The domination game played on unions of graphs

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Abstract

In a graph G, a vertex is said to dominate itself and its neighbors. The Domination game is a two player game played on a finite graph. Players alternate turns in choosing a vertex that dominates at least one new vertex. The game ends when no move is possible, that is when the set of chosen vertices forms a dominating set of the graph. One player (Dominator) aims to minimize the size of this set while the other (Staller) tries to maximize it. The game domination number, denoted by γ_g, is the number of moves when both players play optimally and Dominator starts. The Staller-start game domination number γ_g' is defined similarly when Staller starts. It is known that the difference between these two values is at most one [4, 9]. In this paper, we are interested in the possible values of the domination game parameters γ_g and γ_g' of the disjoint union of two graphs according to the values of these parameters in the initial graphs.

We first describe a family of graphs that we call no-minus graphs, for which no player gets advantage in passing a move. While it is known that forests are no-minus, we prove that tri-split graphs and dually chordal graphs also are no-minus. Then, we show that the domination game parameters of the union of two no-minus graphs can take only two values according to the domination game parameters of the initial graphs. In comparison, we also show that in the general case, up to four values may be possible.

Key words: domination game, game domination number, disjoint union
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1 Introduction

In a graph G, a vertex is said to dominate itself and its neighbors. The set of vertices dominated by v is called its closed neighborhood and is denoted by $N[v]$. A set of vertices is a dominating set if every vertex is dominated by some vertex in the set. The domination number is the

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minimum cardinality of a dominating set. Domination is a classical topic in graph theory. For more details we refer to the books by Haynes et al. [7, 8].

The domination game was introduced by Brešar, Klavžar and Rall in [4]. It is played on a finite graph G by two players, Dominator and Staller. They alternate turns in adding a vertex to a set S, provided that this added vertex v dominates at least one new vertex, i.e. $N[S] \subsetneq N[S \cup \{v\}]$. The game ends when there are no more possible moves, that is, when the chosen vertices form a dominating set. Dominator's goal is that the game finishes in as few moves as possible while Staller tries to keep the game going as long as she can. There are two possible variants of the game, depending on who starts the game. In Game 1, Dominator starts, while in Game 2, Staller starts. The game domination number, denoted by $\gamma_g(G)$, is the total number of chosen vertices in Game 1 when both players play optimally. Similarly, the Staller-start game domination number $\gamma'_g(G)$ is the total number of chosen vertices in Game 2 when both players play optimally.

Variants of the game where one player is allowed to pass a move once were also considered in [4, 6, 9] (and possibly elsewhere). In the Dominator-pass game, Dominator is allowed to pass one move, while in the Staller-pass game, Staller is. We denote respectively by γ_g^{dp} and $\gamma_g'^{dp}$ the size of the set of chosen vertices in Game 1 and 2 where Dominator is allowed to pass once, and by γ_g^{sp} and $\gamma_g'^{sp}$ the size of the set of chosen vertices in Games 1 and 2 where Staller is allowed to pass a move. Note that passing does not count as a move in the game domination number, and the value of these games is the number of chosen vertices.

An interesting question about the domination game is how the number of chosen vertices in Game 1 and Game 2 compare on the same graph. Clearly, there are some graphs where Game 1 uses less moves than Game 2. Stars are examples of such graphs. On the other hand, some other graphs give an advantage to the second player. The cycle C_6 on six vertices is an example of such a graph. Nevertheless, results from Brešar et al. [4] and from Kinnersley et al. [9] give a bound to the difference, with the following:

Theorem 1.1 ([4],[9]) *For any graph G, $|\gamma_g(G) - \gamma'_g(G)| \leq 1$*

It should be noted that this result is obtained by applying a very useful principle from [9], known as the continuation principle. For a graph $G = (V, E)$ and a subset of vertices $S \subseteq V$, we denote by $G|S$ the *partially dominated graph* G where the vertices of S are considered already dominated in the game. Kinnersley et al. proved:

Theorem 1.2 (Continuation principle [9]) *Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.*

Note also that Theorem 1.1 naturally extends to partially dominated graphs.

In this paper, we continue the study of the relation between $\gamma_g(G)$ and $\gamma'_g(G)$. We say that a partially dominated graph $G|S$ realizes a pair $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ if $\gamma_g(G|S) = k$ and $\gamma'_g(G|S) = \ell$. A consequence of Theorem 1.1 is that the only realizable pairs are of the form $(k, k+1)$, (k, k) and $(k, k-1)$. It is known that all these pairs are indeed realizable. Examples of graphs of each of these three types are given in [4, 5, 9, 10]. Accordingly, we say that a partially dominated graph $G|S$ is a $(k, +)$ (resp. $(k, =)$, $(k, -)$) if $\gamma_g(G|S) = k$ and $\gamma'_g(G|S) = k + 1$ (resp. $\gamma_g(G|S) = k$ and $\gamma'_g(G|S) = k$, $\gamma_g(G|S) = k$ and $\gamma'_g(G|S) = k - 1$). By PLUS we denote the family of all graphs that are $(k, +)$ for some positive k. Similarly we define EQUAL and MINUS.

The initial question that motivated our study is the following:

Question 1 *Knowing the family of two graphs G and H, what can we infer on the game domination number of the disjoint union $G \cup H$?*

It should be observed that this question is partially motivated by combinatorial game theory (CGT). Combinatorial games can be classified into four classes according to who wins the game when each player starts [1, 2]. More importantly, from the classes where two games belong, the class of the union (called sum in CGT) of these two graphs can often be deduced.

Unfortunately, in the Domination game, we can not deduce similar information in general. Of course the pair realized by the union of two graphs $G \cup H$ is related to the pairs realized by the graphs G and H, though different pairs can be attained, as we show in Section 3. However, we identified a family of graphs, that we call *no-minus graphs*, for which much more can be said.

We say a graph G is a no-minus graph if for any subset of vertices S, $\gamma_g(G|S) \leq \gamma'_g(G|S)$; that is $G|S$ is not in MINUS. Intuitively, the idea is that no player should get any advantage by passing in a no-minus graph. Kinnersley et al. [9] already proved that forests satisfy the property of no-minus graphs.

