Full Information from Measured ADC Test Data using Maximum Likelihood Estimation

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Abstract

ADC testing is often done using sine wave excitation (see e.g. IEEE standard 1241). A sine wave is fitted to the measured data in least squares sense, and the residuals are analyzed further. In recent papers, it has been recognized that even more (and more precise) information can be extracted by the solution of the maximum likelihood equations. This is an improvement to the usual three-parameter and four-parameter fits. In this paper practical implementation of this algorithm is suggested. Then, theoretical background is overviewed. Further investigations lead to the statement that the same principle can be extended to any measurement which uses an excitation signal which can be described with a few parameters. A candidate for this is an exponential signal, with 3 parameters: e.g. start value, end (steady-state) value, and time constant. The maximum likelihood (ML) equations yield a solution for these too, more accurate than least squares (LS) fitting. Reasonable approximations make the ML problem solvable in practice.

1. Introduction

One of the most general principles applied in calibration is to use high-precision excitation signals and/or high-precision instrumentation. The error of the calibration measurement is desired to be by an order of magnitude smaller than that of the device under test. However, this is often a too strict requirement even when a medium-precision analog-to-digital converter is tested. Therefore, the problem is usually circumvented in ADC testing by applying a sine wave as an excitation signal to the ADC. Although parameters of the sine wave are still cumbersome to be precisely measured, the accepted procedure executes a least-squares fit to the output of the ADC. The estimated parameters of the sine wave are then used to evaluate the error samples and characterize the ADC [1].

This procedure works well, however, it is still not optimal in the sense that

- least squares is not optimal for treating quantization errors,
- ADC nonlinearities are not properly handled with the least squares fit,
- eventual overload of the ADC is not modeled by the customary LS fit,
- sine wave is not the only possibility for the excitation signal,
- the sine wave has an excess weight and improper error form for the samples close to the peaks.

We are going to tackle a part of these problems in the following sections.

1.1. Excitation Signal

The sine wave is very popular because it can be described by a few parameters only (amplitude,

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phase, frequency and maybe a dc component), furthermore it can be produced with the desired purity, and its quality is measurable by using a spectrum analyzer. Nevertheless, in ADC testing its probability density function (PDF) is not always desirable because of the large peaks at its edges:

$$f(x) = \frac{1}{\pi\sqrt{(A^2 - (x-\mu)^2)}}, \quad (1)$$

where $\mu - A < x < \mu + A$ and μ is mean value and A is amplitude of the sine wave.

These peaks correspond to the peaks of the sine wave which are very flat and therefore around these the signal does not excite the ADC with a sufficient variation. Moreover, the amplitude range is strictly limited to $(-A, A)$. A possibility to circumvent these difficulties is a systematic overload of the ADC at both ends. The overloaded samples need to be neglected in the fit, along with the samples which, after recovery, represent biased transient recovering response of the circuitry due to the previous overload. This is possible in the LS fit, as well as in maximum likelihood estimation, to be discussed below.

An important question is whether the use of a sine wave is the ultimate solution, or some other signals are still possible. Here is what is made use from the properties of the sine wave:

- it can be represented by a few parameters,
- it is rather smooth and excites the ADC at many amplitudes,
- its local behavior is different from time instant to time instant,
- the form is precisely defined by 3 parameters, and eventual changes in the parameters do not mask usual DNL/INL patterns,
- it can be generated quite simply with small error,
- usual spectral analysis can be used to supervise purity of the sine wave,
- its PDF is close to uniform (at least at the central part),
- its PDF has a known closed form, using the signal parameters,
- since the test performed at one single frequency, ADC parameters can be described at given frequency.

According to the above arguments, an attractive signal form would also be piecewise linear (e.g. triangle wave, generated using a simple integrator) which corresponds to several requirements above, except for small error, easy quality check, and changing behavior. Therefore, a piecewise linear signal is not really advisable. Instead, a possibility which allows to fulfill most above criteria is an exponential signal [7], especially when two exponential signals are applied consecutively in the two directions [12]. We will see in the following that this is a valid alternative of the sine wave, and fits into the same estimation framework.

