Loading and Unloading of Thick-Walled Cylindrical Pressure Vessels of Strain-Hardening Material

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A thick-walled closed-end cylinder of isotropic, homogeneous and strain-hardening material is considered in this study. Loading is assumed to consist of a temperature gradient as well as an internal pressure. Unloading is completely elastic without considering a Bauschinger effect. A generalized plane strain case in which the material obeys Von Mises yield criterion is studied. Using the yield criterion, critical conditions for a wide range of loading combinations and thickness ratios are investigated. After the critical condition is established, load is increased beyond the critical values and calculations are made for plastic stresses and strains and progress of plastic zone using an incremental theory of plasticity. Residual stresses are obtained as the cylinder is unloaded from a 25 and 50-percent overstrained condition. Reverse yielding is not considered while the residual stresses at the onset of reverse yielding are calculated. Loading function is assumed to follow the stress-strain curve of SUS 304 at a constant temperature of 400°C, which is selected from the experimental work of earlier researchers.

Introduction

Thermoelastoplastic and residual stress analysis in thick-walled cylindrical pressure vessels are important in two major aspects of design: design for strength and design for fracture. There are some closed-form solutions for elastoplastic and thermoelastoplastic stress distributions in cylindrical pressure vessels available in literature. For a perfectly plastic material which obeys Tresca's criterion, Hill (1950) established a closed-form solution for radial and tangential stress components. He also introduced a numerical procedure based on Reuss's equation to calculate axial stress. Bland (1956), using Tresca's criterion, has established stress and displacement equations for a tube of a linear-hardening material subjected to pressure and steady-state heat flow. Whalley (1956) considered elastic and plastic behavior of thick-walled cylinders of brittle and perfectly plastic material subjected to internal and external pressure along with an arbitrary temperature distribution. There are also closed-form solutions and simulations for residual stress distributions available in the literature. Chen (1986) proposed a theoretical model for high-strength steel based on the experimental results of Milligan et al. (1966). The model used a perfectly plastic loading condition and a linear hardening with the unloading function including Bauschinger effect. Using this model, a closed-form solution for calculating residual stresses and strains with reverse yielding was obtained. Hussain et al. (1980) showed that an active thermal load can be used to produce thermal stresses equivalent to autofrettage residual stresses. Rees (1987) considered a closed-end cylinder of hardening and nonhardening material subjected to internal pressure. He assumed that axial plastic strain is zero and axial stress is the average of radial and hoop stresses. He compared residual stress distributions from two different material models with strain-hardening effects.

The aim of this paper is to present a numerical solution based on incremental theory of plasticity and Mendelson's (1968) method of successive elastic solutions for elastoplastic and residual stress distributions in a closed end cylinder of isotropic strain hardening material subjected to an internal pressure and a temperature gradient.

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Theoretical Analysis

To present a general dimensionless solution, it is convenient to introduce the following nondimensional quantities:

$$S_r=\sigma_r/\sigma_o \quad S_\theta=\sigma_\theta/\sigma_o \quad S_z=\sigma_z/\sigma_o \quad \tau_a=(E\alpha T_a)/(1-\nu)\sigma_o$$

$$P_a=p_a/\sigma_o \quad P_b=p_b/\sigma_o \quad \rho=r/a \quad \tau_b=(E\alpha T_b)/(1-\nu)\sigma_o$$

$$\epsilon_r=\mathcal{E}_r/\mathcal{E}_o \quad \epsilon_\theta=\mathcal{E}_\theta/\mathcal{E}_o \quad \epsilon_z=\mathcal{E}_z/\mathcal{E}_o \quad \mathcal{E}_o=\sigma_o/E \quad \beta=b/a$$

The equilibrium and compatibility relations in dimensionless form are written as follows:

$$\frac{dS_r}{d\rho}+\frac{S_r-S_\theta}{\rho}=0$$

$$\frac{d\epsilon_\theta}{d\rho}+\frac{\epsilon_\theta-\epsilon_r}{\rho}=0 \qquad (1)$$