We proceed as follows. In the next section, we first give early results about no-minus graphs, then we present other classes of graphs that we can prove are no-minus, and finally we describe the pairs that can be realized by the union of no-minus graphs. We then consider the general case and the possible values realized by the union of two graphs in the general case showing that the situation is not as good.

2 About no-minus graphs

2.1 Early properties of no-minus graphs

To begin with no-minus graphs, we first need to prove what we claimed was the intuitive definition of a no-minus, i.e. that it is not helpful to be allowed to pass in such games. In [4], Brešar et al. proved the following in general:

Lemma 2.1 ([4]) *Let G be a graph. We have $\gamma_g(G) \leq \gamma_g^{sp}(G) \leq \gamma_g(G)+1$ and $\gamma_g(G)-1 \leq \gamma_g^{dp}(G) \leq \gamma_g(G)$.*

Though the authors of [4] did not prove it, the exact same proof technique (using the imagination strategy) can give the following inequalities, for partially dominated graphs and for both Games 1 and 2.

Lemma 2.2 *Let G be a graph, S a subset of vertices of G. We have*

$$\begin{aligned}
\gamma_g(G|S) &\leq \gamma_g^{sp}(G|S) \leq \gamma_g(G|S)+1\,,\\
\gamma'_g(G|S) &\leq \gamma_g'^{sp}(G|S) \leq \gamma'_g(G|S)+1\,,\\
\gamma_g(G|S)-1 &\leq \gamma_g^{dp}(G|S) \leq \gamma_g(G|S)\,,\\
\gamma'_g(G|S)-1 &\leq \gamma_g'^{dp}(G|S) \leq \gamma'_g(G|S)\,.
\end{aligned}$$

We now prove that passing is useless in no-minus graphs:

Proposition 2.3 *Let G be a no-minus graph. For any subset S of vertices, we have $\gamma_g^{sp}(G|S) = \gamma_g^{dp}(G|S) = \gamma_g(G|S)$ and $\gamma_g'^{sp}(G|S) = \gamma_g'^{dp}(G|S) = \gamma'_g(G|S)$.*

Proof: By Lemma 2.2, we already have $\gamma_g^{dp}(G|S) \leq \gamma_g(G|S) \leq \gamma_g^{sp}(G|S)$ and $\gamma_g'^{dp}(G|S) \leq \gamma'_g(G|S) \leq \gamma_g'^{sp}(G|S)$. Suppose a partially dominated no-minus graph $G|S$ satisfies $\gamma_g^{dp}(G|S) < \gamma_g(G|S)$. We use the imagination strategy to reach a contradiction.

Consider a normal Dominator-start game played on $G|S$ where Dominator imagines he is playing a Dominator-pass game, while Staller plays optimally in the normal game. Since

$\gamma_g^{dp}(G|S) < \gamma_g(G|S)$, the strategy of Dominator includes passing a move at some point, say after x moves are played. Let X be the set of dominated vertices at that point. Since Dominator played optimally the Dominator-pass domination game (but not necessarily Staller), if he was allowed to pass that move the total number of moves in the game would be no more than $\gamma_g^{dp}(G|S)$. We thus have the following inequality:

$$x + \gamma_g'(G|X) \leq \gamma_g^{dp}(G|S)\,.$$

Now, remark that since Staller played optimally in the normal game, we have that

$$x + \gamma_g(G|X) \geq \gamma_g(G|S)\,.$$

Adding the fact that G is a no-minus, so that $\gamma_g(G|X) \leq \gamma_g'(G|X)$, we reach the following contradiction:

$$\gamma_g(G|S) \leq x + \gamma_g(G|X) \leq x + \gamma_g'(G|X) \leq \gamma_g^{dp}(G|S) < \gamma_g(G|S)\,.$$

Similar arguments complete the proof for the Staller-pass and/or Staller-start games. ∎

The next lemma also expresses a fundamental property of no-minus graphs. It is an extension of a result on forests from [9], the proof is about the same.

Lemma 2.4 *Let G be a graph. If $S \subseteq V(G)$ and $\gamma_g(G|X) \leq \gamma_g'(G|X)$ for every $X \supseteq S$, then $\gamma_g((G \cup K_1)|S) \geq \gamma_g(G|S) + 1$ and $\gamma_g'((G \cup K_1)|S) \geq \gamma_g'(G|S) + 1$.*

Proof: Given a graph G and a set S satisfying the hypothesis, we use induction on the number of vertices in $V \setminus S$. If $V \setminus S = \emptyset$, the claim is trivial. Suppose now that $S \subsetneq V$ and that the claim is true for every $G|X$ with $S \subsetneq X$.

Consider first Game 1. Let v be an optimal first move for Dominator in the game on $(G \cup K_1)|S$. If v is the added vertex, then $\gamma_g((G \cup K_1)|S) = \gamma_g'(G|S) + 1 \geq \gamma_g(G|S) + 1$ by our (no-minus like) assumption on $G|S$, and the inequality follows. Otherwise, let $X = S \cup N[v]$. By the choice of the move and induction hypothesis, we have $\gamma_g((G \cup K_1)|S) = 1 + \gamma_g'((G \cup K_1)|X) \geq 1 + \gamma_g'(G|X) + 1$. Since v is not necessarily an optimal first move for Dominator in the game on $G|S$, we also have that $\gamma_g(G|S) \leq 1 + \gamma_g'(G|X)$ and the result follows.

Consider now Game 2. Let w be an optimal first move for Staller in the game on $G|S$, and let $Y = S \cup N[w]$. By optimality of this move, we have $\gamma_g'(G|S) = 1 + \gamma_g(G|Y)$. Playing also w in $G \cup K_1|S$, Staller gets $\gamma_g'((G \cup K_1)|S) \geq 1 + \gamma_g((G \cup K_1)|Y) \geq 2 + \gamma_g(G|Y)$ by induction hypothesis, and the implied inequality follows. ∎

2.2 More no-minus graphs

In this section, we propose other families of graphs that are no-minus. But first we start with the following observation about MINUS, that will prove useful.