1.2. Maximum Likelihood Estimation - A Common Framework

According to the standards, the parameters of the sine wave are determined from the least squares fit of the ADC output data [8]. This is a very simple and robust method, but it certainly does not utilize all information present in the signal. According to estimation theory, whenever quite general conditions are fulfilled, maximum likelihood estimation is the "best" in the sense that it is asymptotically unbiased, and it is asymptotically efficient. It coincides with least squares when the samples from the model (the excitation signal of which the parameters are determined) are corrupted by additive white Gaussian observation noise. Least squares generally also has favorable properties [2], but without the above conditions fulfilled, maximum likelihood outperforms it. Therefore, it is reasonable to look also at ML at least to see the ultimate performance we might want to achieve or approximate.

It is obvious that even in the case of an ideal quantizer, the observation error due to quantization is not Gaussian. Moreover, we need to handle ADC nonlinearity − this is what we would like to characterize by the test.

The signal is described in parametric form. Cosine and sine functions are used to make the equation linear in the parameters A, B:

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + C \quad (2)$$

where ω is the angular frequency and t denotes time. For a single (falling) exponential, linearity also holds:

$$x(t) = (F_0 - F_\infty)e^{-t/\tau} + F_\infty \quad (3)$$

where F_0 is the maximal of ADC input full scale range, F_∞ is final voltage of the exponential signal.

Let us denote in the following the parameters of the signal to be determined by p: $p = [A, B, C]^T$, or $p = [A, B, C, \omega]^T$, or $p = [F_0, F_\infty, \tau]^T$.

The output of the analog-to-digital converter is recorded. This has the consequence that the observations have discrete distribution. Therefore, we need to formulate the ML criterion in the discrete domain, using the quantizer characteristic. Before doing that, we also need to find a way to describe the uncertainty around the comparison levels. The usual way to describe this is noise added to the sample of the input signal before quantization. Thus, the observed discrete-time signal can be described as

$$z_k = Q(x(t_k) + n_k) \tag{4}$$

where Q denotes the quantizer characteristic (this is a deterministic function), and n_k is a Gaussian noise sample, with zero mean and variance σ: $n_k = N(0, \sigma)$. By this, we handle the quantization error properly by Q. Furthermore, somewhat erroneous measured samples become compatible with the model by this: without noise the imperfect samples would be considered impossible (probability zero), thus the ML cost function would be degenerate and cannot be optimized.

For simplicity, let us denote the vector of the comparison levels of the quantizer (T_l, $l = 1, \ldots, M-1$) by T. The number of the possible ADC outputs is $M = 2^B$.

Now we are ready to formulate the likelihood function for independent noise samples:

$$L(p, \sigma, T) = \prod_{k=1}^{N} P(z_k = s_k | p, \sigma, T), \tag{5}$$

where $P(.)$ denotes the probability of the given event, s_k is the actually measured sample at t_k, and $|$ denotes the condition. For a given set of parameters, the probability $P(z_k = s_k|p, \sigma, T)$ can be evaluated by integration of the normal distribution between the corresponding comparison levels. It cannot be given in a closed form, but numerically it is treatable. Since (5) can be evaluated, it can also be maximized via the parameters, and thus the ML estimates can be obtained. This can be done for a sine wave [4], [3], or for an exponential signal, or for a combination of positive-slope and negative-slope (rising and falling) exponentials:

$$x_1(t) = (F_{0,1} - F_{\infty,1})e^{-t/\tau} + F_{\infty,1},$$
$$F_{0,1} < F_{\infty,1}, \text{ for } t_{1,1} \le t \le t_{1,2}$$

and

$$x_2(t) = (F_{0,2} - F_{\infty,2})e^{-t/\tau} + F_{\infty,2},$$
$$F_{0,2} > F_{\infty,2}, \text{ for } t_{2,1} \le t \le t_{2,2}.$$

Notice that τ is the same in the two equations. This can be exploited in the fit, minimizing (5) [13].