Stress-strain relations are

$$\epsilon_r = S_r - \nu(S_\theta + S_z) + (1-\nu)\tau + \epsilon_r^p$$

$$\epsilon_\theta = S_\theta - \nu(S_r + S_z) + (1-\nu)\tau + \epsilon_\theta^p$$

$$\epsilon_z = S_z - \nu(S_\theta + S_r) + (1-\nu)\tau + \epsilon_z^p \quad (2)$$

where ϵ_r, ϵ_θ, and ϵ_z are total strains and ϵ_r^p, ϵ_θ^p, and ϵ_z^p are total plastic strains. The boundary conditions are

$$S_r = -P_a \quad \text{at } \rho = 1$$

$$S_r = -P_b \quad \text{at } \rho = \beta \quad (3)$$

In generalized plane strain case, the axial strain ϵ_z is constant, which can be determined from the end conditions. The condition of closed-end cylinder can be mathematically expressed as follows:

$$2\int_1^\beta S_z \rho d\rho = P_a \quad (4)$$

Temperature distribution for steady-state outward heat flow is

$$\tau = \ln\frac{1}{\beta}\left(\tau_a \ln\frac{\beta}{\rho} - \tau_b \ln\frac{1}{\rho}\right) \quad (5)$$

With the foregoing boundary and end conditions and the specified temperature distribution, the dimensionless solution for the stresses are as follows:

$$S_r^p = U(\rho, \mathcal{E}_r^p, \mathcal{E}_\theta^p) + F(\rho, \beta, \Theta, P_b) + G(\rho, \beta)P_a$$

$$S_\theta^p = V(\rho, \mathcal{E}_r^p, \mathcal{E}_\theta^p) + H(\rho, \beta, \Theta, P_b) + R(\rho, \beta)P_a$$

$$S_z^p = W(\rho, \mathcal{E}_r^p, \mathcal{E}_\theta^p) + M(\rho, \beta, \Theta) + N(\beta)P_a \quad (6)$$

where functions U, F, G, V, H, R, W, M, and N are defined in Appendix A. In terms of these functions, the elastic solution is rewritten in the following form:

$$S_r = F(\rho, \beta, \Theta, P_b) + G(\rho, \beta)P_a$$

$$S_\theta = H(\rho, \beta, \Theta, P_b) + R(\rho, \beta)P_a$$

$$S_z = M(\rho, \beta, \Theta) + N(\beta)P_a \quad (7)$$

Critical Condition

Loading conditions in which yielding may start in the cylinder thickness is called critical conditions. When yielding starts at a point, Von Mises yield criterion needs to be satisfied at that point. The dimensionless form of Von Mises criterion is