Observation 2.5 *If a partially dominated graph $G|S$ is a $(k, -)$, then for any legal move u in $G|S$, the graph $G|(S \cup N[u])$ is a $(k - 2, +)$.*

Proof: Let $G|S$ be a $(k, -)$ and u be any legal move in $G|S$. By definition of the game domination number, we have $k = \gamma_g(G|S) \leq 1 + \gamma_g'(G|(S \cup N[u]))$. Similarly, $k - 1 = \gamma_g'(G|S) \geq 1 + \gamma_g(G|(S \cup N[u]))$. By Theorem 1.1, we get that

$$k - 1 \leq \gamma_g'(G|(S \cup N[u])) \leq \gamma_g(G|(S \cup N[u])) + 1 \leq k - 1$$

and so equality holds throughout this inequality chain. Thus $G|(S \cup N[u])$ is a $(k-2,+)$, as required. ∎

It was conjectured in [5] and proved in [9] that forests are no-minus graphs. We now propose two other families of graphs that are no-minus. The first is the family of tri-split graphs that we introduce here. It is a generalization of split graphs.

Definition 2.6 *We say a graph is tri-split if and only if its set of vertices can be partitioned into three disjoint sets $A \neq \emptyset$, B and C with the following properties*

$$\forall u \in A \ \forall v \in A \cup C: \ uv \in E(G),$$
$$\forall u \in B \ \forall v \in B \cup C: \ uv \notin E(G).$$

We prove the following.

Theorem 2.7 *Connected tri-split graphs are no-minus graphs.*

Proof: Let G be a tri-split graph with the corresponding partition (A, B, C), let $S \subseteq V(G)$ be a subset of dominated vertices, and consider the game played on $G|S$. If the game on $G|S$ ends in at most two moves, then clearly $\gamma_g(G|S) \leq \gamma'_g(G|S)$. From now on, we assume that $\gamma_g(G|S) \geq 3$.

Observe that Dominator has an optimal strategy playing only in A (in both Game 1 and Game 2). Indeed, any vertex u in B dominates only itself and some vertex in A (at least one by connectivity). Any neighbor v of u in A dominates all of A and v, so is a better move than u for Dominator by the continuation principle. Similarly, the neighborhood of any vertex in C is included in the neighborhood of any vertex in A. So we now assume Dominator plays only in A in the rest of the proof. Though we do not need it, a similar argument using the continuation principle would also allow us to observe that Staller has an optimal strategy where she plays only vertices in $B \cup C$.

Suppose we know an optimal strategy for Dominator in Game 2. We propose a (imagination) strategy for Game 1 guaranteeing it will finish no later than Game 2. Let Dominator imagine a first move $v_0 \in B \cup C$ by Staller (not necessarily optimal) and play the game on $G|S$ as if playing in $G|(S \cup N[v_0])$. Staller plays Game 1 optimally on $G|S$ not knowing about Dominator's imagined game. Note that after Dominator's first move, the only difference between the imagined game and the real game is that v_0 is dominated in the first but possibly not in the second. Indeed, all the neighbors of v_0 belong to $A \cup C$, which are dominated by Dominator's first move (in A by our assumption). Therefore, any move played by Dominator in his imagined game is legal in the real game, though Staller may eventually play a move in the real game that is illegal in the imagined game, provided it newly dominates only v_0. If she does so and the game is not finished yet, then Dominator imagines she played any legal move v_1 in B instead and continues. This may happen again, leading Dominator to imagine a move v_2 and so on. Denote by v_i the last such vertex before the game ends, we thus have that v_i is the only vertex possibly dominated in the imagined game but not in the real game.

Assume now that the imagined game is just finished. Denote by $k_{\mathcal{I}}$ the total number of moves in this imagined game. Note that the imagined game looks like a Game 2 where Dominator played optimally but possibly not Staller. We thus have that $k_{\mathcal{I}} \leq \gamma'_g(G|S)$. At that point, either the real game is finished or only v_i is not yet dominated. So the real game finishes at latest with the next move of any player, and the number of moves in the real game $k_{\mathcal{R}}$ satisfies $k_{\mathcal{R}} \leq k_{\mathcal{I}} - 1 + 1$. Moreover, in the real game, Staller played optimally but possibly not Dominator, so $k_{\mathcal{R}} \geq \gamma_g(G|S)$. We now can conclude the proof bringing together all these inequalities into

$$\gamma_g(G|S) \leq k_{\mathcal{R}} \leq k_{\mathcal{I}} \leq \gamma_g'(G|S)\,.$$

■

The second family of graphs we prove to be no-minus is the family of dually chordal graphs, see [3]. Let G be a graph and v one of its vertices. A vertex $u \in N[v]$ is a maximum neighbor of v if for all $w \in N[v]$, we have $N[w] \subseteq N[u]$ (i.e. $N[u]$ contains all vertices at distance at most 2 from v). A vertex ordering $v_1, \dots, v_n$ is a *maximum neighborhood ordering* if for each $i \leq n$, v_i has a maximum neighbor in $G_i = G[\{v_1, \dots, v_i\}]$, the induced subgraph of G on the set of vertices $\{v_1, \dots, v_i\}$. A graph is dually chordal if it has a maximum neighborhood ordering. Note that forests and interval graphs are dually chordal. We first need a little statement on maximal neighborhood orderings that will prove useful later on.

Lemma 2.8 *Let G be a dually chordal graph. There exists a maximum neighborhood ordering $v_1, \dots, v_n$ of G such that if v_i's only maximum neighbor in $G_i = G[\{v_1, \dots, v_i\}]$ is itself, then v_i is isolated in G_i.*

Proof: Let G be a dually chordal graph and consider $v_1, \dots, v_n$ a maximum neighborhood ordering of G with a minimum number of vertices v_i non isolated in G_i but whose only maximum neighbor is itself. If there are no such vertices, we are done. Suppose by way of contradiction that there are some, and let v_k be such a vertex of maximum index.

