Solvable Affine Term Structure Models

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Abstract

Pricing of contingent claims in the Affine Term Structure Models (ATSM) can be reduced to the solution of a set of Riccati-type Ordinary Differential Equations (ODE), as shown in Duffie, Pan and Singleton (2000) and in Duffie, Filipović and Schachermayer (2001). We discuss the solvability of these equations. While admissibility is a necessary and sufficient condition in order to express their general solution as an analytic series expansion, we prove that, when the factors are restricted to have continuous paths, these ODE admit a fundamental system of solutions if and only if all the positive factors are independent. Finally, we classify and solve all the consistent polynomial term structure models admitting a fundamental system of solutions.

Keywords: Affine Terms Structure Models, Riccati ODE, Lie algebra, Fundamental System of Solutions.

1 Introduction

The class of multifactor Affine Term Structure Models (ATSM hereafter) as introduced by Duffie and Kan (1996), combines some financial appealing properties:

1. The sensitivities of the zero coupon yield curve to the stochastic factors are deterministic, as discussed in Brown and Schaefer (1994);

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2. Explicit parametric restrictions, called *admissibility conditions,* grant the existence of a regular affine process, as discussed in Dai and Singleton (2000) and in Duffie, Filipović and Schachermayer (2001); in particular correlations between positive factors are allowed.

3. The pricing problem can be reduced to the solution of a system of ODE as discussed in Duffie, Pan and Singleton (2000). In fact, the explicit expression of the conditional discounted characteristic function of the factors can be specified in terms of the solutions of (deterministic) Riccati equations for any admissible ATSM (see e.g. Duffie et al. 2001).

In this paper we discuss the solvability of such Riccati ODE. Although they have been well discussed and classified in many books (see e.g. Reid 1972), in the multi dimensional context most of the interest has been devoted to a particular subset called Matrix Riccati ODE, which are completely integrable. Unfortunately, Riccati ODE arising within admissible ATSM do not belong to this class in general.

We provide a systematic classification of the necessary and sufficient conditions in order to obtain

i) an analytic series solution

ii) a Fundamental System of Solution (FSS), i.e. a closed form expression in terms of a finite number of parameters.

In particular, we find that when the factors are restricted to have continuous paths, the existence of a FSS is in contradiction with the presence of correlations between positive factors. Finally, extending the above analysis, we provide a systematic classification and solution of all general polynomial term structure models which admit a FSS.

The paper is organized as follows: in Section 2 we introduce the pricing problem in ATSM and we discuss a reduced *normal form* togheter with the admissibility issue. In Section 3 we show that the admissibility of the ATSM implies that the series expansion solution is analytic, we compute explicitly the series coefficients and we discuss the necessary and sufficient parametric restrictions for the Riccati ODE in order to admit a FSS. Finally we find the explicit solution for all consistent separable polynomial term structures. In Appendix A we review two approaches in the simplest one-dimensional case, while Appendix B shows an example of "quasi closed form" solution for ATSM not admitting a FSS.

2 Stochastic dynamics of factors and the pricing problem

The specification of an ATSM can be given in full generality even in presence of jump diffusion processes according to the approach of Duffie et al. (2001). Here we restrict our treatment to the more familiar setup of Duffie and Kan (1996) model in the specification proposed in Dai and Singleton (2000), which is essentially the same of Duffie, Pan and Singleton (2000) model when factors are restricted to have continuous paths. This restriction is adopted in order to simplify the exposition and to disclose the applicative follow-up of the above general results within the more traditional notation and framework for financial applications.

Definition 1 *A Term Structure Model is Affine (ATSM) if the short interest rate r_t is an affine combination of factors*

$$r_t = \delta^0 + \delta' Y_t, \quad t \geq 0$$

for some $\delta^0 \in \mathbb{R}$ and $\delta \in \mathbb{R}^n$, and the dynamics of the factors under the risk neutral measure $\mathbb{Q}$ satisfies the following SDE:

$$\begin{aligned} dY_t &= \mathcal{K}(\theta - Y_t)\,dt + \Sigma diag\left[(\alpha_i + \beta_i' Y_t)^{1/2}\right] dW_t, \quad t \geq 0, \\ Y_0 &= y \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where W_t is an n-dimensional standard Brownian motion and

$$\begin{aligned} \theta, \alpha &\in \mathbb{R}^n, \\ \mathcal{K}, \beta &\in M_n, \\ \Sigma &\in GL_n \end{aligned}$$

where α_i indicates the $i-th$ element of the vector α, β_i indicates the $i-th$ column of matrix β, and given a vector $z \in \mathbb{R}^n$, $diag\,(z_i) \in M_n$ is the diagonal matrix with the elements of the vector z along the diagonal (the prime $'$ denotes transposition). Σ is required to be invertible as in Duffie and Kan (1996) excluding the presence of deterministic factors.

The rank of β is defined to be to $m \leq n$: in the notation of Dai and Singleton (2000), m classifies the families $\mathbb{A}_m(n)$ of admissible models parametrized by m, n. Without loss of generality we assume that the upper left minor of order m is non singular.

Being interested in pricing contingent claims, we follow Duffie, Pan and Singleton (2000) and compute the discounted characteristic function of the factors Y_t, conditional on the information at time $t \leq T$:

$$\Psi_Y(v, Y_t, t, T) = \mathbb{E}_t\left[\exp\left(-\int_t^T r(s)\,ds\right)\exp(v'Y_T)\right], \qquad v \in \mathbb{C}^n.$$

Under technical integrability conditions, Duffie, Pan and Singleton (2000) have shown that the previous function can be explicitly written as an exponential affine function of the factors:

$$\Psi_Y(v, Y_t, t, T) = \exp\left(\mathcal{A}(t) + \mathcal{B}(t)'\,Y_t\right), \quad t \leq T, \tag{2}$$

where $\mathcal{A} \in \mathbb{C}, \mathcal{B} \in \mathbb{C}^n$ satisfy the following complex-valued backward ODE

$$\begin{aligned} -\frac{d}{dt}\mathcal{B}(t) &= -\mathcal{K}'\mathcal{B}(t) + \frac{1}{2}\sum_{i=1}^n \left[(\Sigma'\mathcal{B}(t))_i\right]^2 \beta_i - \delta, \\ -\frac{d}{dt}\mathcal{A}(t) &= -\theta\mathcal{K}'\mathcal{B}(t) + \frac{1}{2}\sum_{i=1}^n \left[(\Sigma'\mathcal{B}(t))_i\right]^2 \alpha_i - \delta^0, \end{aligned} \tag{3}$$

with boundary conditions

$$\begin{aligned} \mathcal{B}(T) &= v, \\ \mathcal{A}(T) &= 0. \end{aligned}$$

Remark 1 *An important contingent claim is the zero-coupon bond, whose price is given by ($t \leq T$)*

$$\begin{aligned} P(t,T) &= \mathbb{E}_t\left[\exp\left(-\int_t^T r(s)\,ds\right)\right] \\ &= \exp\left(\mathcal{A}(t) + \mathcal{B}(t)\,Y_t\right), \end{aligned}$$

where $\mathcal{A}, \mathcal{B}$ satisfy eq.s (3) with boundary conditions

$$\begin{aligned} \mathcal{B}(T) &= 0, \\ \mathcal{A}(T) &= 0. \end{aligned}$$

When dealing with bonds, the sensitivities $\mathcal{A}, \mathcal{B}$ are usually expressed as functions of time to maturity: $\tau = T - t$. In fact when parameters are time independent, prices depend only on the combination $\tau = T - t$ and Riccati ODE become a genuine initial value problem at $\tau = 0$. Under the change of variables $t \rightarrow \tau$ the left hand side of eq.(3) gets a factor -1.

Remark 2 *An alternative way to define the ATSM consists in requiring that the characteristic function in eq.(2) has the exponential affine form as in Duffie et al. (2001).*

Remark 3 *In the alternative approach of Elliott and van der Hoek (2001), it is shown that the solution of the Riccati (3) can be expressed through the expected value, under the forward measure, of the stochastic Jacobian, and such expected value turns out to be deterministic for ATSM. As discussed in Grasselli and Tebaldi (2004), however, this approach applied to multifactor ATSM leads to a non linear equation which is equivalent and perfectly consistent with our algebraic results when dealing with the solvability issue.*

2.1 Symmetry reduction and Normal Form for ATSM

Each ATSM is identified by the following vector of parameters (δ^0, δ, $\mathcal{K}$, θ, Σ, $\{\alpha_i, \beta_i\}_{1\le i\le n}$). As discussed in Dai and Singleton (2000), an affine change of variables:

$$Y \to X = LY + \vartheta \qquad L \in GL_n, \vartheta \in \mathbb{R}^n \tag{4}$$

leaves unaffected all the prices, while the parameters are changed according to:

$$(\delta^0, \delta, \mathcal{K}, \theta, \Sigma, \{\alpha_i, \beta_i\}_{1\le i\le n}) \to$$

$$(\delta^0 - \delta' L^{-1}\vartheta, \left(L^{-1}\right)' \delta, L\mathcal{K}L^{-1}, L\theta + \vartheta, L^{-1}\Sigma, \left\{\left(\alpha - \beta' L^{-1}\vartheta\right)_i, \left(\beta' L^{-1}\right)_i\right\}_{1\le i\le n})$$

It is thus possible to reduce the discussion of a whole class of models, those differing at most for an affine change of variables, to a single representative element which we will call hereafter *normal form*.