1.2.1. Implementation

The maximum likelihood estimation is calculated by minimization of a non-linear function (5). Several methods can be applied, either using the gradient of the cost function (a version of Levenberg-Marquardt) or using values of the cost function only (e.g. Nelder-Mead simplex search). In our program, the transition levels were taken from the histogram, and the signal parameters were found by the Nelder-Mead method. This is usually slow, but for a few parameters as here it was fast enough and robust.

2. Fast calculations

If the number of bits in the ADC is B, there are $2^B - 1$ comparison levels to be estimated. Thus the maximization of (5) involves a large number of parameters. While this is possible in theory, the value of B is many times limited in practice to about $B = 6$-8. This is often less than we need. Moreover, the likelihood function is a nonlinear function of the parameters, thus starting values are needed to start optimizing iteration. Consequently, we need an effective way to determine the transition levels, or at least good enough starting values.

In general, histogram test is used to characterize the differential nonlinearities of the ADC [5]. This seems to be possible even in our case. The knowledge of the differential nonlinearity is equivalent to the knowledge of the comparison levels. The histogram test is a very robust procedure to obtain the transition levels. Moreover, in the case of sine waves, the so-called normalized transition levels can be calculated without the actual parameters of the sine wave [5], [6].

Concerning the information on the transition levels, one can argue that the total relevant information seems to be present in the histogram, since this contains number of the samples above and below a certain transition level, and there is no further information on these in the digital samples. At present we cannot rigorously prove but we are convinced that the histogram is (at least approximately) sufficient statistic concerning the transition levels. If so, transition levels can be determined separately from the histogram, and more involved minimization is only necessary to determine the signal and noise parameters.

However, still there is a problem: the measured histogram is modified by the PDF of the signal, thus INL can be evaluated from the histogram only by using the knowledge of the signal. On the other hand, the signal parameters (or at least a part of them) can be estimated from (5) only by knowing the INL values T. Therefore, only an iterative procedure is possible. If the following steps converge (as they do in practical cases), they lead to a reasonable maximizer parameter set:

1. determine the histogram,
2. determine the signal parameters as well as possible,
3. correct the histogram if it is necessary to achieve transition levels, using the PDF of the signal+noise,
4. minimize L by the signal parameters and σ in (5),
5. if stop criterion does not meet, go to 3 and continue.

It is worth noting that if a sine wave is applied, and the noise is not very large (e.g. $\sigma < 2\,\mathrm{LSB}$, see [5]), in step 3 the transition levels can be directly obtained from the cumulative histogram by the transformation $-A_X \cos(\pi H_c(k)) + C$, where A_x is the amplitude and C is the dc level, see [5], [6], thus iteration is not necessary. This is a consequence of the parametric form of the sine wave: its CDF does not change with changing the frequency ω or the phase ϕ.

When determining the signal parameters, the two cases (sine or exponential) need to be discussed separately.

2.1. Sine Wave

The starting values can be obtained from an LS fit to the non-overloaded samples (maybe also excluding the samples around the peaks, see [8], and the ones with recovery transients after overload). If the frequency is known, this is a linear LS problem, if not, this is nonlinear LS. The starting value for ω can be determined by using Interpolated FFT [10], [11]. Starting from the above starting values, the ML optimizer can be found.

An alternative possibility is to make a least squares fit to the measured histogram. Start from the above values and fit (1) to the histogram in LS sense, numerically. This is certainly faster than the above ML method, however, if σ is not very small, this needs numerical adjustment of the PDF to make a proper fit, since the signal PDF needs to be convolved first with the noise PDF. Moreover, although LS fit seems to be "logical", there is no guarantee how well it will perform compared to the theoretically optimal maximum likelihood estimate.

2.2. Exponential Signal

The time constant is the only parameter which nonlinearly appears in the likelihood function. As we will see later (e.g. (9) and (10)), knowledge of the end value F_∞ allows simple determination of τ, thus F_∞ can also be determined first. This offers an alternative (but equivalent) calculation.

The starting value of τ can be determined in a few simple ways.