$$(S_r - S_\theta)^2 + (S_\theta - S_z)^2 + (S_z - S_r)^2 = 2 \quad (8)$$

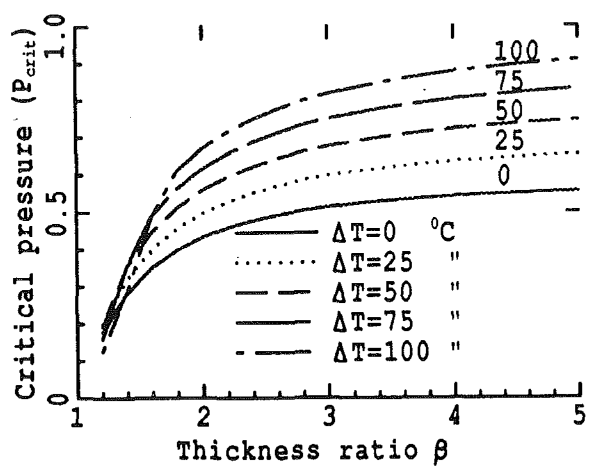

Fig. 1 Critical inner pressure versus β

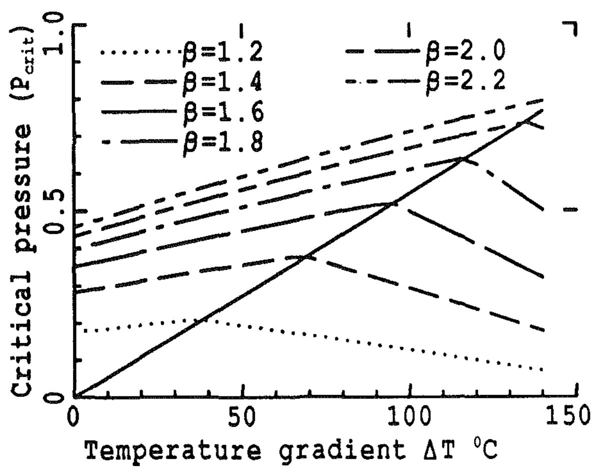

Fig. 2 Critical inner pressure versus ΔT

Substituting Eq. (7) into Eq. (8), the following equation for critical condition is obtained:

$$A(\rho, \beta)P_a^2 + B(\rho, \beta, \Theta, P_b)P_a + C(\rho, \beta, \Theta, P_b) = 0 \quad (9)$$

Variables ρ, β, Θ, P_a, and P_b are related together in Eq. (9). Any combination of these variables which satisfies Eq. (9) can produce the critical condition for plastic yielding. Numerical results of this equation are shown in Figs. 1 and 2. Variables A, B and C in Eq. (9) are defined as follows:

Nomenclature

a, b = inside, outside radii
E = modulus of elasticity
P_a, P_b = dimensionless pressures
p_a, p_b = inner, outer pressures
S_r, S_θ, S_z = dimensionless stress components
S_r^p, S_θ^p, S_z^p = dimensionless plastic stresses
T_a, T_b = inner, outer temperatures
α = coefficient of thermal expansion
β = thickness ratio (b/a)
ΔT = temperature gradient ($T_a - T_b$)
$\Delta\epsilon_e^p$ = effective plastic strain increment
$\Delta\epsilon_{r,\theta,z}^p$ = dimensionless strain increments
$\Delta\mathcal{E}_{r,\theta,z}^p$ = plastic strain increments
$\epsilon_r, \epsilon_\theta, \epsilon_z$ = dimensionless total strains
ϵ_e^p = effective plastic strain
$\epsilon_r^p, \epsilon_\theta^p, \epsilon_z^p$ = dimensionless plastic strains
$\mathcal{E}_r, \mathcal{E}_\theta, \mathcal{E}_z$ = total strain components
$\mathcal{E}_r^p, \mathcal{E}_\theta^p, \mathcal{E}_z^p$ = plastic strain components
Θ = $\tau_a - \tau_b$
ν = Poisson's ratio
ρ = dimensionless radius
ρ_c = elastic-plastic interface
σ_e = effective stress
σ_o, ϵ_o = yield stress, yield strain
$\sigma_r, \sigma_\theta, \sigma_z$ = stress components
τ_a, τ_b = dimensionless temperatures

Subscripts

a = inner
b = outer
c = elastic-plastic interface
e = effective
r, θ, z = cylindrical coordinates