Let us now discuss the change of variables that relates a generic ATSM with the corresponding normal form.

Definition 2 *(Normal Form) Consider a symmetry transformation (L, ϑ) of the type given in eq.(4). Let us fix $L = \Sigma^{-1}$ and $\vartheta \in \mathbb{R}^n$ any solution to the system of equations*

$$\begin{aligned} \beta_i' \Sigma \vartheta &= \alpha_i, \ i = 1, .., m, \\ \left\{\Sigma^{-1}\mathcal{K}\left(\theta - \vartheta\right)\right\}_i &= 0, \ i = m+1, .., n. \end{aligned}$$

Such a transformation maps the original factors' dynamics (1) into the Normal Form ATSM, whose factors' dynamics becomes

$$\begin{aligned} dX_t &= \left(AX_t + A^0\right) dt + diag\left[S_{ii}^{1/2}\right] dW_t, \quad t \ge 0, \\ S_{ii} &= \left(C_i X_t + C_i^0\right), \\ X_0 &= x, \end{aligned} \tag{5}$$

(C_i denotes the $i-th$ row of the matrix C, $m = rank\,(C)$) while the short rate is given by

$$r_t = \gamma^0 + \gamma' X_t,$$

where the parameters $\phi = (\gamma, \gamma^0, A, A^0, C, C^0)$ are defined as follows:

$$\begin{aligned}
X_t &= \Sigma^{-1} Y_t, \quad t \geq 0, \\
\gamma^0 &= \delta^0 - \delta' \Sigma \vartheta, \\
\gamma &= \Sigma' \delta, \\
A &= -\Sigma^{-1} \mathcal{K} \Sigma \in M_n, \\
A^0 &= \Sigma^{-1} \mathcal{K} (\theta - \vartheta) \in \mathbb{R}^n, \\
C &= \beta' \Sigma \in M_n, \\
C^0 &= \alpha - \beta' \Sigma \vartheta,
\end{aligned}$$

where $\left(C^0\right)_i = 0$, $i = 1, .., m$ and $\left(A^0\right)_i = 0, i = m+1, .., n$

The Riccati ODE providing the generalized conditional characteristic function

$$\begin{aligned}
\Psi_X (u, X_t, t, T) &= \exp\left(\mathcal{V}^0(t) + \mathcal{V}(t)' X_t\right), \quad t \leq T, \\
u &= \Sigma v
\end{aligned}$$

become ($i = 1, ..n$):

$$\begin{aligned}
-\frac{d}{dt}\mathcal{V}_i(t) &= \sum_{j=1}^{n} (A')_{ij}\, \mathcal{V}_j(t) + \frac{1}{2}\sum_{j=1}^{n} (C')_{ij}\, \mathcal{V}_j^2(t) - \gamma_i, \qquad \mathcal{V}(T) = u, \quad (6) \\
-\frac{d}{dt}\mathcal{V}^0(t) &= \sum_{j=1}^{n} A_j^0 \mathcal{V}_j(t) + \frac{1}{2}\sum_{j=1}^{n} \left(C_j^0\right) \mathcal{V}_j^2(t) - \gamma^0, \qquad \mathcal{V}^0(T) = 0,
\end{aligned}$$

with

$$\begin{aligned}
\gamma &= \Sigma' \delta \in \mathbb{R}^n, \\
\gamma^0 &= \delta^0 - \delta' \Sigma \vartheta \in \mathbb{R}.
\end{aligned} \tag{7}$$

Remark 4 *The zero-coupon bond price expression in terms of the reduced factors will be:*

$$\begin{aligned}
P(t, T) &= \exp\left(\mathcal{V}^0(t) + \mathcal{V}'(t)\, X_t\right) \\
\mathcal{V}_i(T) &= 0 \quad i = 0, .., n.
\end{aligned} \tag{8}$$

2.2 Admissibility

So far we did not address the issue of the *admissibility,* introduced in Dai and Singleton (2000). Duffie et al. (2001) completely characterized the conditions for admissibility in the "canonical" state space $\mathcal{D} = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$. Observe that an affine transformation of the factors does not affect the existence of their flow, but changes their domain of admissibility. By introducing the normal form, we reduce any model to an equivalent one whose natural domain is the canonical state space $\mathcal{D}$. Then we can identify the parametric restrictions imposed by the admissibility conditions on the normal form. The results are essentially the same obtained for the class of canonical maximal ATSMs, as originally defined in Dai and Singleton (2000): on the other hand the introduction of the normal form will clarify the relationship between their classification and the results of Duffie et al. (2001) about the class of admissible regular affine processes, when processes are constrained to have continuous paths.

Let us begin with some definitions.

Definition 3 *A time-homogeneous Markov process with state space* $\mathcal{D} = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$ *(the first* m *components of the process correspond to the positive factors) and semigroup* (P_t)

$$P_t f(x) = \int_{\mathcal{D}} f(\xi) p_t(x, d\xi)$$

is called regular affine if, for every $t \in \mathbb{R}_+$,

- *the characteristic function* $f_u(x)$ *of* $p_t(x, .)$ *has exponential-affine dependence on* x,
- *the process is stochastically continuous, and*
- *the right-hand derivative*

  $$\partial_t^+ P_t f_u(x) \mid_{t=0} = \mathcal{A} f_u(x)$$

  exists, for all $x \in \mathcal{D}$ *and is continuous in* $u = 0$ *(see Duffie et al. 2001 for technical details). By definition* $\mathcal{A}$ *is the infinitesimal generator of the semigroup.*

Duffie et al. (2001) give the necessary and sufficient conditions on the form of the infinitesimal generator in order to guarantee that the process is regular affine in the canonical domain: in this case the parameters are said to be *admissible.*

Proposition 4 *(Duffie, Filipović and Schachermayer 2001) An ATSM is admissible in the domain* $\mathcal{D}$ *iff the generator* $\mathcal{A}$ *for any* $f \in C_c^2(\mathcal{D})$*(the Banach space of* $f \in C^2(\mathcal{D})$ *with compact support) has the following functional form:*

$$\begin{aligned}\mathcal{A}f(x) &= \frac{1}{2}\sum_{k,l=1}^{n}\left(\phi_{kl}^{0}+\sum_{i=1}^{m}\phi_{kl}^{i}x_i\right)\frac{\partial^2 f(x)}{\partial x_k \partial x_l}+ \\ &\quad +\sum_{k=1}^{n}\left(\Omega_k^0+\sum_{i=1}^{n}\Omega_{ki}x_i\right)\frac{\partial f(x)}{\partial x_k}-\left(\eta^0+\sum_{i=1}^{n}\eta_i x_i\right)f(x)\end{aligned}$$

where:

- *drift matrix* Ω

$$\Omega=\begin{pmatrix}\Omega_{m\times m}^{BB} & 0_{m\times(n-m)} \\ \Omega_{(n-m)\times m}^{DB} & \Omega_{(n-m)\times(n-m)}^{DD}\end{pmatrix}, \tag{9}$$

  and the out of diagonal elements of Ω^{BB} *are restricted to be nonnegative,*

- $\Omega^0 \in \mathcal{D}$,

- $\phi^i, \kappa \in Sem^n$ *the set of semidefinite positive matrices,* κ *is non zero only in the lower diagonal square block,* κ^{DD}*, of order* $n-m$*, while* $\phi_{k,l}^i = \alpha_i \delta_{k,i}\delta_{i,l}$ *for* $i,k,l = 1..m$ *and* $\alpha_i \in \mathbb{R}$.

Definition 5 *An ATSM will be said "admissible in its natural domain" if the corresponding Normal Form ATSM is admissible within the canonical domain* $\mathcal{D}$.

Remark 5 *Notice that the notion of canonical domain is not invariant under affine transformations of the factors, to the opposite of the definition of admissibility in its natural domain. In particular, only the domain of definition of the normal factors* X_t *is restricted to be the canonical one* $\mathcal{D}$*, while the domain of the factors*Y_t *is* $\mathcal{D}^Y = L^{-1}(\mathcal{D}-\vartheta)$*. Observe that our definition of admissibility significantly enlarges the set of well defined models we can deal with; in fact, L can explicitly depend on time (time-inhomogeneous models) and even on some external sources of risk (independent of the factors). Hence this extension significantly extends the possible parametrizations for the observed dynamic correlations among the yields without leaving the class of ATSM.*

We are thus ready to state the parametric restrictions on the normal form:

Proposition 6 *The normal form corresponding to an admissible ATSM in its natural domain is specified by the parameter set* $\phi = (\gamma, \gamma^0, A, A^0, C, C^0)$ *with:*

- *the drift matrix A as in eq.(9)*
- $A^0 \in \mathcal{D}$
- C *and* C^0 *given by:*

$$\begin{aligned} C &= \begin{pmatrix} \mathbb{I}_{m\times m} & 0_{m\times(n-m)} \\ C^{DB}_{(n-m)\times m} & 0_{(n-m)\times(n-m)} \end{pmatrix}, \\ C^0 &= \begin{pmatrix} 0_{m\times 1} \\ 1_{(n-m)\times 1} \end{pmatrix}, \end{aligned} \tag{10}$$

- $\gamma^0 \in \mathbb{R}$, $\gamma \in \mathbb{R}^n$.

