}(s)\right) ,$$

is a proper wavelet satisfying Eq. (38). Notice also that the components ψ^i, $i = 1, \ldots p$, of the mother wavelet, are suitably translated and re-normalized Haar functions.

In the case $p = 1$, corresponding to the classical Allan variance, the mother wavelet is exactly given by the Haar function, as was already pointed out in [4].

Though the wavelet representation could be convenient in many respects, the Haar mother wavelet does not satisfy one of the conditions which are usually required in order to study the convergence of the estimator (see condition (W2) in [18]). Moreover, there is a fundamental difference between the two methods: in the wavelet setting the log-regression is done over the scale parameter τ_0 for p fixed, while the MAVAR log-regression given in (21) is performed over p and for τ_0 fixed.

5.2. **The modified Hadamard variance.** Further generalizing the notion of the MAVAR, one can define the modified Hadamard variance (MHVAR): For fixed integers p, Q and $\tau_0 \in \mathbb{R}$, set

$$\sigma^2(\tau_0, p, Q) := \frac{1}{Q!\tau_0^2 p^2}\mathbb{E}\left[\left(\frac{1}{p}\sum_{i=1}^{p}\sum_{q=0}^{Q} c(Q,q)X_{t_{i+qp}}\right)^2\right],$$

where $c(Q,q) = (-1)^q \frac{Q!}{q!(Q-q)!}$. Notice that for $Q = 2$ we recover the modified Allan variance. The MHVAR is again a time-domain quantity which has been introduced in [8] for the analysis of the network traffic. A standard estimator for this variance, is given by

$$\begin{aligned}\widehat{\sigma}_n^2(\tau_0, p, Q) &:= \frac{1}{Q!\tau_0^2 p^4 n_{Q,p}}\sum_{h=1}^{n_{Q,p}}\left(\sum_{i'=h}^{p+h-1}\sum_{q=0}^{Q} c(Q,q)X_{i'+qp}\right)^2 \\ &= \frac{1}{n_{Q,p}}\sum_{k=0}^{n_{Q,p}-1}\left(\frac{1}{\sqrt{Q!}\tau_0 p^2}\sum_{i=1}^{p}\sum_{q=0}^{Q} c(Q,q)X_{k+i+qp}\right)^2\end{aligned}$$

for $p = 1, \ldots, [n/(Q+1)]$ and $n_{Q,p} = n - (Q+1)p + 1$.

Similarly to the analysis performed for the MAVAR, let us set

$$d_{p,k} = d(\tau_0, p, Q, k) := \frac{1}{\sqrt{Q!}\tau_0 p^2}\sum_{i=1}^{p}\sum_{q=0}^{Q} c(Q,q)X_{k+i+qp}$$

so that we can write

$$\sigma^2(\tau_0, p, Q) = \mathbb{E}\left[d_{p,0}^2\right] \qquad \text{and} \qquad \widehat{\sigma}_n^2(\tau_0, p, Q) = \frac{1}{n_{Q,p}}\sum_{k=0}^{n_{Q,p}-1} d_{p,k}^2.$$

This suggests that the convergence results, similar to Theorems 4.1 and 4.2, can be achieved also for the MHVAR and its related log-regression estimator.

5.3. **The case of stationary processes.** In applications, MAVAR and MHVAR are also used in order to estimate the memory parameter of long-range dependent processes. This general case is not included in our analysis (which is restricted to the fractional Brownian motion) and it requires a more involved investigation. To our knowledge, there are no theoretical results along this direction.

Appendix A.

In this appendix we recall the Wick's rule for jointly Gaussian random variables and some facts used in the above proofs.

Wick's rule. Let us consider a family $\{Z_i\}$ of jointly Gaussian random variables with zero-mean. The Wick's rule is a formula that provides an easy way to compute the quantity $\mathbb{E}(Z_\Lambda) := \mathbb{E}\left(\prod_{i\in\Lambda} Z_i\right)$, for any index-set Λ (see, e.g. [12]).

Since the Z_i's are zero-mean random variables, if Λ has odd cardinality we trivially get $\mathbb{E}(Z_\Lambda) = 0$. We then assume that $|\Lambda| = 2k$, for some $k \geq 1$. To recall the Wick's rule, it is convenient to introduce the following graph representation. To the given index-set Λ we associated a vertex-set V indexed by the distinct elements of Λ, and to every vertex $j \in V$ we attached as many half-edges as many times the index j appears in Λ. In particular there is a bi-univocal correspondence between the set of half-edges and Λ, while $|V| \leq |\Lambda|$. Gluing together two half-edges attached to vertices i and j, we obtain the edge (i, j). Performing this operation recursively over all remaining half-edges, we end-up with a graph G, with vertex set $V(G)$ and edge-set $E(G)$. Let $\mathcal{G}_\Lambda$ denote the set of graphs (possibly not distinguishable) obtained by performing this "gluing procedure" in all possible ways. Notice that the set $\mathcal{G}_\Lambda$ is in bi-univocal correspondence with the set of pairings with ordered pairs of the elements of Λ.

With this notation, and for all index-sets Λ with even cardinality, the *Wick's rule* for a family $\{Z_i\}$ of jointly centered Gaussian random variables, provides the identity

$$\mathbb{E}(Z_\Lambda) = \sum_{G \in \mathcal{G}_\Lambda} \prod_{(i,j) \in E(G)} \mathbb{E}(Z_i Z_j) . \tag{39}$$

Now we recall some facts used in the proof of Theorem 4.1.
Denote by $\rho[A]$ the spectral radius of a matrix $A = \{a_{i,j}\}_{1 \leq i,j \leq n}$, then

$$\rho[A] \leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{i,j}| . \tag{40}$$

Moreover the following lemmas hold.