We first observe that in G_k, there are no vertices at distance 2 from v_k. Indeed, if a vertex u in G_k is adjacent to both v_k and another vertex u', then by definition of a maximum neighbor, v_k is also adjacent to u'. So v_k is adjacent to all the vertices in its component. Now we claim that the ordering $v_k, v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n$ is also a maximum neighborhood ordering of G. All vertices v_i where $i > k$, and all vertices of index less than k but not in the component of v_k in G_k have the same vertex as a maximum neighbor. The vertex v_k itself can be chosen as the maximum neighbor of all vertices of index less than k which are in v_k's component in G_k. Let v_i be a vertex who is its only maximum neighbor in the new ordering. Necessarily, v_i was already its only maximum neighbor in the initial ordering. Also, unless v_i was in the component of v_k in G_k, the neighborhood of v_i in G_i has not changed. Nevertheless, v_k itself used to be its own maximum neighbor and to be non isolated in G_k, but now is isolated. So this new ordering $w_1, \dots, w_n$ contains less vertices w_i non isolated in $G[\{w_1, \dots, w_i\}]$ but whose only maximum neighbor is itself. This contradicts our initial choice of the ordering. ■

Theorem 2.9 *Dually chordal graphs are no-minus graphs.*

Proof: We prove the result by induction on the number of non-dominated vertices. Let G be a dually chordal graph with $v_1, \dots, v_n$ a maximum neighborhood ordering of $V(G)$ where no vertex v_i is its own maximum neighbor unless v_i is isolated in G_i. Let $S \subseteq V(G)$ be a subset of dominated vertices and denote by j the largest index such that v_j is not in S. We suppose by way of contradiction that $G|S$ is a $(k, -)$, and note that necessarily $k \geq 3$. Let v_i be a maximum neighbor of v_j in G_j. Let u be an optimal move for Staller in $G|(S \cup N[v_i])$ and let $X = S \cup N[v_i] \cup N[u]$. By Observation 2.5, $G|(S \cup N[u])$ and $G|(S \cup N[v_i])$ are both $(k-2, +)$, so $\gamma_g(G|(S \cup N[u]) = k-2$ and $\gamma_g'(G|(S \cup N[v_i])) = k-1$. By optimality of u, we get that

$$k - 1 = \gamma_g'(G|(S \cup N[v_i])) = \gamma_g(G|X) + 1\,.$$

Let v_ℓ be a vertex adjacent to v_j, we prove by induction on ℓ that $N[v_\ell] \subset (S \cup N[v_i])$. If $\ell \leq j$, then v_i being a maximum neighbor of v_j in G_j, vertices adjacent to v_ℓ are either in $N[v_i]$ or have index larger than j and thus are in S, and the claim is true. Assume now $\ell > j$. Let v_m be a maximum neighbor of v_ℓ in G_ℓ with smallest index. Since v_ℓ is not isolated in G_ℓ, $m < \ell$. Since v_j is adjacent to v_ℓ and $j \leq \ell$, v_j is adjacent to v_m. Then by induction, $N[v_m] \subset S \cup N[v_i]$. Hence all neighbors $v_{\ell'}$ of v_ℓ with $\ell' \leq \ell$ are in $N[v_m] \subset S \cup N[v_i]$ and vertices $v_{\ell'}$ with $\ell' > \ell$ are in S, and finally $N[v_\ell] \subset (S \cup N[v_i])$.

This implies that the vertex u is not a neighbor of v_j, otherwise playing u would not be legal in $G|(S \cup N[v_i])$. Therefore, by continuation principle (Theorem 1.2),

$$\gamma_g(G|(S \cup N[u])) \geq \gamma_g(G|(X \setminus \{v_j\}))\,.$$

Moreover, because all vertices at distance at most two from v_j are dominated in $G|X$, we get that $\gamma_g(G|(X \setminus \{v_j\})) = \gamma_g((G \cup K_1)|X)$. Now using induction hypothesis to apply Lemma 2.4, we get

$$\gamma_g(G|(X \setminus \{v_j\})) \geq \gamma_g(G|X) + 1\,.$$

We thus conclude that

$$k - 2 = \gamma_g(G|(S \cup N[u])) \geq \gamma_g(G|(X \setminus \{v_j\})) \geq \gamma_g(G|X) + 1 = k - 1,$$

which leads to a contradiction. Therefore, $G|S$ is not in MINUS and this concludes the proof. ■

2.3 Realizations by unions of two no-minus graphs

In this section, we are interested in the possible values that the union of two no-minus graphs may realize, according to the realizations of the components. We in particular show that the union of two no-minus graphs is always also no-minus.

We first prove a very general result that will allow us to compute most of the bounds obtained later.

Theorem 2.10 *Let $G_1|S_1$ and $G_2|S_2$ be two partially dominated graphs and x be any legal move in $G_1|S_1$. We have*

$$\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \geq \min\left(\gamma_g(G_1|S_1) + \gamma_g^{dp}(G_2|S_2), \gamma_g^{dp}(G_1|S_1) + \gamma_g(G_2|S_2)\right), \quad (1)$$

$$\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq 1 + \max\begin{pmatrix} \gamma'_g(G_1|(S_1 \cup N[x])) + \gamma_g'^{sp}(G_2|S_2) \\ \gamma_g'^{sp}(G_1|(S_1 \cup N[x])) + \gamma'_g(G_2|S_2) \end{pmatrix}, \quad (2)$$

$$\gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq \max\left(\gamma'_g(G_1|S_1) + \gamma_g'^{sp}(G_2|S_2), \gamma_g'^{sp}(G_1|S_1) + \gamma'_g(G_2|S_2)\right), \quad (3)$$

$$\gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) \geq 1 + \min\begin{pmatrix} \gamma_g(G_1|(S_1 \cup N[x])) + \gamma_g^{dp}(G_2|S_2) \\ \gamma_g^{dp}(G_1|(S_1 \cup N[x])) + \gamma_g(G_2|S_2) \end{pmatrix}. \quad (4)$$

Proof: To prove all these bounds, we simply describe what a player can do by using a *strategy of following*, i.e. always answering to his opponent's moves in the same graph if possible.

