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Technical Brief

Forward Position Analysis of the 3-DOF Module of the TriVariant: A 5-DOF Reconfigurable Hybrid Robot

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This paper deals with the forward position analysis of a 3-DOF parallel mechanism module, which forms the main body of a 5-DOF reconfigurable hybrid robot named TriVariant. The TriVariant is a modified version of the Tricept robot, achieved by integrating one of the three active limbs into the passive one. The analytical method is employed to obtain the forward position solutions. It shows that the forward position analysis of the TriVariant is much simpler than that of the Tricept.
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1 Introduction

Forward position analysis is one of the important issues in the development of parallel mechanisms. Approaches for solving the problem can be classified into two categories, i.e., analytical approach and numerical algorithm. The analytical approach is focused on finding the complete set of solutions using the procedure that can usually be implemented in two steps: (1) formulate the kinematic constraints into a set of nonlinear equations, and (2) generate an end polynomial equation having a single unknown by means of certain elimination methods. As a result, all configurations can be found by solving the end polynomial equation [1–7]. The complexity of the polynomial approach depends upon the geometry of the object and proper choice of elimination techniques. The numerical approach can be used to find the solutions close to the initial estimate using root search algorithms or optimization techniques [8].

This paper deals with the forward position analysis of the 3-DOF parallel mechanism module which forms the main body of a newly invented hybrid robot named TriVariant (Fig. 1) [9]. The research interests will be focused on finding the number of its forward position solutions in comparison with the corresponding results of the Tricept [10–12]. As shown in Fig. 1, the module consists of a base and a mobile platform connected by three kinematic chains (limbs). Two are identical U$\underline{\text{P}}$S limbs and the other is a U$\underline{\text{P}}$ limb whose output link is fixed with the mobile platform. Here, U, P and S represent the universal, prismatic, and spherical joints respectively, and $\underline{\text{P}}$ denotes that the corresponding joint is active. The TriVariant can be considered as a simplified version of the Tricept robot, achieved by integrating one of the three active limbs into the passive one while keeping the required type and degrees of freedom. For simplicity, we use "Tricept" and "TriVariant" to refer to the 3-DOF modules in what follows.

2 System Description

Figure 2 shows a schematic diagram of the TriVariant, where B_i for $i=1, 2, 3$ represent the centers of the U joints and A_i for $i=1, 2$ the centers of the S joints. A_3 is the intersection of the axial axis of the U$\underline{\text{P}}$ limb and its normal plane in which A_i ($i=1, 2$) is located. Since limb B_3A_3 is fixed with the mobile platform, A_3 can also be considered as the reference point of the mobile platform. Similar to Tricept 605 [9], it is assumed that the rotation axes of the outer ring of the U-joints of the three limbs are parallel and located in the same plane.

A fixed coordinate system $B_3-x_3y_3z_3$ is established with the center of the U joint of the U$\underline{\text{P}}$ limb being taken as the origin B_3, the rotation axis of its outer ring being the x_3 axis and the z_3 axis being vertically placed as shown. Meanwhile, a body-fixed coordinate system $A_3-u_3v_3w_3$ is attached to the U$\underline{\text{P}}$ limb. The w_3 axis is coincident with the axial axis of the limb, pointing from B_3 to A_3. The u_3 axis is parallel to the cross product of two vectors along y_3 and w_3 axes, and the v_3 axis satisfies the right-hand rule. Thus the orientation matrix $\boldsymbol{R}_3$ of $A_3-u_3v_3w_3$ with respect to $B_3-x_3y_3z_3$ can be formulated by rotating angles ψ_3 about the x_3 axis, and θ_3 about the v_3 axis.

$$\begin{aligned}\boldsymbol{R}_3 &= \mathrm{Rot}(x_3,\psi_3)\mathrm{Rot}(v_3,\theta_3)\\ &= \begin{bmatrix} \mathrm{c}\theta_3 & 0 & \mathrm{s}\theta_3 \\ \mathrm{s}\psi_3\mathrm{s}\theta_3 & \mathrm{c}\psi_3 & -\mathrm{s}\psi_3\mathrm{c}\theta_3 \\ -\mathrm{c}\psi_3\mathrm{s}\theta_3 & \mathrm{s}\psi_3 & \mathrm{c}\psi_3\mathrm{c}\theta_3 \end{bmatrix}\\ &= [\boldsymbol{u}_3\ \boldsymbol{v}_3\ \boldsymbol{w}_3] \end{aligned} \tag{1}$$

where s# and c# denote sin(#) and cos(#), and $\boldsymbol{u}_3$, $\boldsymbol{v}_3$, and $\boldsymbol{w}_3$ are

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Fig. 1 The TriVariant robot

the unit vectors of the u_3, v_3, and w_3 axes, respectively.