Superscripts

p = plastic

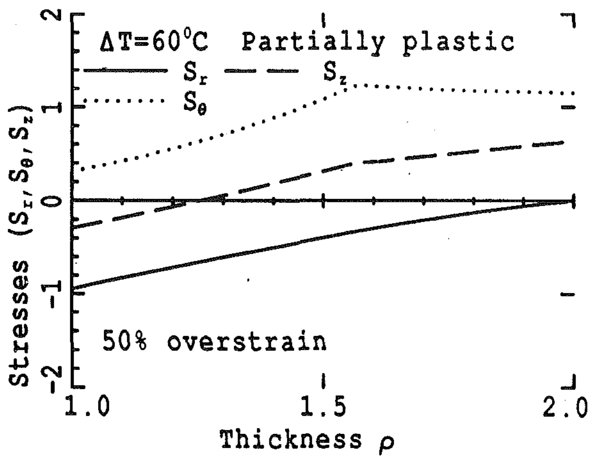

Fig. 3 Stress distribution at 50 percent overstrain

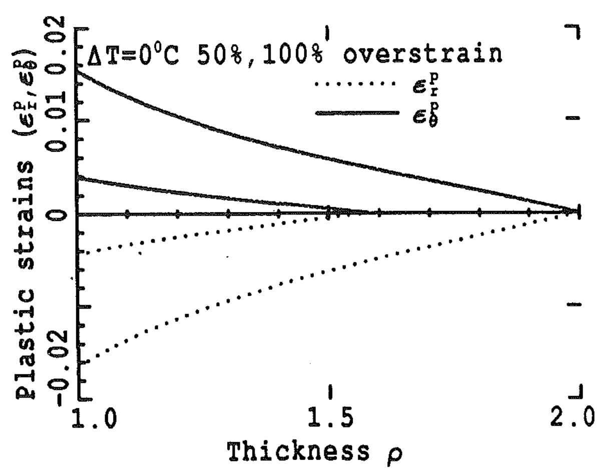

Fig. 5 Plastic strain distribution

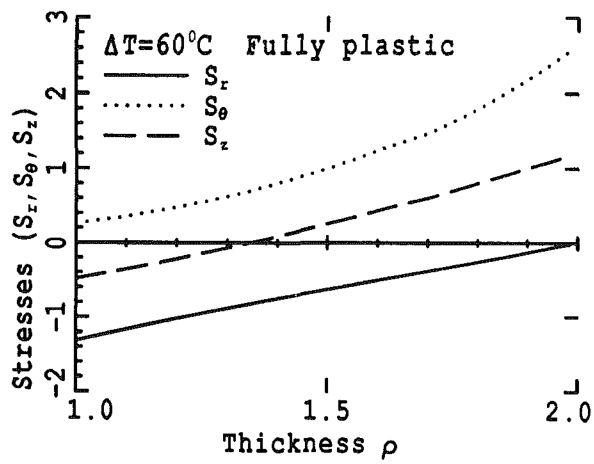

Fig. 4 Fully plastic stress distribution

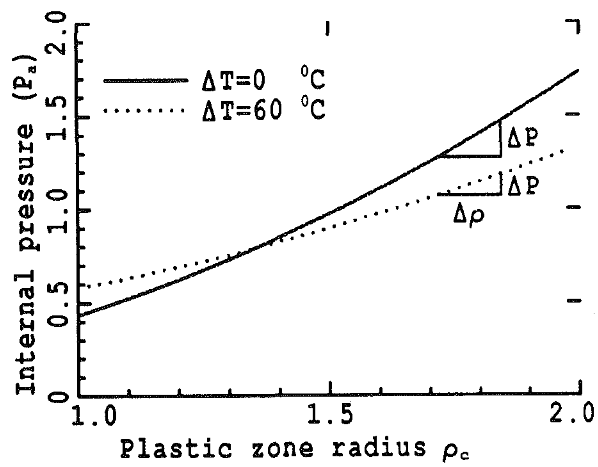

Fig. 6 Progress of plastic zone

$$A(\rho, \beta) = 2(G^2 + R^2 + N^2 - G*R - R*N - N*G)$$

$$B(\rho, \beta, \Theta, P_b) = 4(F*G + H*R + M*N) - 2(F*R + F*N + H*G + H*N + M*R + M*G)$$

$$C(\rho, \beta, \Theta, P_b) = 2(F^2 + H^2 + M^2 - F*H - H*M - M*F - \sigma_o^2)$$