2.2.1. Calculation from Samples

Take 4 equidistant samples at $t_1 \le t_2 \le t_3 < t_4$, respectively, which are reasonably apart from each other, and $t_2 - t_1 = t_4 - t_3 = \Delta t$. From the corresponding samples (see (3)), simple algebra gives

$$\hat{\tau} = (t_3 - t_1)\frac{1}{\ln\left(\frac{x_2 - x_1}{x_4 - x_3}\right)}. \tag{6}$$

For a double exponential, this can be done for both parts and the results can be averaged.

This is an elementary solution, however, it is sensitive to noise, since only 4 samples are used. The effect of the noise can be decreased by averaging several estimates by shifting the time instants, e.g. like:

$$\hat{\tau} = \frac{1}{K}\sum_{n=0}^{K-1} T_d \frac{1}{\ln\left(\frac{x_{t_n+\Delta t} - x_{t_n}}{x_{t_n+\Delta t+T_d} - x_{t_n+T_d}}\right)}. \tag{7}$$

Knowing the time constant, the samples can be fitted by using linear LS (see (3)), and the value of the time constant can be refined by using nonlinear LS.

2.2.2. Calculation from the histogram

An alternative solution can be built on the form of the CDF. Since the cumulative histogram is the running sum of the regular (code) histogram, its values contain the sum of the histogram values, left from the actual point, thus these are averages. Therefore, samples of the CDF can be used for reasonable calculations. E.g. for a negative slope, one can calculate:

$$F(x) = \int_{-\infty}^{x} f(z)dz = \int_{x_{\min}}^{x} \frac{C_{\mathrm{tr}}}{z - F_\infty} dz$$
$$= \left[C_{\mathrm{tr}} \ln(z - F_\infty)\right]_{x_{\min}}^{x}$$
$$= C_{\mathrm{tr}}(\ln(x - F_\infty) - \ln(x_{\min} - F_\infty)) \qquad (8)$$

for $x_{\min} \le x \le x_{\max}$. C_{tr} can be calculated from the condition $F(x_{\max}) = 1$:

$$C_{\mathrm{tr}} = \frac{1}{\ln(x_{\max} - F_\infty) - \ln(x_{\min} - F_\infty)}$$

$$F(x) = \frac{\ln(x - F_\infty) - \ln(x_{\min} - F_\infty)}{\ln(x_{\max} - F_\infty) - \ln(x_{\min} - F_\infty)}. \qquad (9)$$

$F(x_{\min})$ clearly equals 0. The only unknown is F_∞ which can be numerically determined e.g. from the equation $F(x_1) = P_1$, where P_1 can be chosen e.g. $P_1 = 0.5$, and the corresponding value of x_1 is taken from the cumulative histogram.

The solution can be made even more accurate by fitting (9) to the histogram in LS sense [9]. However, the "best" (albeit slower) solution is to return for the last refinement to the time samples, and solve the ML problem. The starting value of τ can be calculated for this as [9]

$$\tau = C_{\mathrm{tr}} t_1, \qquad (10)$$

where t_1 is the observation length from which the histogram is obtained. Since C_{tr} is a function of F_∞, see (9), determinations of F_∞ and τ are essentially equivalent.

This is again a question of sufficient statistics: the histogram contains somewhat less information with respect to the signal parameters than the samples themselves, thus it can be expected that the ML solution is somewhat more accurate than LS fit of the PDF to the histogram.

3. Numerical results

This section is devoted to present the numerical results obtained by running the method introduced. First, a simulation result is presented to illustrate the improved accuracy of the suggested method. After this, measured data are used as input of the algorithm.

3.1. Simulated Data

A sine wave is generated with amplitude 2.021 V, dc level 0V and frequency $f = 23$ Hz, $f_s = 1$ kHz. $N = 512000$ samples are generated. The ADC has 12 bits, with input range ±2.0 V. The INL of the ADC is set to the pattern shown in Figure 1. In Figure 2 the calculated normalized histogram of the ADC is plotted.

