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PROJECTIVE CONVEXITY IN COMPUTATIONAL KINEMATIC GEOMETRY

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ABSTRACT

In recent years, there is an increasing interest in developing geometric algorithms for kinematic computations. The aim of this paper is to present the notion of projective convexity as a key element for a new framework for kinematic geometry, that allows for the development of more elegant and efficient algorithms for geometric computations with kinematic applications. The resulting framework, called computational kinematic geometry, is developed by combining the oriented projective geometry with the kinematic geometry of rigid body motions.

Introduction

Geometric analysis has much of its roots in kinematics and has long been the foundation for many methods in kinematic analysis and synthesis. Much of the existing work on kinematic geometry, however, deals primarily with rigid-body motions generated by mechanisms and robot manipulators (see, for example, Hunt, 1978, Bottema and Roth, 1979; McCarthy, 1990; Erdman and Sandor, 1997). Research in kinematics over the last decade has been oriented towards on an area called *Computational Kinematics*. Although its definition and scope is still a subject for debate, computational kinematics deals with the development and application of general computational algorithms and numerical methods for solving a broad class of problems that arise from the analysis and synthesis of mechanisms and manipulators (Angeles et al., 1993; Merlet and Ravani, 1995; Park and Iurascu, 2001).

In more recent years, there is an increasing interest in developing analytically defined parametric motions by bring together the fields of kinematics and Computer Aided Geometric Design (CAGD). These parametric motions are typically defined by combining Bézier or B-spline representation in CAGD with representations of rigid-body displacements in kinematics such as quaternions, dual quaternions, Lie groups and Lie algebra (Ge and Ravani, 1994a and 1994b; Park and Ravani, 1995; Kim and Nam, 1995; Ge and Kang, 1995; Etzel and McCarthy, 1996; Jüttler and Wagner, 1996; Zefran and Kumar, 1996; Srinivasan and Ge, 1996 and 1998). Applications of these freeform motions include motion animation in computer graphics, spatial navigation in virtual reality as well as path planning in robotics and 5-axis CNC machining.

Another area of research that has strong connection to kinematic geometry is the geometric analysis of swept volumes of an object under a rigid-body motion. This research topic has attracted intense research effort in Robotics and CNC machining research commmunity due to its obvious applications in motion planning in robotics and CNC machining (see, for example, Abdel-Malek et al., 1998 and 2001; Blackmore et al., 1997 and 1999; Lee, 1997 and 1998). In difference to the classical kinematics, which deals with movements of unbounded infinite spaces (including linear elements such as points, infinite lines, and infinite planes embedded in them), the problem of swept volume analysis focuses on trajectories and swept volumes of a bounded object which may have curved boundaries.

The aim of this paper is to present the notion of projective convexity in computational kinematic geometry. It is developed by combining the classical projective geometry with the kinematic geometry of rigid body motions. Geometric algorithms become more general and elegant when developed from the perspective of projective geometry.

The organization of the paper is as follows. Section 1 summarizes the homogeneous coordinates of points, planes, and lines as well as the notion of convexity in affine geometry. Section 2 studies the notion of convexity of a set of spherical displacements in an oriented image space of spherical kinematics. Section 3 studies the notion of convexity of a set of spatial displacements in an oriented image space of spatial kinematics. In both section 2 and 3, special attention is given to clarify the meaning of ”line-segments”, which is essential for studying convexity in the image spaces.

1 Geometric Fundamentals

1.1 Homogeneous Coordinates of Points

It is well known that a point P in Euclidean three-space E^3 with Cartesian coordinates (x_1, x_2, x_3) can be identified with a point in projective three-space P^3 using homogeneous coordinates:

$$\mathbf{P} = (wx_1, wx_2, wx_3, w),$$

where $w \neq 0$ is a real-number weight. Regardless of the value for w, the homogeneous vector $\mathbf{P}$ represents one and the same point. When $w = 1$, the homogeneous coordinates $(x_1, x_2, x_3, 1)$ are said to be normalized.

1.2 Homogeneous Coordinates of Lines

A line in E^3 passing through a point $\mathbf{x}$ and with unit direction vector $\mathbf{u}$ can be described by so-called Plücker line coordinates:

$$\mathbf{L} = (w\mathbf{u}, w\mathbf{u}^0), \tag{1}$$

where $\mathbf{u}^0 = \mathbf{x} \times \mathbf{u}$ and $w \neq 0$ is a positive weight. The pair of Plücker vectors satisfy the Plückerian condition:

$$\mathbf{u} \cdot \mathbf{u}^0 = 0. \tag{2}$$

Regardless of the value for $w > 0$, $\mathbf{L}$ represents one and the same oriented line. When $w = 1$, the Plücker coordinates are said to be normalized.