Proof. The conditions on the drift are identical to those discussed in Dai Singleton (2000) and in Duffie et al. (2001). In our framework the conditional covariance has the form

$$\begin{aligned} Cov_t\left(dX^k, dX^l\right) &= \left(\sum_{i=1}^{m} C_{ki}X^i + C_k^0\right)\delta_{l,k} \\ l,k,i &= 1...n, \end{aligned}$$

while admissibility conditions imply:

$$Cov_t\left(dX^k, dX^l\right) = \sum_{i=1}^{m} \phi_{kl}^i X^i + \kappa_{kl}, \quad t \geq 0.$$

The restrictions on C and C^0 are then implied by imposing that the two expressions are equal. Without loss of generality, we can assume $\alpha_i = 1$, $i = 1,..,m$ in $\phi_{k,l}^i$ and that the non zero eigenvalues of the matrix κ are equal to 1. It turns out that

$$\begin{aligned} C_{k,i}\delta_{k,l} &= \phi_{k,l}^i = \delta_{i,l}\delta_{i,k} \quad k,i,l = 1,..,m, \\ C_{k,i}\delta_{k,l} &\in Sem^n \rightarrow C_{k,i} \geq 0 \quad k = m+1,..,n \quad i = 1,..,m, \end{aligned}$$

thus the upper m dimensional matrix C^{BB} must be the identity, while the lower block C^{DB} is an unconstrained combination of nonnegative elements. The conditions on constant terms read

$$C_{l+m}^0\delta_{l,k} = \kappa^{DD}, \; l,k = 1,..,n-m$$

and thus $C^0_{l+m} \geq 0$, since $\kappa \in Sem^n$. ■

The *canonical* form, defined in Dai and Singleton (2000), fits precisely in such subset of models, thus we have the following relation:

$$\{\text{Canonical Form Models}\} \subset \{\text{Admissible Models}\} \cap \{\text{Normal Form Models}\}.$$

The specification of the additional restrictions which completely specify the class of canonical models, among the normal admissible models, are required only for the uniqueness of the econometric specification: for their discussion we refer to Appendix C in Dai and Singleton (2000).

3 Solvability of the Riccati ODE

In this section we discuss the *existence of a closed form solution* for the Riccati equations (6). As a first fact observe that from the analytical point of view results on existence, uniqueness and regularity of the solutions of Riccati ODE are easily deduced by the Cauchy's theorem (local and global) and are essentially discussed in Duffie et al. (2001) as we will clarify later. On the contrary, the existence of a "closed form" solution deserves a deeper analysis. In fact the definition itself of a closed form solution is a subject of discussion: from the analytic point of view proving the existence of an analytic function which solves the ODE is completely satisfactory, from the geometric and mechanical point of view the existence of a "closed form" is usually related with the integrability of the flow, which implies the *existence of a change of coordinates that linearizes the solution flow.* Remarkably in our financial context there's a natural definition of a "closed form solution": we search the conditions under which it is possible to *identify any solution of the ODE using a finite number of parameters.* It is clear that from the econometric point of view the fulfillement of this property extends crucially the range of econometric estimation techniques which can be used. A parametrization of the solution set involving only a finite number of parameters is called a Fundamental System of Solutions (FSS) and the Lie-Scheffers theorem, as we will see in the last part of the section, identifies the parametric restrictions on the ODE in order to grant the existence of such an expression for the solution. Within this section we will assume time independent coefficients, thus we will always consider the time to maturity parametrization $\tau = T - t$, and we will concentrate our interest on the solutions for $\mathcal{V}(\tau) \in \mathbb{R}^n$. As usual the solution for $\mathcal{V}^0(\tau)$ can be obtained through explicit integration.

3.1 Series Expansion Solution

In this subsection we recall some basic facts from elementary analysis which allow us to provide a series expansion for the solution of (6) in the general (non constrained) case. For any ODE in normal form whose vector field is analytic, the Cauchy-Kovalevsky theorem (see e.g. Walcher 1991, p. 27) ensures the existence of a *local* analytic solution of (6) which can be expanded in a power series with respect to time to maturity. The series converges in a neighborhood of the initial point $\mathcal{V}(0) = 0 \in \mathbb{R}^n$:

$$\mathcal{V}(\tau) = \sum_{k=0}^{\infty} g_k(\mathcal{V}(0))\,\tau^k, \tag{11}$$

and the expression of the coefficients $g_k(\mathcal{V}(0))$, $k \in \mathbb{N}$, can be recursively deduced from the defining equation (6). The vector field for any ATSM is quadratic, thus while the local theorem holds without any restriction, the global extension is not granted, but again referring to Duffie et al. (2001) we can state the following

Proposition 7 *The solution of any Riccati ODE arising from a ATSM is analytic in the whole natural domain of definition and thus the series (11) is convergent for any time to maturity τ iff the corresponding ATSM is admissible in the same domain.*

Proof. For existence and uniqueness of the flow see Duffie et al. (2001), Prop. 6.1. The analyticity of the solution is implied by the restriction to ATSM with continuous paths. In this case the vector field is quadratic and *a fortiori* analytic, henceforth also the solution will possess the same property ■

We provide now the explicit methodology to determine the series expansion coefficients.

Corollary 8 *The coefficients $g_k(\mathcal{V}(0))$ in eq.(11) are recursively defined by (i=1,..,n):*

$$\begin{aligned}
(g_0)_i &= (\mathcal{V}(0))_i \\
(g_1)_i &= \sum_{j=1}^{n} (A')_{ij} (\mathcal{V}(0))_j + \frac{1}{2}\sum_{j=1}^{n} (C')_{ij} (\mathcal{V}(0))_j - \gamma_i \\
&\dots \\
(g_{k+1})_i &= \frac{1}{k+1}\left\{\sum_{j=1}^{n} A'_{ij}(g_k)_j + \frac{1}{2}\sum_{j=1}^{n}(C')_{ij}\left[\sum_{n,l:n+l=k+1}(g_n)_j (g_l)_j\right]\right\}.
\end{aligned}$$

Proof. The expression for the coefficients is obtained computing the iterated derivatives of the ODE expression (6). ■

Numerical computation of the series expansion coefficients is one among the possible approaches to determine the solution for a generic admissible ATSM. From our numerical checks it appears that the computation of a truncated series is by far the most efficient numerical method. It overperforms numerical integration using the Runge-Kutta finite difference approach and the truncation error of the polynomial approximation becomes irrelevant for any reasonable maturity with a mild number of coefficients. The truncation error can be easily determined using the residual estimate for analytic functions.

3.2 Existence of a Fundamental System of Solutions

As we discussed in the previous section, the admissibility condition provides strong parametric restrictions on the possible ATSM. The problem now is to address the following important question: under which parametric restrictions can the model be solved in *closed form*?

As a first step we observe that, due to the admissibility conditions, the last $m - n$ equations are linear and can be solved independently of the first m nonlinear equations. Once their solution is given, the system of the first m equations becomes a non homogeneous quadratic system.

Before proceeding in the investigation of the non linear quadratic equations we must formalize the notion of Fundamental System of Solutions (FSS) which is due to Lie (see e.g. Walcher 1986):

Definition 9 *An ODE is said to possess a Fundamental System of Solutions (FSS) if it is possible to express the solution as an analytic function Φ of a finite number d of special solutions $\mathcal{V}^1(\tau), ..., \mathcal{V}^d(\tau)$ and as a finite number r of parameters $k^1(\mathcal{V}(0)), ..., k^r(\mathcal{V}(0))$ depending on the initial conditions:*

$$\mathcal{V}(\mathcal{V}(0), \tau) = \Phi\left(\mathcal{V}^1(\tau), ..., \mathcal{V}^d(\tau), k^1(\mathcal{V}(0)), ..., k^r(\mathcal{V}(0))\right)$$

The econometric follow up of the above definition is clear: such property allows for a fully parametric identification of the solution in terms of a finite number of parameters. We give now some examples in order to clarify the concept of FSS.

Example 10 *(Linear ODE) Any linear (possibly non homogeneous) ordinary differential equation admits a FSS. Consider for example the ODE (6) associated to a constant volatility n dimensional model ($C = 0$): in this case* *the function Φ turns out to be affine, and any solution can be written in the form*

$$\begin{aligned} & \Phi^{Aff}\left(\mathcal{V}^{P}(\tau), \mathcal{V}^{1}(\tau), ..., \mathcal{V}^{n}(\tau), k^{1}(\mathcal{V}(0)), ..., k^{n}(\mathcal{V}(0))\right) \\ = & \mathcal{V}^{P}(\tau) + \sum_{i=1}^{n} k^{i}(\mathcal{V}(0))\left(\mathcal{V}^{i}(\tau) - \mathcal{V}^{P}(\tau)\right) \end{aligned}$$

where $\mathcal{V}^{P}(\tau)$ is a particular solution to the non homogeneous equation, while the differences $\mathcal{V}^{i}(\tau) - \mathcal{V}^{P}(\tau)$, $i = 1, .., n$ are solutions of the corresponding homogeneous equation.