3 Forward Position Analysis

Forward position analysis of the TriVariant is concerned with the determination of the position vector of A_3 given the limb lengths and the geometry of both base and platform. The closed-loop constraint equation for each limb can be generated as follows:

$$\boldsymbol{r}_3 = q_3\boldsymbol{w}_3 \tag{2}$$

$$\boldsymbol{r}_3 = \boldsymbol{b}_i + q_i\boldsymbol{w}_i - \boldsymbol{a}_i, \quad i=1,2 \tag{3}$$

where q_i and $\boldsymbol{w}_i$ are the length and unit vector of the ith limb, $\boldsymbol{b}_i$ and $\boldsymbol{a}_i$ are the position vectors of B_i and A_i, respectively, and

$$\boldsymbol{b}_i = (b_{ix}\ b_{iy}\ 0)^{\mathrm{T}}, \quad \boldsymbol{a}_{i0} = (a_{i0x}\ a_{i0y}\ 0)^{\mathrm{T}}, \quad \boldsymbol{a}_i = \boldsymbol{R}_3\boldsymbol{a}_{i0} \tag{4}$$

$\boldsymbol{a}_{i0}$ denotes the position vector of A_i measured in $A_3-u_3v_3w_3$.

3.1 General Case. Without losing generality, the end polynomial equation for the TriVariant having general geometry given by Eq. (4) will be formulated to obtain all possible configurations for a given set of limb lengths.

Rewrite Eq. (3) as

$$q_i\boldsymbol{w}_i = \boldsymbol{r}_3 + \boldsymbol{a}_i - \boldsymbol{b}_i, \quad i=1,2 \tag{5}$$

Taking the Euclidean norm on both sides of Eq. (5) and keeping in mind $\boldsymbol{r}_3^{\mathrm{T}}\boldsymbol{a}_i=0$, leads to

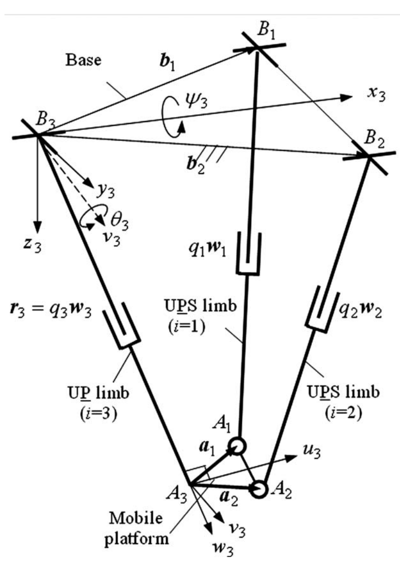

Fig. 2 Schematic diagram of the TriVariant

$$q_i^2 = q_3^2 + a_i^2 + b_i^2 - 2\boldsymbol{r}_3^{\mathrm{T}}\boldsymbol{b}_i - 2\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{b}_i, \quad i=1,2 \tag{6}$$

where a_i and b_i are the lengths of $\overline{A_3A_i}$ and $\overline{B_3B_i}$, respectively. Substituting Eqs. (1), (2), and (4) into Eq. (6) yields

$$d_{i1} + d_{i2}\mathrm{s}\psi_3 + d_{i3}\mathrm{c}\psi_3 = 0, \quad i=1,2 \tag{7}$$