Lemma A.1. *(*[17]*)*
Let (V_n) be a sequence of centered Gaussian random vectors and denote by C_n its covariance matrix. Let (A_n) be a sequence of deterministic symmetric matrices such that

(1) $\lim_{n \to +\infty} \mathrm{Var}[V_n^T A_n V_n] = \lambda^2 \in [0, +\infty)$
(2) $\lim_{n \to +\infty} \rho[A_n] \rho[C_n] = 0.$

Then $V_n^T A_n V_n - \mathbb{E}[V_n^T A_n V_n]$ converges in distribution to the normal law $\mathcal{N}(0, \lambda^2)$.

Lemma A.2. *(*[15]*)*
Let $m \geq 2$ be an integer and C be a $m \times m$ covariance matrix. Let r be an integer such that $1 \leq r \leq m-1$. Denote by C_1 the top left submatrix with size $r \times r$ and by C_2 the bottom right submatrix with size $(m-r) \times (m-r)$, i.e.

$$C_1 = [C_{i,j}]_{1 \leq i,\, j \leq r} \qquad \textit{and} \qquad C_2 = [C_{i,j}]_{r+1 \leq i,\, j \leq m}$$

Then $\rho[C] \leq \rho[C_1] + \rho[C_2]$.

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References

[1] Albeverio, S., Jorgensen, P.E.T. and Paolucci, A.M. (2010). On Fractional Brownian Motion and Wavelets. *Complex Anal. Oper. Theory*.

[2] Allan, D. W. and Barnes, J. A. (1981). A Modified Allan Variance with Increased Oscillator Characterization Ability. *Proc. 35th Annual Frequency Control Symposium*.

[3] Abry, P., Flandrin, P., Taqqu, M.S. and Veitch, D. (2000). Wavelets for the analysis, estimation and synthesis of scaling data. *Park, K., Willinger, W. (Eds.), Self-Similar Network Traffic and Performance Evaluation*, 39–88, *Wiley (Interscience Division), New York*.

[4] Abry, P. and Veitch, D.(1998). Wavelet analysis of long-range-dependent traffic. *IEEE Trans. Inform. Theory* **44** , no. 1, 2–15.

[5] Beran J. (1994). Statistics for Long Memory Processes. Chapman and Hall, London.

[6] Bernier L. G. (1987). Theoretical Analysis of the Modified Allan Variance. *Proc. 41st Annual Frequency Control Symposium*.

[7] Bregni, S. (2002). Characterization and Modelling of Clocks, in *Synchronization of Digital Telecommunications Networks*, John Wiley & Sons.

[8] BREGNI, S. and JMODA, L. (2008). Accurate Estimation of the Hurst Parameter of Long-Range Dependent Traffic Using Modified Allan and Hadamard Variances. *IEEE Transactions on Communications* **56**, No. 11, 1900–1906.

[9] BREGNI, S. and ERANGOLI, W. (2005). Fractional noise in experimental measurements of IP traffic in a metropolitan area network. *Proc. IEEE GLOBECOM*.

[10] BREGNI, S. and PRIMERANO, L. (2004). The modified Allan variance as timedomain analysis tool for estimating the Hurst parameter of long-range dependent traffic. *Proc. IEEE GLOBECOM*.

[11] FLANDRIN, P. (1989). On the spectrum of fractional Brownian motions. *IEEE Trans. Inform. Theory* **35**, no. 1, 197–199.

[12] GLIMM, J. and JAFFE A (1987). Quantum physics. A functional integral point of view. Second edition. *Springer-Verlag, New York*.

[13] MANDELBROT, B. B. and VAN NESS, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10**, 422-437.

[14] MASRY, E. (1993). The wavelet transform of stochastic processes with stationary increments and its application to fractional Brownian motion. *IEEE Trans. Inform. Theory* **39**, no. 1, 260–264.

[15] MOULINES, E. , ROUEFF, F. and TAQQU, M.S. (2007). Central limit theorem for the log-regression wavelet estimation of the memory parameter in the Gaussian semi-parametric context. *Fractals* **15**, no. 4, 301–313.

[16] MOULINES, E., ROUEFF, F. and TAQQU, M.S. (2007). On the spectral density of the wavelet coefficients of long-memory time series with application to the log-regression estimation of the memory parameter. *J. Time Ser. Anal.* **28**, no. 2, 155–187. MR2345656

[17] MOULINES, E. , ROUEFF, F. and TAQQU, M.S. (2008). A wavelet Whittle estimator of the memory parameter of a nonstationary Gaussian time series. *Ann. Statist.* **36**, no. 4, 1925–1956.

[18] ROUEFF, F. and TAQQU, M.S. (2009). Asymptotic normality of wavelet estimators of the memory parameter for linear processes. *J. Time Series Anal.* **30**, no. 5, 534–558. MR2560417

[19] TEWFIK, A. H. and KIM, M. (1992). Correlations structure of the discrete wavelet coefficients of fractional Brownian motion. *IEEE Trans. Inform. Theory* **38**, no. 2, part 2, 904–909.

[20] YAGLOM, A. M. (1987). Correlation theory of stationary and related random functions. Vol. I. Basic results. *Springer Series in Statistics. Springer-Verlag, New York*.

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