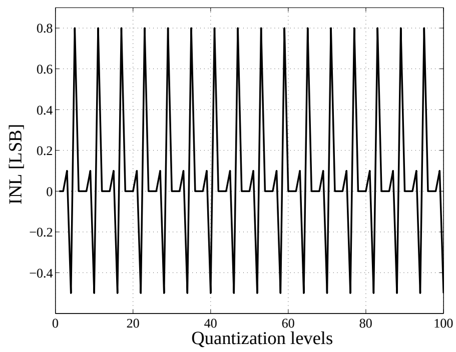

Figure 1: INL of the simulated ADC as a function of the transition level vector. The presented pattern is repeated in the whole domain.

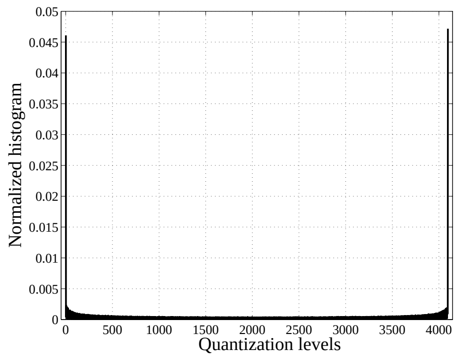

Figure 2: Normalized histogram of the ADC output data.

In the case of least squares (LS) fitting the error of the amplitude estimation is $A_{\mathrm{est}} - A = -2.8131$ LSB, and the dc error is -0.0875 LSB. The amplitude error of the presented method is $A_{\mathrm{est}} - A = -0.0072$ LSB and dc error is -0.023 LSB. Both errors are significantly decreased by using the maximum likelihood (ML) method.

3.2. 12-bit ADC, exponential excitation

The second example uses measured data in which exponential excitation signal is applied. A 12-bit

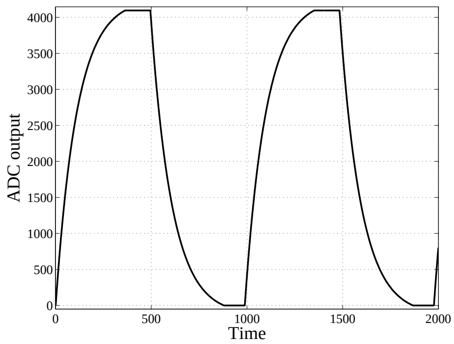

Figure 3: Exponential excitation signal. It contains alternating falling and rising parts.

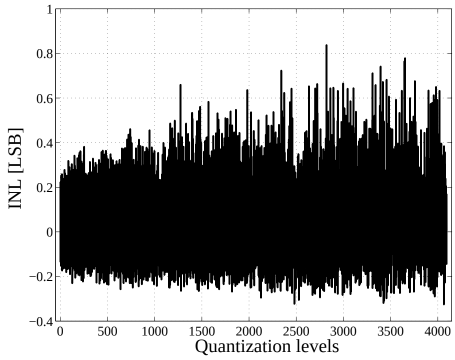

Figure 4: Estimated INL of the ADC using exponential signal as excitation.

AD converter collected $M = 1000000$ samples ($f_s = 10\,\text{kHz}$). Measuring device was a multi-function board by National Instruments: NI USB-6008. Part of the measured signal is plotted in Figure 3. Frequency of the excitation signal was 10.123 Hz. Input range of ADC was $[-1, 1]$ V (-1 V represented by code 0, 1 V represented by code 4095).

Following the algorithm in [13] fitting the histogram results in $F_{\infty,\mathrm{f}} = -1.0649\,\text{V}$ ($B_f = -132.92$) and $F_{\infty,\mathrm{r}} = 1.078\,\text{V}$ ($B_r = 4258.82$). The estimated INL can be seen in Figure 4.

LS fitting of parametric model in time domain and using the estimated INL values minimization of the ML cost function were performed. Numerical outputs are summarized in Table 1. In the estimation of $F_{0,r}$ the ML solution contains a small improvement.