Loading Beyond the Critical Loading Condition

If inner pressure is increased beyond the critical calculated value, some part of the cylinder will deform plastically. Plastic stresses are shown to be functions of total plastic strains (Eq. (6)). Plastic strains are path-dependent, and, therefore, the increments of plastic strains must be integrated or added together along the loading path to give the total plastic strains. Increments of plastic strains are related to the stresses and uniaxial stress-strain curve through Prandtl-Reuss equation. In this case

$$\Delta\epsilon_r^p = \frac{\Delta\epsilon_e^p}{S_e}\left[S_r - \frac{1}{2}(S_\theta + S_z)\right]$$

$$\Delta\epsilon_\theta^p = \frac{\Delta\epsilon_e^p}{S_e}\left[S_\theta - \frac{1}{2}(S_z + S_r)\right] \quad (10)$$

in which effective stress S_e and effective plastic strain increment $\Delta\epsilon_e^p$ are defined as

$$S_e = \frac{1}{\sqrt{2}}\left[(S_r - S_\theta)^2 + (S_\theta - S_z)^2 + (S_z - S_r)^2\right]^{1/2} \quad (11)$$

$$\Delta\epsilon_e^p = \frac{\sqrt{2}}{3}\left[(\Delta\epsilon_r^p - \Delta\epsilon_\theta^p)^2 + (\Delta\epsilon_\theta^p - \Delta\epsilon_z^p)^2 + (\Delta\epsilon_z^p - \Delta\epsilon_r^p)^2\right]^{1/2} \quad (12)$$

Considering incompressibility for the material, axial plastic strain increment can be written as follows:

$$\Delta\epsilon_z^p = -(\Delta\epsilon_r^p + \Delta\epsilon_\theta^p) \quad (13)$$

Effective stress and effective plastic strain are related together through the stress-strain curve. In this study, loading function is the stress-strain curve of SUS 304 at 400°C, which is selected from the experimental results of Niitsu and Ikegami (1990)

$$S_e = 1 + \frac{K}{E}(\epsilon_e^p)^\gamma \quad (14)$$

Constant K and the exponent γ for various constant temperatures are tabulated in Niitsu and Ikegami (1990). A computer program based on incremental theory of plasticity for calculation of plastic stresses and strains has been developed. Numerical procedures will be explained later in this paper. Some results of plastic stress-strain distribution and progress of plastic zone are shown in Figs. 3, 4, 5, and 6.

Residual Stresses

Residual stresses are obtained by superposing a completely elastic stress system due to pressure $(-P_a)$ and temperature $(-\Theta)$, denoted by prime, on the plastic stress system due to pressure P_a and temperature Θ. Denoting residual stress with double prime, we write