where

$$d_{i1} = c_i + 2b_{ix}(q_3\mathrm{s}\theta_3 + a_{i0x}\mathrm{c}\theta_3)$$

$$d_{i2} = 2b_{iy}(a_{i0x}\mathrm{s}\theta_3 - q_3\mathrm{c}\theta_3)$$

$$d_{i3} = 2a_{i0y}b_{iy}$$

$$c_i = q_i^2 - q_3^2 - a_i^2 - b_i^2$$

Substituting trigonometric identities

$$\mathrm{s}\psi_3 = \frac{2t_\psi}{1+t_\psi^2}, \quad \mathrm{c}\psi_3 = \frac{1-t_\psi^2}{1+t_\psi^2}, \quad \mathrm{s}\theta_3 = \frac{2t_\theta}{1+t_\theta^2}, \quad \mathrm{c}\theta_3 = \frac{1-t_\theta^2}{1+t_\theta^2}$$

$$t_\psi = \tan(\psi_3/2), \quad t_\theta = \tan(\theta_3/2)$$

into Eq. (7) leads to

$$\kappa_{i2}t_\psi^2 + \kappa_{i1}t_\psi + \kappa_{i0} = 0, \quad i=1,2 \tag{8}$$

where

$$\kappa_{i2} = e_{i1} - e_{i3}, \quad \kappa_{i1} = 2e_{i2}, \quad \kappa_{i0} = e_{i1} + e_{i3}$$

$$e_{i1} = (c_i - 2a_{i0x}b_{ix})t_\theta^2 + 4q_3b_{ix}t_\theta + (c_i + 2a_{i0x}b_{ix})$$

$$e_{i2} = 2b_{iy}(q_3t_\theta^2 + 2a_{i0x}t_\theta - q_3), \quad e_{i3} = 2a_{i0y}b_{iy}$$

Multiplying t_ψ on both sides of Eq. (8) results in two complementary equations which, together with Eq. (8), can be written in a matrix form

$$\boldsymbol{K}\boldsymbol{t} = 0 \tag{9}$$

where

$$\boldsymbol{K} = \begin{bmatrix} 0 & \kappa_{12} & \kappa_{11} & \kappa_{10} \\ \kappa_{12} & \kappa_{11} & \kappa_{10} & 0 \\ 0 & \kappa_{22} & \kappa_{21} & \kappa_{20} \\ \kappa_{22} & \kappa_{21} & \kappa_{20} & 0 \end{bmatrix}$$

$$\boldsymbol{t} = (t_\psi^3\ t_\psi^2\ t_\psi^1\ 1)^T$$

The necessary condition for Eq. (9) to have nontrivial solutions is

$$\det \boldsymbol{K} = 0 \tag{10}$$

Expanding Eq. (10) yields an eighth-order polynomial equation because all terms in the equation are quadratic functions of t_θ.

$$\kappa_{12}\kappa_{20}(\kappa_{12}\kappa_{20} - \kappa_{11}\kappa_{21} - 2\kappa_{10}\kappa_{22}) + \kappa_{10}\kappa_{21}(\kappa_{12}\kappa_{21} + \kappa_{11}\kappa_{22}) + \kappa_{22}(\kappa_{10}^2\kappa_{22} + \kappa_{11}^2\kappa_{20}) = 0 \tag{11}$$

When t_θ is determined, multiplying κ_{22} for $i=1$ and κ_{12} for $i=2$ on both sides of Eq. (8) and doing subtraction, results in the explicit expression of t_ψ in terms of t_θ

$$t_\psi = \frac{\kappa_{12}\kappa_{20} - \kappa_{10}\kappa_{22}}{\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}} \tag{12}$$

This leads to the determination of ψ_3 and θ_3 using the trigonometric identities. It can be seen that the above procedure leads to a two-equation system in echelon form, the first equation is of eighth order and the remaining is linear. The result is exactly coincident with the forward position analysis of the fully-parallel mechanisms for the orientation of a rigid body with a fixed point [5].

It was reported in [12] that the order of the end polynomial equation of the Tricept is 24, meaning that it may have at most 24 real solutions. Whilst, it is easy to see from Eq. (11) that the order

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of the end polynomial equation of the TriVariant having a general geometry is 8, resulting in at most 8 real solutions though the type and degrees of freedom of two robots are identical. The difference in the number of solutions will be explained in Sec. 5.