Let us first consider Game 1 in $G_1 \cup G_2|S_1 \cup S_2$ and what happens when Staller adopts the strategy of following. Assume first that the game in G_1 finishes before the game in G_2. Then Staller can ensure with her strategy that the number of moves in G_1 is at least $\gamma_g(G_1|S_1)$. However, when G_1 finishes, Staller may be forced to play in G_2 if Dominator played the final move in G_1. This situation somehow allows Dominator to pass once in G_2, but no more. So Staller can ensure that the number of moves in G_2 is no less than $\gamma_g^{dp}(G_2|S_2)$. Thus, in that case, the total number of moves is no less than $\gamma_g(G_1|S_1) + \gamma_g^{dp}(G_2|S_2)$. If on the other hand the game in G_2 finishes first, we similarly get that the number of moves is then no less than $\gamma_g^{dp}(G_1|S_1)+\gamma_g(G_2|S_2)$. Since she does not decide which game finishes first, Staller can guarantee that

$$\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \geq \min\left(\gamma_g(G_1|S_1) + \gamma_g^{dp}(G_2|S_2), \gamma_g^{dp}(G_1|S_1) + \gamma_g(G_2|S_2)\right).$$

The same arguments with Dominator adopting the strategy of following in Game 2 ensure that

$$\gamma'_g(G_1 \cup G_2|S_1 \cup S_2) \leq \max\left(\gamma'_g(G_1|S_1) + \gamma'^{sp}_g(G_2|S_2), \gamma'^{sp}_g(G_1|S_1) + \gamma'_g(G_2|S_2)\right).$$

Let us come back to Game 1. Suppose Dominator plays some vertex $x \in V(G_1)$ and then adopts the strategy of following. Then he can ensure that $\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq 1 + \gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2 \cup N[x]))$ and thus

$$\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq 1 + \max\left(\begin{array}{c} \gamma'_g(G_1|(S_1 \cup N[x])) + \gamma'^{sp}_g(G_2|S_2), \\ \gamma'^{sp}_g(G_1|(S_1 \cup N[x])) + \gamma'_g(G_2|S_2) \end{array}\right).$$

The same is true for Staller in Game 2 and gives Inequality (4). ∎

In the case of the union of two no-minus graphs, these inequalities allow us to give rather precise bounds on the possible values realized by the union. The first case is when one of the components is in EQUAL.

Theorem 2.11 *Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs. If $G_1|S_1$ is a $(k,=)$ and $G_2|S_2$ is a $(\ell,\star)$ (with $\star \in \{=,+\}$), then the disjoint union $(G_1 \cup G_2)|(S_1 \cup S_2)$ is $(k+\ell,\star)$.*

Proof: We use inequalities from Theorem 2.10. Note that since G_1 and G_2 are no-minus graphs, we can apply Proposition 2.3 and get that the Staller-pass and Dominator-pass game number of any partially dominated graph are the same as the corresponding non-pass game numbers.

For Game 1, let Dominator choose an optimal move x in $G_2|S_2$, for which we get $\gamma'_g(G_2|(S_2 \cup N[x])) = \ell - 1$. Applying Inequalities (1) and (2) interchanging the role of G_1 and G_2, we then get that

$$k + \ell \leq \gamma_g(G_1 \cup G_2|S_1 \cup S_2) \leq 1 + k + \ell - 1.$$

For Game 2, Staller can also choose an optimal move x in $G_2|S_2$ for which $\gamma_g(G_2|S_2 \cup N[x]) = \gamma'_g(G_2|S_2) - 1$, and applying Inequalities (3) and (4), we get that $\gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) = \gamma'_g(G_1|S_1) + \gamma'_g(G_2|S_2)$. This proves that $(G_1 \cup G_2)|(S_1 \cup S_2)$ is indeed a $(k+\ell,\star)$.

∎

In the second case, when both of the components are in PLUS we prove the following assertion.

Theorem 2.12 *Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs such that $G_1|S_1$ is $(k,+)$ and $G_2|S_2$ is $(\ell,+)$. Then*

$$k + \ell \leq \gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k + \ell + 1,$$
$$k + \ell + 1 \leq \gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k + \ell + 2.$$

In addition, all bounds are tight.

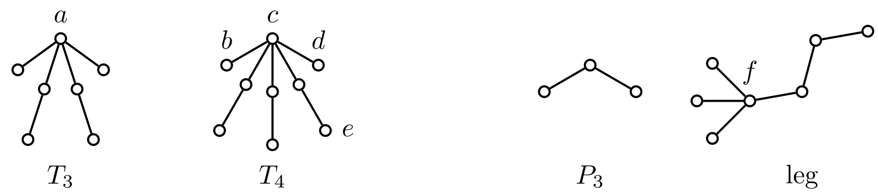

Figure 1: The trees T_3 and T_4, the graph P_3 and the leg

Proof: Similarly as in the proof before, taking x as an optimal first move for Dominator in $G_1|S_1$ and applying Inequalities (1) and (2), we get that $k+\ell \leq \gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k+\ell+1$. Also, taking x as an optimal first move for Staller in $G_1|S_1$ and applying Inequalities (3) and (4), we get that $k+\ell+1 \leq \gamma_g'((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k+\ell+2$.

We now propose examples showing that these bounds are tight. Denote by T_i the tree made of a root vertex r of degree $i+1$ adjacent to two leaves and $i-1$ paths of length 2. Figure 1 shows the trees T_3 and T_4. Note that the domination number of T_i is $\gamma(T_i) = i$. For the domination game, T_i realizes $(i, i+1)$. We claim that for any k, ℓ, $\gamma_g(T_k \cup T_\ell) = k+\ell+1$. Note that if x is a leaf adjacent to the degree $i+1$ vertex r in some T_i, then i vertices are still needed to dominate $T_i|N[x]$. Then a strategy for Staller so that the game does not finish in less than $k+\ell+1$ moves is to answer any move from Dominator in the other tree by choosing such a leaf (e.g. in Figure 1, answer to Dominator's move on a with b). Then two moves are played already and still $k+\ell-1$ vertices at least are needed to dominate the graph.