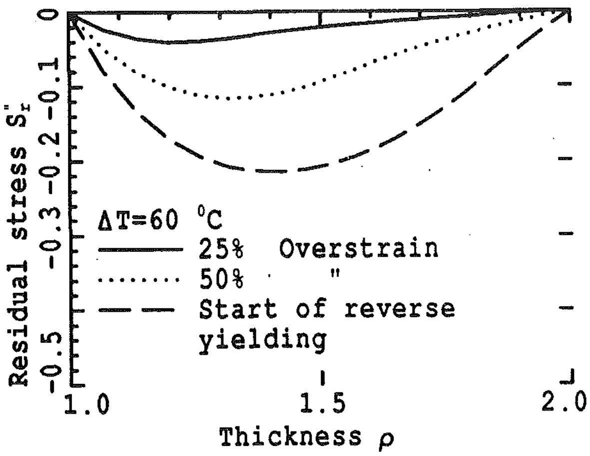

Fig. 7 Residual radial stresses

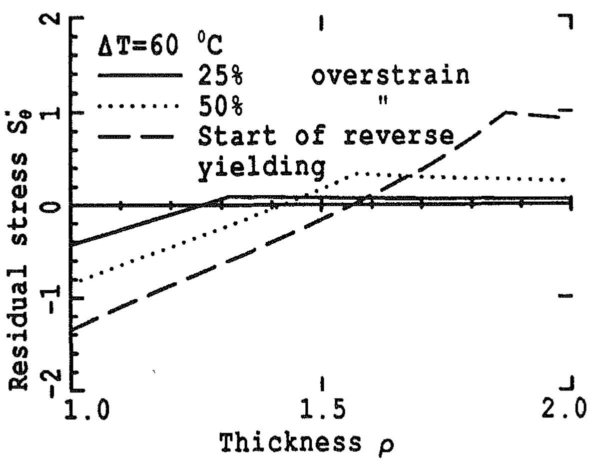

Fig. 8 Residual hoop stresses

$$S_r'' = S_r^p + S_r'$$

$$S_\theta'' = S_\theta^p + S_\theta'$$

$$S_z'' = S_z^p + S_z' \quad (15)$$

Reverse yielding will occur if the following equation is satisfied:

$$(S_r'' - S_\theta'')^2 + (S_\theta'' - S_z'')^2 + (S_z'' - S_r'')^2 = 2 \quad (16)$$

Residual stress distributions of 25 and 50-percent overstrain and the distribution at the onset of reverse yielding are shown in Figs. 7 and 8.

Numerical Procedure

Step by step procedure is explained as follows:

1 For a fixed temperature gradient, the critical pressure (P_{crit}) and the radius at which plastic yielding begins are calculated from Eq. (9).

2 Assuming final pressure is P_f, the loading path is divided into N steps and each pressure increment is $\Delta p = (P_f - P_{\text{crit}})/N$. The internal pressure at the ith loading step is then

$$P_{a,i} = P_{\text{crit}} + i\Delta p$$

3 Initial values are assumed for radial and tangential plastic strain increments $\Delta\epsilon^p_{r,ij}$ and $\Delta\epsilon^p_{\theta,ij}$ and are added to the accumulated plastic strains obtained from the previous loading step at any increment of radius. In the initial loading step, the accumulated plastic strains are zero. The radial and tangential plastic strains are

$$\epsilon^p_{r,ij} = \sum_{k=1}^{i-1} \Delta\epsilon^p_{r,kj} + \Delta\epsilon^p_{r,ij}$$

$$\epsilon^p_{\theta,ij} = \sum_{k=1}^{i-1} \Delta\epsilon^p_{\theta,kj} + \Delta\epsilon^p_{\theta,ij}$$

The subscripts i and j refer to the loading step and the layer along radius, respectively. The plastic strain increment in axial direction is obtained from incompressibility condition, which is

$$\Delta\epsilon^p_{z,ij} = -(\Delta\epsilon^p_{r,ij} + \Delta\epsilon^p_{\theta,ij})$$

In this study, the initial values of -0.00003 and $+0.00004$ are assumed for the radial and tangential plastic strain increments, respectively.

4 The effective plastic strain increment is then calculated as

$$\Delta\epsilon^p_{e,ij} = \frac{\sqrt{2}}{3}\left[(\Delta\epsilon^p_{r,ij} - \Delta\epsilon^p_{\theta,ij})^2 + (\Delta\epsilon^p_{\theta,ij} - \Delta\epsilon^p_{z,ij})^2 + (\Delta\epsilon^p_{z,ij} - \Delta\epsilon^p_{r,ij})^2\right]^{1/2}$$

5 From the stress-strain curve, the effective stress is calculated at each loading step for each increment of radius

$$S_{e,ij} = 1 + \frac{K}{E}(\epsilon^p_{e,ij})^\gamma$$

in which $\epsilon^p_{e,ij} = \Sigma\Delta\epsilon^p_{e,ij}$.