3.2 A Special Case. As shown in Fig. 2, if $B_3-x_3y_3z_3$ is placed in such a way that either $\overline{B_3B_1}$ or $\overline{B_3B_2}$ is coincident with the x_3 axis, i.e., the rotation axis of the outer ring of the U joint of the U$\underline{\text{P}}$ limb, the position vector of B_i in $B_3-x_3y_3z_3$ becomes

$$\boldsymbol{b}_i = (b_{ix} \quad b_{iy} \quad 0)^{\mathrm{T}} = (b_i \quad 0 \quad 0)^{\mathrm{T}} \tag{13}$$

For simplicity, let $i=1$, then $b_{1x}=b_1$ and $b_{1y}=0$, leading to $d_{12}=d_{13}=0$. As a result, Eq. (7) for $i=1$ is simplified as

$$d_{11} = A_1 \mathrm{s}\theta_3 + B_1 \mathrm{c}\theta_3 + C_1 = 0 \tag{14}$$

where

$$A_1 = 2q_3 b_1, \quad B_1 = 2a_{10x} b_1, \quad C_1 = c_1$$

Thus, θ_3 can be explicitly obtained by

$$\theta_3 = 2 \arctan\left[\frac{-A_1 \mp \sqrt{A_1^2 + B_1^2 - C_1^2}}{C_1 - B_1}\right] \tag{15}$$

On this basis, since $b_{2y} \neq 0$, Eq. (7) for $i=2$ becomes

$$A_2 s\psi_3 + B_2 \mathrm{c}\psi_3 + C_2 = 0 \tag{16}$$

where

$$A_2 = 2b_{2y}(a_{20x}\mathrm{s}\theta_3 - q_3 \mathrm{c}\theta_3), \quad B_2 = 2a_{20y} b_{2y}$$

$$C_2 = c_2 + 2b_{2x}(q_3 \mathrm{s}\theta_3 + a_{20x}\mathrm{c}\theta_3)$$

This means that ψ_3 can also be explicitly achieved by

$$\psi_3 = 2 \arctan\left[\frac{-A_2 \mp \sqrt{A_2^2 + B_2^2 - C_2^2}}{C_2 - B_2}\right] \tag{17}$$

It is easy to see that for this special case the TriVariant may have at most 4 real forward position solutions. Obviously, the same conclusion can be drawn when $\overline{B_3B_2}$ is coincident with the x_3 axis.

4 An Example

The dimensions of the TriVariant given in [13] is employed for the forward position analysis, with $\Delta B_1B_2B_3$ and $\Delta A_1A_2A_3$ shown in Fig. 2 being equilateral triangles, and the position vectors of B_i in $B_3-x_3y_3z_3$ and A_i in $A_3-u_3v_3w_3$ being given by

$$\boldsymbol{b}_1 = (519.62 \quad -300 \quad 0)^T$$

$$\boldsymbol{b}_2 = (519.62 \quad 300 \quad 0)^T$$

$$\boldsymbol{a}_{10} = (103.92 \quad -60 \quad 0)^T$$

$$\boldsymbol{a}_{20} = (103.92 \quad 60 \quad 0)^T$$

In order to justify the results of the forward position analysis, the position vector of A_3 is arbitrarily chosen within the workspace as follows:

$$\boldsymbol{r}_3 = (450 \quad 350 \quad 850)^T \tag{18}$$

This allows a set of limb lengths to be determined by the inverse kinematic analysis as a reference for the forward kinematics.

$$q_1 = 1011.69, \quad q_2 = 790.19, \quad q_3 = 1023.47$$

For the given geometry, it is easy to see that Eqs. (11) and (12) have to be used for the forward position analysis because neither $\overline{B_3B_1}$ or $\overline{B_3B_2}$ is coincident with the x_3 axis. Substituting the specified data into Eq. (11) yields an eighth-order polynomial equation,

Table 1 Solutions of forward kinematics of the TriVariant

No.	ψ_3(deg)	θ_3(deg)	$\boldsymbol{r}_3$
1	−22.38	26.08	$(450.00 \quad 350.00 \quad 850.00)^T$
2	−156.39	30.37	$(517.45 \quad 353.65 \quad -809.12)^T$
3	156.39	138.03	$(684.38 \quad 304.77 \quad 697.30)^T$
4	22.38	142.32	$(625.59 \quad 308.42 \quad -749.01)^T$

$$11.9981t_\theta^8 - 67.6058t_\theta^7 + 72.9114t_\theta^6 + 137.9001t_\theta^5 - 219.6885t_\theta^4 - 22.7164t_\theta^3 + 134.9889t_\theta^2 - 52.6414t_\theta + 5.7647 = 0 \tag{19}$$