Similarly, if $k \geq 2$, for any ℓ, $\gamma_g'(T_k \cup T_\ell) = k+\ell+2$. Staller's strategy would be to start on a leaf adjacent to the root of T_k (e.g. b in Figure 1). Then whatever is Dominator's answer (optimally a), Staller can play a second leaf adjacent to a root (d). Then either Dominator answers to the second root (c) and at least $k+\ell-2$ moves are required to dominate the other vertices, or he tries to dominate a leaf already (say e) and Staller can still play the root (c), leaving $k+\ell-3$ necessary moves after the five initial moves. Observe that this value of $\gamma_g'(T_k \cup T_\ell)$ actually implies the value of $\gamma_g(T_k \cup T_\ell)$ by the previous bounds and Theorem 1.1.

To prove that the lower bounds are tight, it is enough to consider the path on three vertices P_3 and the leg drawn in Figure 1. The leg is the tree consisting of a claw whose degree three vertex is attached to a P_3. The path P_3 realizes $(1, 2)$ and the leg realizes $(3, 4)$. Checking that the union is indeed a $(4, 5)$ is left to the reader. By replacing the path $P_3 = T_1$ by the graph T_k, and attaching $\ell-3$ paths of length two to the vertex f in the leg, we get a general construction tightening the lower bounds of Theorem 2.12 for any $k \geq 1$ and $\ell \geq 3$. ∎

The next corollary directly follows from the above theorems.

Corollary 2.13 *No-minus graphs are closed under disjoint union.*

Note also that thanks to Corollary 2.13, we can extend the result of Theorem 2.7 to all tri-split graphs.

Corollary 2.14 *All tri-split graphs are no-minus graphs.*

3 General case

In this section, we consider the unions of any two graphs. Depending on the parity of the length of the game, we can refine Theorem 2.10 as follows:

Theorem 3.1 *Let $G_1|S_1$ and $G_2|S_2$ be partially dominated graphs.*

- *If $\gamma_g(G_1|S_1)$ and $\gamma_g(G_2|S_2)$ are both even, then*

$$\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \geq \gamma_g(G_1|S_1) + \gamma_g(G_2|S_2) \tag{5}$$

- *If $\gamma_g(G_1|S_1)$ is odd and $\gamma'_g(G_2|S_2)$ is even, then*

$$\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq \gamma_g(G_1|S_1) + \gamma'_g(G_2|S_2) \tag{6}$$

- *If $\gamma'_g(G_1|S_1)$ and $\gamma'_g(G_2|S_2)$ are both even, then*

$$\gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq \gamma'_g(G_1|S_1) + \gamma'_g(G_2|S_2) \tag{7}$$

- *If $\gamma'_g(G_1|S_1)$ is odd and $\gamma_g(G_2|S_2)$ is even, then*

$$\gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) \geq \gamma'_g(G_1|S_1) + \gamma_g(G_2|S_2) \tag{8}$$

Proof: The proof is similar to the proof of Theorem 2.10. For inequality (5), let Staller use the strategy of following, assume without loss of generality that G_1 is dominated before G_2. If Dominator played optimally in G_1, by parity Staller played the last move there and Dominator could not pass a move in G_2, thus he could not manage less moves in G_2 than $\gamma_g(G_2|S_2)$. Yet Dominator may have played so that one more move was necessary in G_1 in order to be able to pass in G_2. Then the number of moves played in G_2 may be only $\gamma_g^{dp}(G_2|S_2)$, but this is no less than $\gamma_g(G_2|S_2) - 1$ and overall, the number of moves is the same. Hence we have $\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \geq \gamma_g(G_1|S_1) + \gamma_g(G_2|S_2)$. The same argument with Dominator using the strategy of following gives inequality (7).

Similarly, for inequality (6), Let Dominator start with playing an optimal move x in $G_1|S_1$ and then apply the strategy of following. Then Staller plays in $(G_1 \cup G_2)|((S_1 \cup N[x]) \cup S_2)$, where $\gamma'_g(G_1|(S_1 \cup N[x])) = \gamma_g(G_1|S_1) - 1$ is even, as well as $\gamma'_g(G_2|S_2)$. Then by the previous argument, $\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq \gamma_g(G_1|S_1) + \gamma'_g(G_2|S_2)$. Inequality (8) is obtained with a similar strategy for Staller. ∎

Using Theorem 2.10 and 3.1, we argue the 21 different cases, according to the type and the parity of each of the components of the union. To simplify the computation, we simply propose the following corollary of Theorem 2.10

Corollary 3.2 *Let $G_1|S_1$ and $G_2|S_2$ be two partially dominated graphs. We have*

$$\begin{aligned}
\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) &\geq \gamma_g(G_1|S_1) + \gamma_g(G_2|S_2) - 1\,, && (9)\\
\gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) &\leq \gamma_g(G_1|S_1) + \gamma'_g(G_2|S_2) + 1\,, && (10)\\
\gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) &\leq \gamma'_g(G_1|S_1) + \gamma'_g(G_2|S_2) + 1\,, && (11)\\
\gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) &\geq \gamma'_g(G_1|S_1) + \gamma_g(G_2|S_2) - 1\,. && (12)
\end{aligned}$$

Proof: To prove these inequalities, we simply apply inequalities of Theorem 2.10 in a general case. We choose for the vertex x an optimal move, getting for example that $\gamma'_g(G_1|(S_1 \cup N[x])) = \gamma_g(G_1|S_1) - 1$. We also use Lemma 2.2 and get for example $\gamma_g^{dp}(G_2|S_2) \geq \gamma_g(G_2|S_2) - 1$. ∎

We now present the general bounds in Table 1, which should be read as follows. The first two columns give the types and parities of the components of the union, where e, e_1 and e_2 denote