Example 11 *(One dimensional Riccati) The simplest non linear example of ODE admitting a FSS is the one dimensional Riccati equation. Consider for example the ODE (6) when $n = 1$ and $C \neq 0$. It is possible to verify (a more constructive proof will be provided in Appendix A) that any solution to this equation can be obtained in terms of three particular solutions, $\mathcal{V}^{P1}(\tau), \mathcal{V}^{P2}(\tau), \mathcal{V}^{P3}(\tau)$, corresponding to initial conditions $\mathcal{V}^{P1}(0), \mathcal{V}^{P2}(0), \mathcal{V}^{P3}(0)$. In fact, for any generic solution $\mathcal{V}(\tau)$ with initial condition $\mathcal{V}(0)$, the following FSS can be found:*

$$\begin{aligned} \mathcal{V}(\tau) & = \Phi^{CIR}\left(\mathcal{V}^{P1}(\tau), ..., \mathcal{V}^{P3}(\tau), k(\mathcal{V}(0))\right) \quad (12) \\ & = \mathcal{V}^{P2}(\tau) + \left[k(\mathcal{V}(0)) \frac{\mathcal{V}^{P3}(\tau) - \mathcal{V}^{P1}(\tau)}{\mathcal{V}^{P3}(\tau) - \mathcal{V}^{P2}(\tau)} - 1\right]^{-1} \left(\mathcal{V}^{P2}(\tau) - \mathcal{V}^{P1}(\tau)\right) \end{aligned}$$

It is interesting to understand the geometric and algebraic origin of these FSS; this can be done within the Lie group theory. Vector fields defining ODE are considered as elements of the Lie algebra which is defined as follows:

Definition 12 *Let $\mathbb{V}$ be a finite dimensional real or complex vector space and $U \subset \mathbb{V}$ open and nonempty. By $A(U, \mathbb{V})$ we denote the vector space of all analytic maps from U into $\mathbb{V}$. With the bracket defined by*

$$[f(\mathcal{V}), g(\mathcal{V})] = \nabla g(\mathcal{V}) f(\mathcal{V}) - \nabla f(\mathcal{V}) g(\mathcal{V}), \quad \mathcal{V} \in \mathbb{V},$$

for $f, g \in A(U, \mathbb{V})$ we have that $(A(\mathbb{V}, \mathbb{V}), [,])$ is a Lie algebra.

Remarkably the Lie-Scheffers theorem provides a direct verification test in order to deduce whether an ODE possesses a FSS in terms of the vector fields generating the ODE.

Theorem 13 *(Lie Scheffers theorem, see e.g. Walcher 1991) The ODE* $\frac{d}{d\tau}\mathcal{V} = F(\mathcal{V},\tau)$ *admits a Fundamental System of Solutions if and only if there exist* $f_1,..,f_r \in A(\mathbb{V},\mathbb{V})$ *and continuous parameters* $\alpha_1,...,\alpha_r : \mathbb{R}\supset I \to \mathbb{R}$, *with* $r < +\infty$, *such that*

$$F(\mathcal{V},\tau) = \sum_{i=1}^{r} \alpha_i(\tau) f_i(\mathcal{V}) \tag{13}$$

lie within a finite dimensional subalgebra $L \subset A(\mathbb{V},\mathbb{V})$.

Example 14 *(Linear ODE continued) Consider first a linear non homogeneous ordinary differential equation of order* n:

$$\frac{d}{d\tau}\mathcal{V} = A\mathcal{V} - \gamma, \quad A \in M_n.$$

Put $\gamma = 0$, *then the corresponding finite dimensional Lie subalgebra of linear vector fields is isomorphic to* $\{M_n, [,]_c\}$, *the Lie Algebra of square matrices* M_n *equipped with the commutator brackets*

$$[A,B]_c = AB - BA, \quad A,B \in M_n.$$

The matrix exponential map at time τ:

$$\begin{aligned} \exp: \quad & M_n \times \mathbb{R}_+ \to GL_n \\ & (A,\tau) \longmapsto \exp(\tau A) \end{aligned}$$

maps any linear vector field $A\mathcal{V}$ *to the FSS of the corresponding ODE:*

$$\begin{aligned} \mathcal{V}(\tau) &= \Phi^{Lin}\left(\exp(\tau A)e_1,...,\exp(\tau A)e_n, \mathcal{V}_1(0),...,\mathcal{V}_n(0)\right) \\ &= \sum_{i=1}^{n} \exp(\tau A) e_i \mathcal{V}_i(0) \end{aligned}$$

where e_i *are the elements of the canonical basis.*

The non homogeneous case ($\gamma \neq 0$) *is a trivial extension: given a particular solution* $\mathcal{V}^P(\tau)$, *for any other solution* $\mathcal{V}(\tau)$, *the difference* $\mathcal{V}(\tau) - \mathcal{V}^P(\tau)$ *solves the related homogeneous equation, thus:*

$$\begin{aligned} \mathcal{V}(\tau) &= \mathcal{V}^P(\tau) + \Phi^{Lin}\left(\mathcal{V}_1(\tau) - \mathcal{V}^P(\tau),...,\mathcal{V}_n(\tau) - \mathcal{V}^P(\tau), k_1(\mathcal{V}(0)),...,k_n(\mathcal{V}(0))\right) \\ &= \Phi^{Aff}\left(\mathcal{V}^P(\tau), \mathcal{V}_1(\tau),...,\mathcal{V}_n(\tau), k_1(\mathcal{V}(0)),...,k_n(\mathcal{V}(0))\right) \end{aligned}$$

The previous example plays a crucial role, in fact it can be shown that every finite dimensional Lie Algebra is isomorphic to the Lie subalgebra of the general linear group GL_n for a finite dimensional vector space V, where $n = dimV$ (see e.g. Duistermaat and Kolk 1999 pg.88). As a consequence, the Lie-Scheffers theorem provides a constructive and powerful scheme in order to produce FSS and to linearize the flow of the ODE: in other words, the existence of a finite dimensional Lie subalgebra implies the existence of an isomorphic system of coordinates under which the ODE become linear. This system is the one which realizes the Lie algebra of vector fields as a Lie subalgebra of matrices GL_n (linear representation).

Example 15 *(One dimensional Riccati continued) The Riccati vector field can be expressed as a linear combination of vector fields generated by polynomials of degree smaller than or equal to* 2*, which is well known to close within the Lie algebra* $SL(2,\mathbb{R})$*. In the Appendix A we review the linearization procedure and the derivation of the FSS for* Φ^{CIR}*.*

Remark 6 *A direct computation of the Lie commutator between two generic polynomials of degree equal to* 2*, in dimension* $n > 1$*, shows that the algebra does not close. Thus the unrestricted algebra of all polynomial vector fields of degree smaller than or equal to* 2*, for* $n > 1$*, generates the infinite dimensional algebra of all polynomial vector fields! In particular, the Riccati equations arising from an admissible ATSM without any parametric restrictions do generate an infinite dimensional Lie algebra.*

We shall now discuss the parametric conditions on the drift matrix A in (6) such that the quadratic ODE corresponds to an admissible ATSM admitting a FSS, and to this aim we need some new concepts: we follow essentially Walcher (1991), who well discussed the most general setup under which it is possible to find a FSS for a quadratic ODE.

Let us consider a particular subalgebra of $A(\mathbb{V},\mathbb{V})$, namely the (infinite dimensional) algebra of polynomial functions $Pol(\mathbb{V})$. We can define $P_k \subset Pol(\mathbb{V})$ as the subspace of homogeneous polynomials of degree k, for $k \geq 0$. Since $[P_j, P_k] \subset P_{j+k}$ for all $j, k \in \mathbb{N}$, it follows that $Pol(\mathbb{V})$ has the natural polynomial grading defined on it, and

$$Pol(\mathbb{V}) = \oplus_{k\in\mathbb{N}} P_k$$

is also called *graded subalgebra* of $A(\mathbb{V},\mathbb{V})$ (see Walcher 1991, p. 118). The subspace P_0 contains all constant maps and can be identified with $\mathbb{V}$, while P_1 can be identified with $Hom(\mathbb{V},\mathbb{V})$.

Remark 7 *It is easy to verify that $P_0 \oplus P_1$ is always a closed subalgebra of $Pol(\mathbb{V})$: this reflects the fact that linear differential equations admit a fundamental system of solutions.*

We are interested in finite dimensional subalgebras of $A(\mathbb{V},\mathbb{V})$, and in particular in graded subalgebras of the type

$$L = L_0 \oplus ... \oplus L_k,$$

with $L_i \subset P_i$, $j = 0, ..., k$, also called *graded transitive subalgebras.*

Hel Braun, in an unpublished manuscript (see Walcher 1991 p. 124), proved that it is possible to solve a Riccati equation of the form:

$$\frac{d}{d\tau}\mathcal{V} = -\gamma(\tau) + A(\tau)\mathcal{V} + \sum_{i=1}^{m} c_i(\tau) q_i(\mathcal{V}),$$

where: $c_1, ..., c_m : \mathbb{R} \supset I \to \mathbb{R}$ are continuous, $\gamma(\tau) \in \mathbb{V} = L_0$, $q_1(\mathcal{V}), ..., q_m(\mathcal{V}) \in L_2$, and $A(\tau) \in L_1$ lie in a graded (finite dimensional) transitive subalgebra of the form:

$$L = L_0 \oplus L_1 \oplus L_2,$$

also called *short graded subalgebra*. It is possible to show that each finite dimensional subalgebra of $Pol(\mathbb{V})$ is reducible to a suitable short graded algebra. The crucial result in order to characterize the set of irreducible subalgebras that are compatible with an admissible ATSM is the proposition that provides the definition of L_1, given the subalgebras L_0 and L_2: for the proof see Walcher (1991).