Solving Eq. (19) using Matlab function "root" results in four real and two pairs of conjugate complex solutions as follows:

$$t_\theta = 0.2316,\ 0.2714,\ 2.6074,\ 2.9308,$$
$$-1.1067 \pm 0.0078i,\ 0.9034 \pm 0.0225i$$

Given a real solution of t_θ, t_ψ can then be solved by Eq. (12). Consequently, the corresponding θ_3, ψ_3, and three components of $\boldsymbol{r}_3$ can finally be determined as shown in Table 1. The configurations of the robot associated with these solutions are shown in Fig. 3.

Examination of Table 1 indicates that solution No. 1 is the one associated with the data given in (18) since it satisfies:

$$\psi_3 \in (-90\ °, 90\ °), \quad \theta_3 \in (-90\ °, 90\ °)$$

5 Discussion

In this section, we explain the difference between the number of solutions of the end polynomial equations of the Tricept and the TriVariant.

In the first place, the forward position problem of the Tricept involves the solution of three unknowns in terms of the orientation angles and length of the UP limb, leading to a 24th-order end polynomial equation as reported in [12]. Whilst, the same problem for the TriVariant involves two unknowns in terms of the orientation angles of the U$\underline{\text{P}}$ limb, leading to an eighth-order end polynomial equation.

Secondly, it concluded in [12] there are at most 24 real solutions for the Tricept and a numerical example showed that 4 pairs of real solutions could be found. One solution in each pair is the

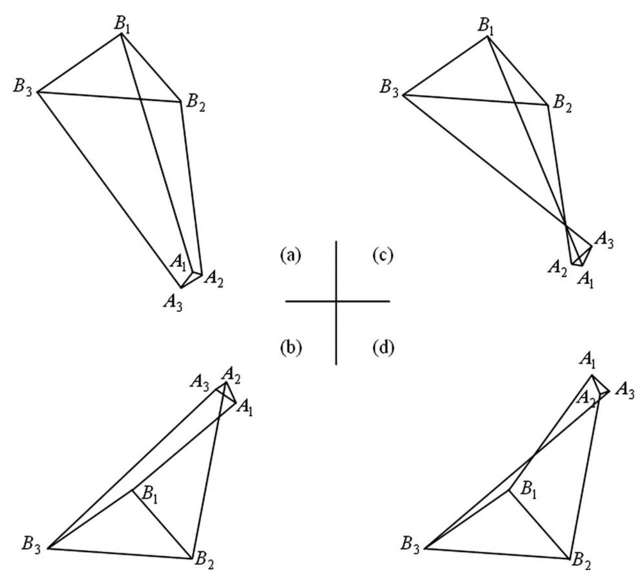

Fig. 3 Solutions of forward kinematics of the TriVariant robot, (*a*)–(*d*) correspond to solutions 1–4, respectively

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negative value of the other and they both constitute a pair of mutual mirror image configuration with respect to the base. This is because the UP limb length for one solution is identical in magnitude but opposite in sign to the other. Note that this is unavoidable in mathematics since the sign of the UP limb length cannot be constrained to be positive in the problem formulation. Note that the mirror image solutions merely make sense in mathematics and they thereby should be regarded as extraneous solutions. As for the TriVariant, no mirror image solutions are found because the U$\underline{\mathrm{P}}$ limb length is positive although its end polynomial equation in general form has at most eight real solutions. Particularly, when the rotation axis of the outer ring of the U joint of U$\underline{\mathrm{P}}$ limb is coincident with $\overline{B_3B_1}$ or $\overline{B_3B_2}$, 4 real solutions can be found in an explicit manner.

6 Conclusions

Utilizing the analytical method this paper deals with the forward position analysis of the 3-DOF parallel mechanism of a newly invented hybrid robot named TriVariant. It is found that the order of the end polynomial equation of the TriVariant having general geometry is 8, leading to at most 8 real solutions. Particularly, 4 real solutions can directly be obtained when the rotation axis of the outer ring of the U joint of the U$\underline{\mathrm{P}}$ limb is coincident with one side of the base triangle. Therefore, we can conclude that the formulation and solution of the forward kinematic problem of the TriVariant are much simpler than those of the Tricept.

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