G_1	G_2	γ_g	γ'_g	for γ_g	for γ'_g
$(o_1,-)$	$(o_2,+)$	$\gamma_g = o_1+o_2-1$	$\gamma'_g = o_1+o_2$	(9),(6*)	(12*),(7)
$(e_1,-)$	$(e_2,+)$	$\gamma_g = e_1+e_2$	$\gamma'_g = e_1+e_2+1$	(5),(10*)	(8*),(11)
$(o_1,-)$	$(o_2,-)$	$\gamma_g = o_1+o_2-1$	$\gamma'_g = o_1+o_2-2$	(9),(6)	(12),(7)
$(e_1,-)$	$(e_2,-)$	$\gamma_g = e_1+e_2$	$\gamma'_g = e_1+e_2-1$	(5),(10)	(8),(11)
$(o_1,=)$	$(o_2,-)$	$\gamma_g = o_1+o_2-1$	$o_1+o_2-1 \le \gamma'_g \le o_1+o_2$	(9),(6)	(12),(11)
$(e_1,=)$	$(e_2,-)$	$\gamma_g = e_1+e_2$	$e_1+e_2-1 \le \gamma'_g \le e_1+e_2$	(5),(10)	(12),(11)
$(e,=)$	$(o,-)$	$e+o-1 \le \gamma_g \le e+o$	$\gamma'_g = e+o-1$	(9),(10)	(12),(7)
$(o,=)$	$(e,-)$	$e+o-1 \le \gamma_g \le e+o$	$\gamma'_g = e+o$	(9),(10)	(8),(11)
$(e,=)$	$(o,+)$	$e+o-1 \le \gamma_g \le e+o$	$e+o \le \gamma'_g \le e+o+1$	(9),(6*)	(12*),(11)
$(o,-)$	$(e,+)$	$e+o-1 \le \gamma_g \le e+o$	$e+o \le \gamma'_g \le e+o+1$	(9),(10*)	(12*),(11)
$(e,-)$	$(o,+)$	$e+o-1 \le \gamma_g \le e+o$	$e+o \le \gamma'_g \le e+o+1$	(9),(10*)	(12*),(11)
$(e,=)$	$(o,=)$	$e+o-1 \le \gamma_g \le e+o$	$e+o \le \gamma'_g \le e+o+1$	(9),(6*)	(8*),(11)
$(o,-)$	$(e,-)$	$e+o-1 \le \gamma_g \le e+o$	$e+o-2 \le \gamma'_g \le e+o-1$	(9),(10)	(12),(11)
$(e_1,=)$	$(e_2,=)$	$e_1+e_2 \le \gamma_g \le e_1+e_2+1$	$e_1+e_2-1 \le \gamma'_g \le e_1+e_2$	(5),(10)	(12),(7)
$(e_1,=)$	$(e_2,+)$	$e_1+e_2 \le \gamma_g \le e_1+e_2+1$	$e_1+e_2+1 \le \gamma'_g \le e_1+e_2+2$	(5),(10*)	(8*),(11)
$(o,=)$	$(e,+)$	$e+o-1 \le \gamma_g \le e+o+1$	$e+o \le \gamma'_g \le e+o+2$	(9),(10*)	(8),(11)
$(o_1,+)$	$(o_2,+)$	$o_1+o_2-1 \le \gamma_g \le o_1+o_2+1$	$o_1+o_2 \le \gamma'_g \le o_1+o_2+2$	(9),(6)	(12),(7)
$(e_1,+)$	$(e_2,+)$	$e_1+e_2 \le \gamma_g \le e_1+e_2+2$	$e_1+e_2+1 \le \gamma'_g \le e_1+e_2+3$	(5),(10)	(8),(11)
$(o_1,=)$	$(o_2,=)$	$o_1+o_2-1 \le \gamma_g \le o_1+o_2+1$	$o_1+o_2-1 \le \gamma'_g \le o_1+o_2+1$	(9),(10)	(12),(11)
$(o_1,=)$	$(o_2,+)$	$o_1+o_2-1 \le \gamma_g \le o_1+o_2+1$	$o_1+o_2 \le \gamma'_g \le o_1+o_2+2$	(9),(10*)	(12*),(11)
$(e,+)$	$(o,+)$	$e+o-1 \le \gamma_g \le e+o+2$	$e+o \le \gamma'_g \le e+o+3$	(9),(10)	(12),(11)

Table 1: Bounds for general graphs.

even numbers and o, o_1, and o_2 denote odd numbers. The next two columns give the bounds on the game domination numbers of the union. In the last two columns, we give the inequalities we use to get these bounds. We add a $*$ to an inequality number when the inequality is used exchanging roles of G_1 and G_2.

Theorem 3.3 *The bounds from Table 1 hold. In particular, we have:*

$$\begin{aligned} \gamma_g(G_1 \cup G_2) - (\gamma_g(G_1) + \gamma_g(G_2)) &\in \{-1, 0, 1, 2\} \\ \gamma'_g(G_1 \cup G_2) - (\gamma'_g(G_1) + \gamma'_g(G_2)) &\in \{-2, -1, 0, 1\} \end{aligned}$$

and all these values are reached.

Note that the entries in Table 1 are sorted by increasing number of different possibilities. In all cases but four, we attained the bounds of Table 1, examples reaching the bounds are given in Table 2 using graphs of Figure 2 or described below. The symbol $\square$ stands for the Cartesian product of graphs and here is considered having priority on the union (so $P_2 \square P_4 \cup P_3$ is actually $(P_2 \square P_4) \cup P_3$), it is actually used only for this graph $P_2 \square P_4$.

Remark that to tighten many of these bounds involving graphs in PLUS and EQUAL, the examples given cannot be no-minus, for consistency with Theorems 2.11 and 2.12. The graphs used in that cases all contain either an induced C_6 (which is $(3,-)$) or $P_2 \square P_4$ (which is $(4,-)$). The realizations of the examples given were computer checked.