6 The radius of elastic-plastic boundary at ith loading step, $\rho_{c,i}$ is found by setting the boundary conditions at this radius. At the plastic zone boundary, Von Mises condition must be satisfied. Suppose the yielding starts from the inside radius; then Eq. (9) can be rewritten in the following form:

$$A(\rho_{c,i}, \beta)S_r^2(\rho_{c,i}) + B[\rho_{c,i}, \beta, \Theta, P_b]S_r^2(\rho_{c,i}) + C[\rho_{c,i}, \beta, \Theta, P_b] = 0$$

For the case in which yielding starts from the outer surface, Eq. (9) is rewritten as

$$A(\rho_{c,i}, \beta)P_{a,i}^2 + B[\rho_{c,i}, \beta, \Theta, S_r(\rho_{c,i})]P_{a,i} + C[\rho_{c,i}, \beta, \Theta, S_r(\rho_{c,i})] = 0$$

7 Since $\rho_{c,i}$ is known the integrals in Eq. (6) can now be evaluated, and, therefore, plastic stresses are calculated.

8 Having the stresses from step 7 and the effective plastic strain and stress from steps 4 and 5, a new and improved approximation is obtained for the latest increment of the plastic strains employing Prandtl-Reuss equations

$$\Delta\epsilon^{p(\text{new})}_{r,ij} = \frac{\Delta\epsilon^p_{e,ij}}{S_{e,ij}}(2S_{r,ij} - S_{\theta,ij} - S_{z,ij})$$

$$\Delta\epsilon^{p(\text{new})}_{\theta,ij} = \frac{\Delta\epsilon^p_{e,ij}}{S_{e,ij}}(2S_{\theta,ij} - S_{r,ij} - S_{z,ij})$$

9 The method is iterated from step 4 until the ith loading step converges.

10 After the ith loading step is converged, the corresponding residual stresses are obtained using Eq. (15).

11 The loading step is advanced to one increment and the numerical procedure is repeated from step 2.

Results and Discussion

A general solution for thermoelastoplastic and residual stress distribution in thick-walled cylinders is presented here. There is no simplification such as zero plastic strain increment in axial direction ($\Delta\epsilon_z^p = 0$), and in this study a generalized plane strain condition is considered. Solution presented here is in dimensionless form, and, therefore, can be used for any loading conditions and thickness ratios. Critical condition for any loading combination can be investigated by Eq. (9). Some features of this equation are shown in Figs. 1 and 2. Critical pressure at various temperature gradients is plotted against thickness ratio in Fig. 1. This figure shows that, except for a small range of low thickness ratios, higher temperature gradients tend to increase critical pressure. Critical pressure for various thickness ratios is plotted against temperature gradient in Fig. 2. In this figure, the maximum points represent conditions in which all the thickness yields simultaneously. All the points to the left and right of the maximum points are conditions in which yielding starts at inner and outer surfaces, respectively. Figures 3 and 4 show the stress distribution at 50 percent overstrain and fully plastic condition of a cylinder of thickness ratio equal to 2. These figures show that during plastic flow in cylinders, the maximum shear stress which is $(S_\theta - S_r)/2$ becomes more uniform throughout the thickness. Plastic strain distribution of 50-percent overstrain and fully plastic vessel are shown in Fig. 5. This figure shows that plastic strains in radial and tangential directions are equal in magnitudes and opposite in signs throughout the thickness. On the other hand, the assumption of zero plastic strain increment in axial direction by some investigators is quite reasonable. Progress of plastic zone with and without the presence of a thermal gradient is shown in Fig. 6. In the presence of a thermal gradient, less pressure differential is needed for an equal progress of plastic zone. Figures 7 and 8 show the residual stress distribution of 25 and 50-percent overstrained cylinder and the distribution at the onset of reverse yielding. The results show that residual hoop stress is highly compressive at inner surface and is tensile at outer surface. Beneficial effect of this residual hoop stress in fracture design of pressure vessels is well established.