Proposition 16 *a) Let $L_0 = \mathbb{V}$, $L_2 = \{q_i(\mathcal{V}), i = 1, .., m\}$ and*

$$L_1 = \{A \in Hom(\mathbb{V},\mathbb{V}), \exists \mu \in \mathbb{R}^m \text{ s.t. } Q(\mathcal{V}, A\mathcal{V}) = AQ(\mathcal{V}) + Q(\mathcal{V})\mu \text{ for all } \mathcal{V} \in \mathbb{V}\} \tag{14}$$

where:

$$\begin{aligned} Q(\mathcal{V}) &= \sum_{i=1}^{m} c_i(\tau) q_i(\mathcal{V}), \\ Q(\mathcal{V}, \mathcal{W}) &= \frac{1}{2}\{Q(\mathcal{V}+\mathcal{W}) - Q(\mathcal{V}) - Q(\mathcal{W})\}; \end{aligned}$$

then

$$L = L_0 \oplus L_1 \oplus L_2$$

is a transitive short graded subalgebra of $Pol(\mathbb{V})$,

b) Conversely, let $L = L_0 \oplus L_1 \oplus L_2$, be a transitive subalgebra of $Pol(\mathbb{V})$. Then L is the algebra given in part a).

Now we have all the elements to find the main result, i.e. the parametric restrictions on the matrix A in (6) in order to obtain a fundamental system of solutions under the admissibility restrictions:

Theorem 17 *The Riccati equations (6) admit a FSS if and only if the matrix A^{BB} is diagonal.*

Proof. If the matrix A^{BB} is diagonal then, due to the special diagonal form of C^{BB} in (10), the first m equations associated to the positive factors become independent and reduce to a sequence of one dimensional problems, where the algebra is well known to close.

Let us prove now the converse.We identify $L_0 = \mathbb{R}^m$,

$$L_2 = \left\{ \sum_{i=1}^{m} \alpha_i e_i \left(\mathcal{V}_i^B \right)^2 ; \alpha_i \in \mathbb{R} \right\},$$

where e_i the $i-$th element the canonical basis. From the previous proposition the set L_1 will be given by (14), i.e.

$$L_1 = \left\{ A^{BB} \in M_m : \exists \mu \in \mathbb{R}^m \quad s.t. \quad Q\left(\mathcal{V}^B, A^{BB}\mathcal{V}^B\right) = A^{BB} Q\left(\mathcal{V}^B\right) + Q\left(\mathcal{V}^B\right)\mu \right\},$$

where:

$$\begin{aligned} Q\left(\mathcal{V}^B\right) &= \sum_{i}^{m} \alpha_i e_i \mathcal{V}_i^2 \partial / \partial \mathcal{V}_i \\ Q\left(\mathcal{V}^B, \mathcal{W}^B\right) &= \frac{1}{2}\left\{ Q\left(\mathcal{V}^B + \mathcal{W}^B\right) - Q\left(\mathcal{V}^B\right) - Q\left(\mathcal{W}^B\right) \right\}. \end{aligned}$$

Suppose that there exists an off-diagonal element of A^{BB} which is non zero, say $A_{kl}, k \neq l$.

We have

$$\left[A_{k,l}^{BB} \mathcal{V}_l^B, \alpha_k \left(\mathcal{V}_k^B \right)^2 \right] = 2\alpha_k A_{kl}^{BB} \mathcal{V}_k^B \mathcal{V}_l^B, \qquad k \neq l$$

and since in the right hand side there is the term $\mathcal{V}_k^B \mathcal{V}_l^B$, it follows that $[L_1, L_2] \subsetneqq L_2$ then the algebra does not close. ■

In conclusion, we have the surprising result that the largest (closed) algebra of autonomous vector fields which admits a fundamental system of solutions is one dimensional.

Corollary 18 *Within ATSM, the presence of correlated positive risk factors requires the estimation of an infinite number of parameters.*

Remark 8 *Our results have been derived under very restrictive assumptions (affine definition of factors and short rate, time-homogeneous Markov setting, admissibility conditions and market completeness). However, the above algebraic arguments can be applied in full generality to discuss the existence of a FSS for the Riccati equations arising from any finite dimensional realization of the Heath, Jarrow and Morton (1992) model (see Filipović and Teichmann 2003). This extension requires a classification of the short graded transitive subalgebras of* $Pol(\mathbb{V})$.

3.3 Separable Polinomial Term Structure Models admitting a FSS

The conditions for the existence of a FSS can be extended without essential increase of complexity to any consistent (see Björk 2003) polynomial term structure model, thanks to the results of Filipović (2002).

Definition 19 *A polynomial term structure model is defined by:*

- *the factors* Z_t *defined in a cone domain* $\mathcal{Z} \subseteq \mathbb{R}^n$, *whose dynamics under the risk neutral measure* $\mathbb{Q}$ *satisfy the following SDE:*

$$dZ_t = b(Z_t)\,dt + \sigma(Z_t)\,dW_t \quad Z_t \in \mathcal{Z},$$

where W_t *is a* n*-dimensional Brownian motion, and the drift* $b(.)$ *and volatility* $\sigma(.)$ *satisfy the growth constraint:*

$$\|b(z)\| + \|\sigma(z)\| \leq C(1 + \|z\|), \quad \forall z \in \mathcal{Z}$$

- *the forward rate curve:*

$$r(Z_t, \tau) = \sum_{|\mathbf{i}|=0}^{d} g_{\mathbf{i}}(\tau)(Z_t)^{\mathbf{i}}, \qquad \tau \geq 0, \tag{15}$$

where the multindex notation has been used: $\mathbf{i} = (i_1, ..., i_m)$, $|\mathbf{i}| = i_1 + ... + i_m$ *and* $z^{\mathbf{i}} = z_1^{i_1} ... z_n^{i_n}$; *here* d *denotes the degree of the polynomial term structure.*

The cases $d = 1$ (ATSM with $Z_t = Y_t$) and $d = 2$ (quadratic term structure models) are the only relevant polynomial term structure models, as proved in the following theorem due to Filipović (2002):

Theorem 20 *Suppose that:*

$$\langle a(z)v, v\rangle \geq k(z)\|v\|^2, \quad \forall v \in \mathbb{R}^n$$

where $a(z)_{k,l} = (\sigma(z)\sigma'(z))_{k,l}$ *and the function* $k(z) : \mathcal{Z} \to \mathbb{R}$ *is such that*

$$\lim_{\|z\|\to\infty} \inf k(z) > 0, \qquad z \in \mathcal{Z},$$

then $d = 1, 2$.

The $d = 1$ situation has been discussed in the previous sections, so let us focus on the case $d = 2$, where, under mild regularity conditions, Z_t are necessarily Ornstein Uhlenbeck (constant volatility) factors (see Filipović 2002).

Let us consider the following specification for the factor dynamics,

$$\begin{aligned} dZ_t &= (A_0 + AZ_t) + dW_t, \\ A_0, Z_t &\in \mathbb{R}^n, \quad A \in M_n \end{aligned}$$

for the short rate,

$$\begin{aligned} r(Z_t) &= Z_t'\Omega_0 Z_t + \Gamma_0' Z_t + \gamma_0, \\ \Gamma_0, \gamma_0 &\in \mathbb{R}^n, \quad \Omega_0 \in Sem^n, \end{aligned}$$

and for the forward rate:

$$\begin{aligned} r(Z_t, \tau) &= Z_t'\Omega(\tau) Z_t + \Gamma(\tau)' Z_t + \gamma(\tau), \\ \Gamma(\tau), \gamma(\tau) &\in \mathbb{R}^n, \quad \Omega(\tau) \in Sem^n, \quad \tau \geq 0. \end{aligned}$$

The corresponding Riccati equations become (see e.g. Leippold and Wu 2001):

$$\frac{d}{d\tau}\Omega(\tau) = \Omega_0 + \Omega(\tau) A + A'\Omega(\tau) - 2\Omega(\tau)^2, \tag{16}$$

$$\frac{d}{d\tau}\Gamma(\tau) = \Gamma_0 + 2\Omega(\tau) A_0 + A'\Gamma(\tau) - 2\Omega(\tau)\Gamma(\tau), \tag{17}$$

$$\frac{d}{d\tau}\gamma(\tau) = \gamma_0 + \Gamma(\tau)' A_0 + Tr(\Omega(\tau)) - \Gamma(\tau)'\Gamma(\tau)/2,$$

subject to the boundary conditions (for bond pricing): $\Omega(0) = 0$, $\Gamma(0) = 0$, $\gamma(0) = 0$.