- P_4 is $(2,=)$
- $BLP = P_2 \square P_4 \cup P_3$ is $(4,+)$
- $BLC = P_2 \square P_4 \cup C_6$ is $(6,=)$

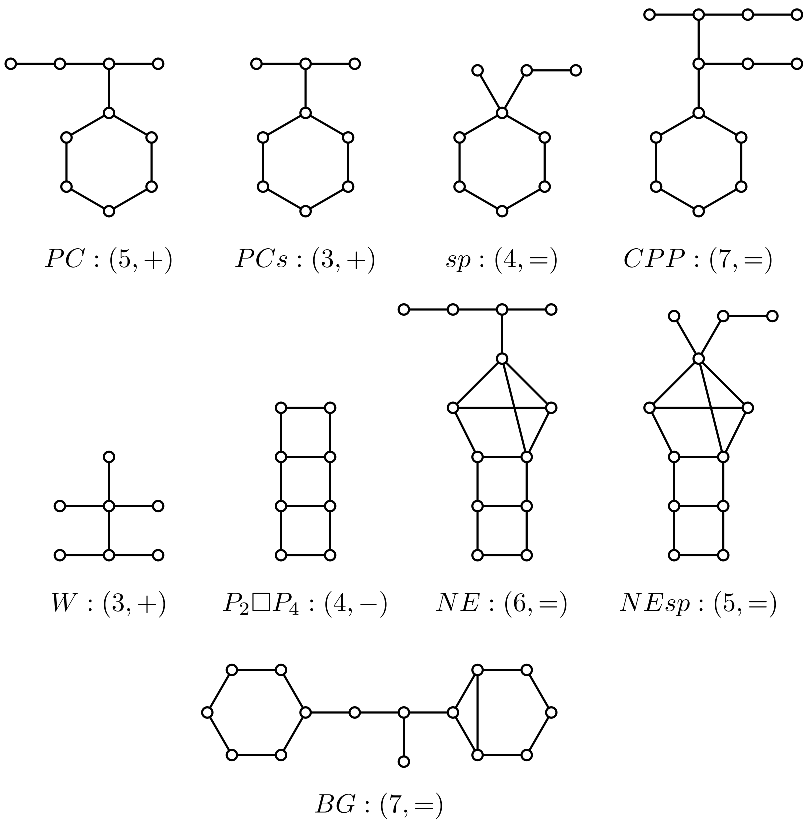

Figure 2: The graphs used in Table 2

G_1	G_2	lower on γ_g	upper on γ_g	lower on γ'_g	upper on γ'_g
$(o_1,-)$	$(o_2,+)$	$C_6 \cup P_3$	$C_6 \cup P_3$	$C_6 \cup P_3$	$C_6 \cup P_3$
$(e_1,-)$	$(e_2,+)$	$P_2 \square P_4 \cup T_2$	$P_2 \square P_4 \cup T_2$	$P_2 \square P_4 \cup T_2$	$P_2 \square P_4 \cup T_2$
$(o_1,-)$	$(o_2,-)$	$C_6 \cup C_6$	$C_6 \cup C_6$	$C_6 \cup C_6$	$C_6 \cup C_6$
$(e_1,-)$	$(e_2,-)$	$P_2 \square P_4 \cup P_2 \square P_4$	$P_2 \square P_4 \cup P_2 \square P_4$	$P_2 \square P_4 \cup P_2 \square P_4$	$P_2 \square P_4 \cup P_2 \square P_4$
$(o_1,=)$	$(o_2,-)$	$K_1 \cup C_6$	$K_1 \cup C_6$	?	$K_1 \cup C_6$
$(e_1,=)$	$(e_2,-)$	$P_8 \cup P_2 \square P_4$	$sp \cup P_2 \square P_4$	$P_8 \cup P_2 \square P_4$	$sp \cup P_2 \square P_4$
$(e,=)$	$(o,-)$	$NE \cup C_6$	$P_8 \cup C_6$	$P_8 \cup C_6$	$P_8 \cup C_6$
$(o,=)$	$(e,-)$	$P_{10} \cup P_2 \square P_4$	$BG \cup P_2 \square P_4$	$P_{10} \cup P_2 \square P_4$	$P_{10} \cup P_2 \square P_4$
$(e,=)$	$(o,+)$	$NE \cup W$	$P_4 \cup T_3$	$NE \cup W$	$P_4 \cup T_3$
$(o,-)$	$(e,+)$	$C_6 \cup BLPK$	$C_6 \cup T_4$	$C_6 \cup BLPK$	$C_6 \cup T_4$
$(e,-)$	$(o,+)$	$P_2 \square P_4 \cup P_{11}$	$P_2 \square P_4 \cup PCs$	$P_2 \square P_4 \cup P_{11}$	$P_2 \square P_4 \cup PCs$
$(e,=)$	$(o,=)$	$NE \cup P_6$	$sp \cup BLCK$	$NE \cup P_6$	$sp \cup BLCK$
$(o,-)$	$(e,-)$	$C_6 \cup (3P_2 \square P_4)$	$(3C_6) \cup P_2 \square P_4$	$C_6 \cup (3P_2 \square P_4)$	$(3C_6) \cup P_2 \square P_4$
$(e_1,=)$	$(e_2,=)$	$NE \cup NE$	$sp \cup sp$	?	$sp \cup sp$
$(e_1,=)$	$(e_2,+)$	$P_4 \cup T_4$	$sp \cup T_4$	$P_4 \cup T_4$	$sp \cup T_4$
$(o,=)$	$(e,+)$	$CPP \cup BLPK$	$K_1 \cup BLP$	$CPP \cup BLPK$	$K_1 \cup BLP$
$(o_1,+)$	$(o_2,+)$	$PC \cup PC$	$T_5 \cup T_5$	$PC \cup PC$	$T_5 \cup T_5$
$(e_1,+)$	$(e_2,+)$	$BLPK \cup BLPK$	$BLP \cup BLP$	$BLPK \cup BLPK$	$BLP \cup BLP$
$(o_1,=)$	$(o_2,=)$	$CPP \cup CPP$	?	?	$NEsp \cup NEsp$
$(o_1,=)$	$(o_2,+)$	$BLCK \cup PC$	$BLCK \cup PCs$	$BLCK \cup PC$	$BLCK \cup PCs$
$(e,+)$	$(o,+)$	$BLWK \cup PC$	$T_4 \cup (C_6 \cup P_3)$	$BLWK \cup PC$	$T_4 \cup (C_6 \cup P_3)$

Table 2: Examples of graphs reaching bounds of Table 1.

- $BLCK = P_2 \square P_4 \cup C_6 \cup K_1$ is $(7,=)$
- $BLPK = P_2 \square P_4 \cup P_3 \cup K_1$ is $(6,+)$
- $BLWK = P_2 \square P_4 \cup W \cup K_1$ is $(8,+)$

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