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APPENDIX A

Functions V, F, G, U, H, R, W, M, and N are written as follows:

$$F(\rho, \beta, \Theta, P_b) = \frac{\Theta}{2(\beta^2-1)\ln\beta}\left[\frac{\beta^2}{\rho^2}\ln\beta + \beta^2 \ln\frac{\rho}{\beta} - \ln\rho\right] + \frac{P_b\beta^2}{\beta^2-1}\left(1-\frac{1}{\rho^2}\right)$$

$$G(\rho, \beta) = \frac{1}{\beta^2-1}\left(1-\frac{\beta^2}{\rho^2}\right)$$

$$H(\rho, \beta, \Theta, P_b) = \frac{\Theta}{2(\beta^2-1)\ln\beta}\left[-\frac{\beta^2}{\rho^2}\ln\beta + \beta^2\ln\frac{\rho}{\beta} - \ln\rho + \beta^2 - 1\right] + \frac{P_b\beta^2}{\beta^2-1}\left(1+\frac{1}{\rho^2}\right)$$

$$R(\rho, \beta) = \frac{1}{\beta^2-1}\left(1+\frac{\beta^2}{\rho^2}\right)$$

$$M(\rho, \beta, \Theta) = \frac{\Theta}{2(\beta^2-1)\ln\beta}\left[+2\beta^2\ln\frac{\rho}{\beta} - 2\ln\rho + \beta^2 - 1\right]$$

$$N(\beta) = \frac{1}{\beta^2-1}$$

$$U(r, \epsilon_r^p, \epsilon_\theta^p) = -\frac{1}{2(1-\nu^2)\rho^2}\left[(1-2\nu)\int_1^\rho (\epsilon_\theta^p + \epsilon_r^p)\rho d\rho + \rho^2\int_1^\rho \frac{\epsilon_\theta^p - \epsilon_r^p}{\rho}d\rho\right] + \frac{1}{2(1-\nu^2)(\beta^2-1)}\left[(1-2\nu)\int_1^\beta (\epsilon_\theta^p + \epsilon_r^p)\rho d\rho + \beta^2\int_1^\beta \frac{\epsilon_\theta^p - \epsilon_r^p}{\rho}d\rho\right]\left(1-\frac{1}{\rho^2}\right)$$

$$V(r, \epsilon_r^p, \epsilon_\theta^p) = \frac{1}{2(1-\nu^2)(\beta^2-1)}\left[(1-2\nu)\int_1^\beta (\epsilon_\theta^p + \epsilon_r^p)\rho d\rho + \beta^2\int_1^\beta \frac{\epsilon_\theta^p - \epsilon_r^p}{\rho}d\rho\right]\left(1+\frac{1}{\rho^2}\right) - \frac{1}{2(1-\nu^2)\rho^2} \times \left[(1-2\nu)\int_1^\rho (\epsilon_\theta^p + \epsilon_r^p)\rho d\rho + \rho^2\int_1^\rho \frac{\epsilon_\theta^p - \epsilon_r^p}{\rho}d\rho - 2\rho^2((1-\nu)\epsilon_\theta^p - \nu\epsilon_r^p)\right]$$

$$W(r, \epsilon_r^p, \epsilon_\theta^p) = \frac{\nu}{(1-\nu^2)(\beta^2-1)}\left[(1-2\nu)\int_1^\beta (\epsilon_\theta^p + \epsilon_r^p)\rho d\rho + \beta^2\int_1^\beta \frac{\epsilon_\theta^p - \epsilon_r^p}{\rho}d\rho\right] + \frac{\nu}{1-\nu^2}\left[\int_1^\beta \frac{\epsilon_\theta^p - \epsilon_r^p}{\rho}d\rho + (1-\nu)\epsilon_\theta^p - \nu\epsilon_r^p\right] + \frac{1}{\beta^2-1}\left[2\int_1^\beta (\epsilon_\theta^p + \epsilon_r^p)\rho d\rho - (\beta^2 - 1)(\epsilon_\theta^p + \epsilon_r^p)\right]$$