Proposition 21 *Any Quadratic Term Structure Model admits a FSS.*

Proof. The Matrix Riccati Equation can be written as a linear combination of generators of the closed short graded algebra $sl\left(2n\right)$ with its natural grading (see Lafortune and Winternitz 1996). Hence, due to the Lie-Scheffers theorem, it admits a FSS. ■

Remark 9 *Notice that the non linear equation for $\Omega\left(\tau\right)$ is a standard Matrix Symmetric Riccati Equation (for a survey see e.g. Reid 1972 or Freiling 2002), and it is well known that its flow corresponds to a linear flow on a Lagrangian Grassmanian manifold (the multi dimensional generalization of projective spaces), hence in this case the linear flow can be found by a standard homogeneization procedure (see at end of this section for the explicit construction of the linear flow).*

Given the solution for $\Omega\left(\tau\right)$, the equation for $\Gamma\left(\tau\right)$ and for $\gamma\left(\tau\right)$ can be solved by simple integration.

We can thus completely classify the Polynomial Term Structure Models whose non linear equations admit a Fundamental System of Solutions. In the following subsection we provide the closed form expression for (6) for all (separable) term structure models admitting a FSS.

3.4 Closed form solution for Term Structures admitting a FSS

3.4.1 Affine models ($d=1$)

Since the existence of a FSS for an ATSM implies that all positive factors are independent, we can write the matrices A and C corresponding to the normal form as follows:

$$A=\begin{pmatrix} diag\left(a_i^{BB}\right)_{m\times m} & 0_{m\times(n-m)} \\ A^{DB}_{(n-m)\times m} & A^{DD}_{(n-m)\times(n-m)}\end{pmatrix},\qquad C=\begin{pmatrix} \mathbb{I}_{m\times m} & 0_{m\times(n-m)} \\ C^{DB}_{(n-m)\times m} & 0_{(n-m)\times(n-m)}\end{pmatrix}.$$

In this case the Riccati ODE (6) become (recall that $t\rightarrow\tau$ leads to a sign -1 in the left side hand)

$$\begin{aligned}\frac{d}{d\tau}\mathcal{V}^B\left(\tau\right) &= -\gamma^B+diag\left(a_i^{BB}\right)\mathcal{V}^B\left(\tau\right)+\left(A^{DB}\right)'\mathcal{V}^D(\tau)+\frac{1}{2}\left(\mathcal{V}^B\left(\tau\right)\right)^2 \qquad (18)\\ &\quad+\frac{1}{2}\left(C^{DB}\right)'\left(\mathcal{V}^D\left(\tau\right)\right)^2\\ \frac{d}{d\tau}\mathcal{V}^D\left(\tau\right) &= -\gamma^D+\left(A^{DD}\right)'\mathcal{V}^D\left(\tau\right), \qquad (19)\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}^B &= (\mathcal{V}_1, \ldots.., \mathcal{V}_m)', \\
\left(\mathcal{V}^B\right)^2 &= \left(\mathcal{V}_1^2, \ldots.., \mathcal{V}_m^2\right)', \\
\mathcal{V}^D &= (\mathcal{V}_{m+1}, \ldots.., \mathcal{V}_n)', \\
\left(\mathcal{V}^D\right)^2 &= \left(\mathcal{V}_{m+1}^2, \ldots.., \mathcal{V}_n^2\right)', \\
\gamma^B &= (\gamma_1, \ldots.., \gamma_m)', \\
\gamma^D &= \left(\gamma_{m+1}, \ldots.., \gamma_n\right)',
\end{aligned}$$

with boundary condition $\mathcal{V}(0) = u$.

In analogy with Duffie et al. (2001), we assume in (6) that $\gamma_i \geq 0$, $i = 1, .., m$, while $\gamma^D = 0$, which is equivalent to require that the short rate is almost surely non negative.

In this case we can propose a procedure in order to find the explicit solution of (6):

1. Solve the linear ODE (19):

$$\mathcal{V}^D(\tau) = \exp\left\{\tau \left(A^{DD}\right)'\right\} \mathcal{V}^D(0) \tag{20}$$

2. Plug (20) into (18) and for all $i = 1, ..., m$ solve the 1-dimensional time-dependent Riccati equations

$$\frac{d}{d\tau}\mathcal{V}_i^B(\tau) = -\widetilde{\gamma}_i(\tau) + a_i^{BB}\mathcal{V}_i^B(\tau) + \frac{1}{2}\mathcal{V}_i^B(\tau)^2, \qquad i = 1, ..., m,$$

where

$$\widetilde{\gamma}_i(\tau) = \gamma_i - \sum_{j=m+1}^{n}\left(\left(A^{DB}\right)'_{ij}\mathcal{V}_j^D(\tau) + \frac{1}{2}\left(C^{DB}\right)'_{ij}\mathcal{V}_j^D(\tau)^2\right).$$

Although the coefficients $\widetilde{\gamma}_i(\tau)$ are time dependent, the FSS property and the linearizability of the flow for the 1-dimensional Riccati ODE (see also Appendix A) allow to express the solution in the form

$$\mathcal{V}_i(\tau) = \frac{M_1^i(\tau)\mathcal{V}_i(0) + M_2^i(\tau)}{M_3^i(\tau)\mathcal{V}_i(0) + M_4^i(\tau)},$$

where

$$\begin{pmatrix} M_1^i(\tau) & M_2^i(\tau) \\ M_3^i(\tau) & M_4^i(\tau) \end{pmatrix} = \exp\left\{\begin{pmatrix} \tau a_1^{BB} & -\int_0^\tau \widetilde{\gamma}_i(\tau')\,d\tau' \\ -\tau/2 & 0 \end{pmatrix}\right\}$$

and the integral

$$\int_0^\tau \widetilde{\gamma}_i(\tau')\, d\tau'$$

can be explicitly performed, since the general expression of $\int_0^\tau \widetilde{\gamma}_i(\tau')\, d\tau'$ will be an affine combination of exponentials and the expressions for the M_j^i, $j = 1, ..., 4$ are obtained through exponentiation of an explicit expression.

Remark 10 *The assumption $\gamma^D = 0$ is not restrictive, since alternatively we can search for a stationary time independent solution $\mathcal{V}^*$ solving the algebraic system $d\mathcal{V}^*(\tau)/d\tau = 0$: then for any solution $\mathcal{V}$ of the original equation the difference $\mathcal{V} - \mathcal{V}^*$ solves the corresponding homogeneous equation.*

3.4.2 Quadratic models $(d = 2)$

The quadratic Gaussian model admits a FSS without any parametric restriction, thus we can provide the formal solution in fully generality (for an alternative approach see Kim 2002, Cheng and Scaillet 2002 who derived independently analogous expressions).

1. As already stated before, the equation (16) for $\Omega(\tau)$ is a symmetric Riccati Matrix differential equation, which corresponds to a linear flow in homogeneous coordinates (the n dimensional generalization of projective spaces, see Reid 1973 and Appendix A for a brief review), so let us describe the homogeneization procedure.

   Put $\Omega(\tau) = F(\tau)^{-1} G(\tau)$, for $F(\tau) \in GL(n), G(\tau) \in M_n$, then:

$$\frac{d}{d\tau}[F(\tau)\Omega(\tau)] - \frac{d}{d\tau}[F(\tau)]\Omega(\tau) = F(\tau)\frac{d}{d\tau}\Omega(\tau),$$

   and from (16) we obtain

$$\frac{d}{d\tau}G(\tau) - \frac{d}{d\tau}[F(\tau)]\Omega(\tau) = (F(\tau)\Omega_0 + G(\tau)A) - (-F(\tau)A' + 2G(\tau))\Omega(\tau).$$

   The previous ODE leads to the system of $(2n)$ *linear* equations:

$$\begin{aligned} \frac{d}{d\tau}G(\tau) &= F(\tau)\Omega_0 + G(\tau)A \\ \frac{d}{d\tau}F(\tau) &= -F(\tau)A' + 2G(\tau), \end{aligned}$$

which can be written as follows:

$$\frac{d}{d\tau}\left(\begin{array}{cc} G(\tau) & F(\tau) \end{array}\right) = \left(\begin{array}{cc} G(\tau) & F(\tau) \end{array}\right)\left(\begin{array}{cc} A & 2\mathbb{I}_n \\ \Omega_0 & -A' \end{array}\right).$$

This is the transposed version of the standard Matrix Riccati Equation. Its solution is simply obtained through exponentiation:

$$\begin{aligned}
\left(\begin{array}{cc} G(\tau) & F(\tau) \end{array}\right) &= \left(\begin{array}{cc} G(0) & F(0) \end{array}\right)\exp\tau\left(\begin{array}{cc} A & 2\mathbb{I}_n \\ \Omega_0 & -A' \end{array}\right) \\
&= \left(\begin{array}{cc} \Omega(0) & \mathbb{I}_n \end{array}\right)\exp\tau\left(\begin{array}{cc} A & 2\mathbb{I}_n \\ \Omega_0 & -A' \end{array}\right) \\
&= \left(\begin{array}{cc} A_1^1(\tau)\Omega(0) + A_1^2(\tau) & A_2^1(\tau)\Omega(0) + A_2^2(\tau) \end{array}\right),
\end{aligned}$$

where

$$\left(\begin{array}{cc} A_1^1(\tau) & A_2^1(\tau) \\ A_1^2(\tau) & A_2^2(\tau) \end{array}\right) = \exp\tau\left(\begin{array}{cc} A & 2\mathbb{I}_n \\ \Omega_0 & -A' \end{array}\right).$$