Alg.	$F_{0,f}$	$F_{\infty,f}$	τ_f	σ_f
LS	4113.24	-132.58	110.97	-
	1.01 V	-1.06 V	0.011 s	
ML	4113.18	-132.25	110.93	3.69
	1.01 V	-1.06 V	0.011 s	0.0018 V
Alg.	$F_{0,r}$	$F_{\infty,r}$	τ_r	σ_r
LS	-19.04	4257.79	110.98	-
	-1.01 V	1.0795 V	0.011 s	
ML	-19.95	4256.4	110.93	4.53
	-1.01 V	1.08 V	0.011 s	0.0022 V

Table 1: Outputs of different algorithms for estimating rising and falling parts of exponential signal. ADC (12 bits) with input range [-1,1] V, f_s =10 kHz, input period: 10.123 Hz.

3.3. 8-bit ADC, Exponential and Sine Wave Excitations

In the next examples measurements reported were done using an 8-bit converter. First, a sine wave as excitation was applied, then using the same converter an exponential signal was applied at the input of ADC. The presented methods were evaluated and finally results were plotted.

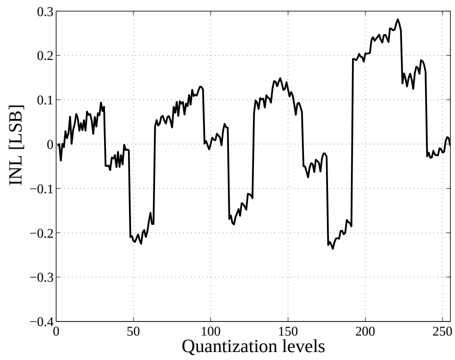

Figure 5: Measured INL of the 8-bit ADC by using of sine wave.

Parameters of the sine wave were estimated by using both the LS and ML methods. The result are summarized in Table 2.

Results of estimating of dc level are almost the same, but amplitude estimators are different.

The same AD converter was tested by using exponential signal as excitation. The INL calculated by the method proposed in [13] (B_f=-263.267 B_r=524.914). As one can see there is small difference between the estimated INL's in Figures 5 and 6, respectively.

Alg.	amplitude	dc level	σ
LS	133.94	-0.934	-
	1.0547 V	-0.0074 V	
ML	135.83	-1.0054	0.1165
	1.0695 V	-0.0079 V	0.00092 V

Table 2: Outputs of different algorithms for estimating parameters of the sine wave signal. ADC (8 bits) with input range [-1,1] V, f_s =811 Hz, input period: 11.111 Hz.

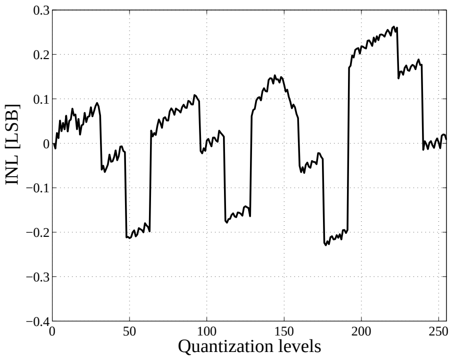

Figure 6: Measured INL of the 8-bit ADC by using of exponential signal.

Both LS and ML estimations were calculated for falling and rising parts of the exponential signals. Results are summarized in Table 3. Because of the small σ difference between outputs of the least squares fit and the maximum likelihood estimator is also small.

4. Conclusions

In this paper it is shown how it is possible to effectively extract all information from the ADC response to sinusoidal or exponential excitation test signals. The results can be improved with respect to the least squares fit. Thus, by using the maximum likelihood principle, ADC testing can be made more accurate. Using fast algorithms the result can even be obtained in a quick way.

Alg.	$F_{0,f}$	$F_{\infty,f}$	τ_f	σ_f
LS	255.19	-264.2656	300.26	-
	1.0093 V	-3.0808 V	0.0300 s	
ML	254.36	-261.57	298.78	0.37
	1.0028 V	-3.0596 V	0.0299 s	0.0029 V
Alg.	$F_{0,r}$	$F_{\infty,r}$	τ_r	σ_r
LS	-0.41	523.91	298.68	-
	-1.0032 V	3.1252 V	0.0299 s	
ML	-0.42	522.99	299.36	0.36
	-1.0033 V	3.1181 V	0.0299 s	0.0028 V

Table 3: Outputs of different algorithms for estimating rising and falling parts of exponential signal. ADC (8 bits) with input range: [-1,1] V.

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