In conclusion we get

$$\Omega(\tau) = \left(A_2^1(\tau)\Omega(0) + A_2^2(\tau)\right)^{-1}\left(A_1^1(\tau)\Omega(0) + A_1^2(\tau)\right). \qquad (21)$$

2. Through the same homogeneization procedure we can also get the expression for $\Gamma(\tau)$. In fact, take again $\Gamma(\tau) = F(\tau)^{-1}\widetilde{\Gamma}(\tau)$, with $\widetilde{\Gamma}(\tau) \in M_n$, then

$$\frac{d}{d\tau}\left[F(\tau)\Gamma(\tau)\right] - \frac{d}{d\tau}\left[F(\tau)\right]\Gamma(\tau) = F(\tau)\frac{d}{d\tau}\Gamma(\tau),$$

and from (17) we obtain

$$\frac{d}{d\tau}\widetilde{\Gamma}(\tau) - \frac{d}{d\tau}\left[F(\tau)\right]\Gamma(\tau) = F(\tau)\Gamma_0 + 2G(\tau)A_0 - \left[-F(\tau)A' + 2G(\tau)\right]\Gamma(\tau),$$

which gives immediately the differential equation for $\widetilde{\Gamma}(\tau)$ in terms of $G(\tau)$, $F(\tau)$:

$$\frac{d}{d\tau}\left[\widetilde{\Gamma}(\tau)\right] = F(\tau)\Gamma_0 + 2G(\tau)A_0.$$

We finally get the expression for $\Gamma(\tau)$:

$$\begin{aligned}
\Gamma(\tau) &= F(\tau)^{-1}\widetilde{\Gamma}(\tau) \\
&= F(\tau)^{-1}\int_0^\tau \left(F(\tau')\Gamma_0 + 2G(\tau')A_0\right)d\tau' \\
&= \left(A_2^1(\tau)\Omega(0) + A_2^2(\tau)\right)^{-1}\int_0^\tau \left(\left(A_2^1(\tau')\Omega(0) + A_2^2(\tau')\right)\Gamma_0\right. \\
&\quad \left. +2\left(A_1^1(\tau')\Omega(0) + A_1^2(\tau')\right)A_0\right)d\tau'.
\end{aligned}$$

3. As usual, $\gamma(\tau)$ can be obtained by direct integration.

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4 Appendix A

In this Appendix A we review different methodologies in order to solve (6) in the one-dimensional case, where the solution of the Riccati ODE is well-known. Both approaches suggest the possibility to linearize the solution flow by doubling the dimension of the problem.

The starting point is the normal form of the SDE driving the single-factor, which is identified with the short interest rate:

$$dX_t = \left(AX_t + A^0\right) dt + \sqrt{X_t} dW_t, \tag{22}$$

where we suppose that $C^0 = 0$ (by construction $\gamma = 1$).

The bond price in terms of the reduced factor will have the expression:

$$P(t,T) = \exp\left(\mathcal{V}^0(T-t) + \mathcal{V}(T-t)X_t\right)$$

where $\mathcal{V}$ solves

$$\frac{d}{d\tau}\mathcal{V}(\tau) = A\mathcal{V}(\tau) + \frac{1}{2}\mathcal{V}^2(\tau) - \gamma, \qquad \mathcal{V}(0) = 0, \tag{23}$$

and the solution (see Cox, Ingersoll and Ross 1985) is given by:

$$\mathcal{V}(\tau) = \frac{2\left(e^{\tau\sqrt{A^2+2\gamma}} - 1\right)}{-\sqrt{A^2+2\gamma} + A + e^{\tau\sqrt{A^2+2\gamma}}\left(\sqrt{A^2+2\gamma} - A\right)}. \tag{24}$$

4.1 Matrix Riccati linearization

In the one-dimensional case, the equation (6) is also a Matrix Riccati equation: in this case it is possible to find out suitable (homogeneous) coordinates such that the solution set is linearly parametrized, i.e. the Riccati becomes a linear ODE that can be solved by quadrature. The intuitive meaning of such procedure can be understood considering the geometric interpretation in the projective extension $\mathbb{P}^1(\mathbb{R})$ of the real line: consider the line as an affine space of points $P \in \mathbb{A}_1(\mathbb{R})$, each point is represented by two coordinates $P^A \approx (x,1)$, a vector in the same space will be represented by $t = (y,0)$. Observe that a rigid translation of a point can be represented as: $P \to P+t$. Projectivization of a space corresponds to get rid of the distinction between points and translations (vectors are special points called points to infinity); in the new space $\mathbb{P}^1$ each point P is represented by a one dimensional linear subspace, say $L(P) \subset \mathbb{R}^2$. Observe that the Affine parametrization of the line corresponds to take a specific local parametrization of the Projective line: to $P^A \approx (x,0)$ we can associate canonically $L\left(P^A\right) = (\lambda x, \lambda)$. Observe that the correspondence in not one to one, because there is a special subspace $L(P^\infty) = (x,0)$ which does not correspond to any point in $\mathbb{A}_1(\mathbb{R})$; rather it represents rigid translations! The affine parametrization separates vectors and points which does not appear in the coordinate free definition of the projective line. These properties have been shown to be crucial in the representation of the generic solution to the Riccati equation.

In our setting, let us consider the Riccati (23) as an ODE written in terms of the affine coordinates

$$(\mathcal{V}(\tau), 1) = \left(\frac{\pi(\tau)}{\lambda(\tau)}, 1\right),$$

such that

$$\frac{d}{d\tau}\mathcal{V}(\tau) = -\frac{\pi(\tau)}{\lambda^2(\tau)}\frac{d}{d\tau}\lambda(\tau) + \frac{1}{\lambda(\tau)}\frac{d}{d\tau}\pi(\tau).$$

If we rewrite the equation in terms of the homogeneous coordinates $(\pi(\tau), \lambda(\tau))$, it becomes:

$$\lambda(\tau)\frac{d}{d\tau}\pi(\tau) - \pi(\tau)\frac{d}{d\tau}\lambda(\tau) = \left(-\gamma\lambda(\tau) + A\pi(\tau)\right)\lambda(\tau) + \left(\frac{1}{2}\pi(\tau)\right)\pi(\tau),$$

which can be written as a linear ODE:

$$\frac{d}{d\tau}\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix} = \begin{pmatrix} A & -\gamma \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix},$$

whose solution is given by

$$\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix} = \exp\left\{\tau\begin{pmatrix} A & -\gamma \\ -\frac{1}{2} & 0 \end{pmatrix}\right\}\begin{pmatrix} \pi_0 \\ \lambda_0 \end{pmatrix}.$$

If we denote

$$\begin{pmatrix} a_1(\tau) & a_2(\tau) \\ a_3(\tau) & a_4(\tau) \end{pmatrix} = \exp\left\{\tau\begin{pmatrix} A & -\gamma \\ -\frac{1}{2} & 0 \end{pmatrix}\right\}, \tag{25}$$

then the solution of the Riccati (23) corresponding to initial condition $(v_0, 1)$ can be written as

$$\mathcal{V}(\tau) = \frac{\pi(\tau)}{\lambda(\tau)} = \frac{a_1(\tau)v_0 + a_2(\tau)}{a_3(\tau)v_0 + a_4(\tau)}. \tag{26}$$

Now one can easily diagonalize the 2×2 matrix and impose the initial conditions in order to obtain (24).

4.2 Integration of the Riccati equation on the Lie group $SL(2, \mathbb{R})$

In this subsection

1. we review the linearization procedure by using the Lie group properties of the Riccati ODE (23) and

2. we discuss the construction of the FSS associated.

The linearizability of the Matrix Riccati ODE is related to the special property of the algebra generated by the vector field

$$\left(-\gamma + A\mathcal{V} + \frac{1}{2}\mathcal{V}^2\right), \tag{27}$$

which can be included in the simplest short graded Lie algebra (see Section 3) with generators:

$$L_0 = \gamma \in \mathbb{R}, L_1 = \mathcal{V}, L_2 = \mathcal{V}^2.$$

It is well-known that they form an irreducible representation of the generators of the Lie Algebra $\mathfrak{sl}(2,\mathbb{R})$ (the tangent space to the Lie Group $SL(2,\mathbb{R})$), in fact they fulfill the defining commutation relations:

$$[L_0, L_1] = -L_0, [L_0, L_2] = -2L_1, [L_2, L_1] = L_2.$$

The consequence is that we can now rewrite our differential equation (23) in a coordinate free way and therefore we can move from the above non linear representation to a more convenient space where the group is linearly represented. The corresponding linear equation on the group can be written as

$$\frac{d}{dt}g(t) = \left(-L_0 + AL_1 + \frac{1}{2}L_2\right)g(t),$$

where the generators L_0, L_1, L_2 have the following linear realization in $M_2(\mathbb{R})$:

$$M_2(L_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M_2(L_1) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, M_2(L_2) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

and in this new space the integration of the Riccati equation becomes equivalent to the integration of the constant coefficients matrix (linear) ODE:

$$\frac{d}{dt}M_2(g(\tau)) = \begin{pmatrix} \frac{1}{2}A & -\gamma \\ -\frac{1}{2} & -\frac{1}{2}A \end{pmatrix} M_2(g(\tau)), \qquad M_2(g(0)) = \mathbf{1}_{2\times 2}. \tag{28}$$

The solutions to this linear system are isomorphic to the solution space of the original non linear Riccati ODE. This in turn identifies uniquely the solution flow:

$$\begin{array}{rcl} M_2[SL(2,\mathbb{R})] \times \mathbb{R} & \rightarrow & \mathbb{R} \\ F(M_2[g(\tau)], \mathcal{V}(0)) & = & \mathcal{V}(\tau) \end{array}$$

For the construction of the FSS we essentially apply Carinena, Nasarre and Marmo (1999). Given a finite number N of particular solutions $\mathcal{V}^{P1}(\tau)\dots\mathcal{V}^{PN}(\tau)$ for the ODE:

$$\begin{aligned} F\left(M_2\left[g(\tau)\right], \mathcal{V}^{P1}(0)\right) &= \mathcal{V}^{P1}(\tau) \\ &\dots \\ F\left(M_2\left[g(\tau)\right], \mathcal{V}^{PN}(0)\right) &= \mathcal{V}^{PN}(\tau), \end{aligned}$$

it is possible to uniquely identify $M_2\left[g(\tau)\right]$ by means of the implicit function theorem:

$$M_2\left[g(\tau)\right] = G\left(\mathcal{V}^{P1}(\tau), .., \mathcal{V}^{PN}(\tau), \mathcal{V}^{P1}(0), .., \mathcal{V}^{PN}(0)\right),$$

thus the FSS is easily obtained:

$$\begin{aligned} \mathcal{V}(\tau) &= F\left(G\left(\mathcal{V}^{P1}(\tau), .., \mathcal{V}^{PN}(\tau), \mathcal{V}^{P1}(0), .., \mathcal{V}^{PN}(0)\right), \mathcal{V}(0)\right) \\ &= \Phi\left(\mathcal{V}^{P1}(\tau), .., \mathcal{V}^{PN}(\tau), \mathcal{V}^{P1}(0), .., \mathcal{V}^{PN}(0), \mathcal{V}(0)\right). \end{aligned}$$

In our special case each element of the group is parametrized by the 2x2 matrix $M_2(g(\tau))$. The Flow F has the form:

$$\mathcal{V}(\tau) = F\left(M_2\left[g(\tau)\right], \mathcal{V}(0)\right) \triangleq \frac{b_1(\tau)\,\mathcal{V}(0) + b_2(\tau)}{b_3(\tau)\,\mathcal{V}(0) + b_4(\tau)}$$

where:

$$M_2\left[g(\tau)\right] = \begin{pmatrix} b_1(\tau) & b_2(\tau) \\ b_3(\tau) & b_4(\tau) \end{pmatrix} = \exp\left\{\tau\begin{pmatrix} \frac{1}{2}A & -\gamma \\ -\frac{1}{2} & -\frac{1}{2}A \end{pmatrix}\right\},$$

so that the function G can be obtained by solving the linear system of equations (here $N = 3$)

$$\begin{pmatrix} \mathcal{V}^{P1}(0) & 1 & \mathcal{V}^{P1}(0)\,\mathcal{V}^{P1}(\tau) & \mathcal{V}^{P1}(\tau) \\ \mathcal{V}^{P2}(0) & 1 & \mathcal{V}^{P2}(0)\,\mathcal{V}^{P2}(\tau) & \mathcal{V}^{P2}(\tau) \\ \mathcal{V}^{P3}(0) & 1 & \mathcal{V}^{P3}(0)\,\mathcal{V}^{P3}(\tau) & \mathcal{V}^{P3}(\tau) \end{pmatrix}\begin{pmatrix} b_1(\tau) \\ b_2(\tau) \\ b_3(\tau) \\ b_4(\tau) \end{pmatrix} = 0$$

in the unknowns $b_i(\tau)$, $i = 1, .., 4$. For non degenerate initial conditions, the solution to the system defines a unique solution space

$$G_\lambda\left(\mathcal{V}^{P1}(\tau), .., \mathcal{V}^{PN}(\tau), \mathcal{V}^{P1}(0), .., \mathcal{V}^{PN}(0)\right) \triangleq \lambda\left(b_1^s(\tau), b_2^s(\tau), b_3^s(\tau), b_4^s(\tau)\right)', \lambda \in \mathbb{R}.$$

The function G is determined up to an irrelevant constant, therefore if we substitute G in the flow F we get:

$$\begin{aligned} \mathcal{V}(\tau) &= \Phi^{CIR}\left(\mathcal{V}^{P1}(\tau), .., \mathcal{V}^{PN}(\tau), \mathcal{V}^{P1}(0), .., \mathcal{V}^{PN}(0), \mathcal{V}(0)\right) \\ &= \frac{b_1^s(\tau)\,\mathcal{V}(0) + b_2^s(\tau)}{b_3^s(\tau)\,\mathcal{V}(0) + b_4^s(\tau)}, \end{aligned}$$

the fundamental system of solutions (12).

5 Appendix B: Quasi closed form solution for ATSM

The aim in this Appendix is to show that the case when the drift matrix A^{BB} in (9) is *triangular*, even if requires just the solution of linear ODE, is crucially different from the diagonal (independent) case. In fact, even if each equation admits a FSS (since we work with one-dimensional subalgebras), the global system does not.

Let be $n = m$ and suppose w.l.o.g. that the matrix A^{BB} is upper triangular.

In this case the determination of the general solution splits in a sequence of one dimensional time dependent problems which can be solved by a recursive procedure.

The first equation in (6) becomes (recall that in equation (6) the matrix A is transposed)

$$\frac{d}{d\tau}\mathcal{V}_1(\tau) = A_{11}\mathcal{V}_1(\tau) + \frac{1}{2}\mathcal{V}_1^2(\tau) - \gamma_1, \qquad \mathcal{V}_1(0) = u_1,$$

and it can be linearized by standard techniques (see the Appendix A). The solution can be written in the form

$$\mathcal{V}_1(\tau) = \frac{a_1^1(\tau)\, u_1 + a_2^1(\tau)}{a_3^1(\tau)\, u_1 + a_4^1(\tau)}, \qquad \tau \geq 0,$$

where

$$\begin{pmatrix} a_1^1(\tau) & a_2^1(\tau) \\ a_3^1(\tau) & a_4^1(\tau) \end{pmatrix} = \exp\left\{\tau \begin{pmatrix} A_{11} & -\gamma_1 \\ -\frac{1}{2} & 0 \end{pmatrix}\right\}.$$

We can now plug $\mathcal{V}_1(\tau)$ into the second one and obtain

$$\frac{d}{d\tau}\mathcal{V}_2(\tau) = A_{22}\mathcal{V}_2(\tau) + \frac{1}{2}\mathcal{V}_2^2(\tau) + \left(A_{12}\mathcal{V}_1(\tau) - \gamma_2\right), \qquad \mathcal{V}_2(0) = u_2,$$

which is still a one-dimensional Riccati (with time-dependent parameters) and it can be solved analogously. The solution can be iteratively found for any $k \leq m$ (recall that from (10) only the first m equations are truly Riccati ODE):

$$\mathcal{V}_k(\tau) = \frac{a_1^k(\tau)\, u_k + a_2^k(\tau)}{a_3^k(\tau)\, u_k + a_4^k(\tau)}, \qquad \tau \geq 0,$$

with

$$\begin{pmatrix} a_1^k(\tau) & a_2^k(\tau) \\ a_3^k(\tau) & a_4^k(\tau) \end{pmatrix} = \exp\left\{\int_0^\tau ds \begin{pmatrix} A_{kk} & -\gamma_k + \sum_{j=1}^{k-1} A_{jk}\mathcal{V}_j(s) \\ -\frac{1}{2} & 0 \end{pmatrix}\right\} \quad (29)$$

$$= \exp\left\{\int_0^\tau ds\Omega^k(s)\right\}, \qquad 1 \leq k \leq m. \quad (30)$$

The above derivation is straightfoward and appears as a "compact" expression for the solution. However, observe that the presence in the matrix exponential of the terms $\mathcal{V}_j(s)$, which depend on the boundary conditions u_j, precludes any possibility to write the generic solution in terms of a FSS. The effect of such a dependence requires the expansion of the exponential in (29) as a time ordered expansion:

$$\sum_{n=1}^{\infty} \int_0^{\tau} ds_1 \int_0^{s_1} ds_2 .. \int_0^{s_{n-1}} ds_n \Omega^k(s_1)\, \Omega^k(s_2) .. \Omega^k(s_n),$$

which is computationally much heavier than the analytic expansion (11). Another verification consists in computing the integrals appearing in (29) by using standard symbolic computation packages like Mathematica: it is easy to check that their expression involves special functions (like the Polylog and Hypergeometric functions) which do not admit closed form. The application of Lie Scheffers theorem provides a direct proof that the resummation of such time series is indeed precluded.