With the introduction of a dual number unit ε such that $\varepsilon^2 = 0$, a pair of vectors $(\mathbf{l}, \mathbf{l}^0)$ can be combined into the following compact form:

$$\hat{\mathbf{l}} = \mathbf{l} + \varepsilon \mathbf{l}^0 = (\hat{l}_1, \hat{l}_2, \hat{l}_3), \tag{3}$$

where $\hat{l}_i = l_i + \varepsilon l_i^0$. The resulting entity is called a dual vector (Bottema and Roth, 1979).

1.3 Convexity in Affine Geometry

The fundamental operation for points in affine geometry is the barycentric combination. Given a set of points $\mathbf{b}_i$ ($i = 0, 1, 2, \ldots, n$) in Euclidean three-space E^3, the following linear combination,

$$\mathbf{b} = \sum_{i=0}^{n} \alpha_i \mathbf{b}_i \tag{4}$$

is called a *barycentric combination* if

$$\sum_{i=0}^{n} \alpha_i = 1.$$

The coefficients α_i are called *barycentric coordinates*.

For example, a barycentric combination of two points $\mathbf{b}_0$ and $\mathbf{b}_1$ is given by

$$\mathbf{b} = \alpha_0 \mathbf{b}_0 + \alpha_1 \mathbf{b}_1, \;\; \alpha_0 + \alpha_1 = 1. \tag{5}$$

Barycentric coordinates defines unambiguously the location of the point $\mathbf{b}$ on a line relative to the points $\mathbf{b}_0$ and $\mathbf{b}_1$. If $\alpha_0 > 0$ and $\alpha_1 > 0$, then the point $\mathbf{b}$ belongs to the open segment (not including the end points) from $\mathbf{b}_0$ to $\mathbf{b}_1$; if one of the coordinates is negative, then the point $\mathbf{b}$ is outside of the closed segment (including the end points) from $\mathbf{b}_0$ to $\mathbf{b}_1$.

When all α_i are nonnegative, the combination (4) is called a *convex combination*. All convex combinations of a point set $\mathbf{b}_i$ define the *convex hull* of the set. The resulting set is a *convex set*, which is characterized by the following: for any two points in the set, the straight line joining them is also contained in the set. This definition is invariant with respect to affine transformations and does not depend on the choice of the coordinate system. An affine transformation leaves barycentric combinations invariant (see Farin 1993).

2 Projective Convexity in Spherical Kinematic Geometry

In this section, we develop the concept of projective convexity in the image space of spherical kinematics. The study of kinematics using kinematic mappings and the associated image spaces dates back to Study (1903), Blaschke (1960), and Müller (1962). A comprehensive review of the work in this area can be found in Bottema and Roth (1979). Ravani and Roth (1984) refined the concept of image spaces for spherical and spatial kinematics and applied to mechanism synthesis. McCarthy (1990) provided another treatment of the image space. Ge and Ravani (1994a, 1994b) applied the notion of image space to freeform motion synthesis. A survey of rational motion design can be found in Röschel (1998). Dooley and B. Ravani (1994) introduced a definition of the convex hull of a set of lines in the context of multiple friction contacts of rigid-body dynamics. Buss and Fillmore (2001) discussed convexity on spheres.

2.1 The Oriented Image Space of Spherical Kinematics

It is well know in kinematics that a spherical displacement, which is a rotation of angle θ about a fixed axis, can be described the so-called Euler parameters, $\mathbf{q} = (q_1, q_2, q_3, q_4)$, where

$$\mathbf{q} = (\mathbf{s}\sin(\theta/2), \cos(\theta/2), \tag{6}$$

and $\mathbf{s} = (s_1, s_2, s_3)$ denotes the unit direction vector along the axis. The Euler parameters so defined are normalized, i.e., $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$. A homogeneous representation of the Euler parameters is given by

$$\mathbf{Q} = w\mathbf{q}, \tag{7}$$

where $w \neq 0$ is the weight. Regardless of the value for w, $w\mathbf{q}$ and $\mathbf{q}$ represent one and the same spherical displacement. The homogeneous coordinates, $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4)$, can also be used to define a point in the projective three-space P^3. Ravani and Roth (1984) referred to this space as the *Image Space of Spherical Kinematics*. They have shown that a change in the reference frames attached the moving and fixed spaces would leave the quadratic form $w^2 = Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2$ invariant. This implies that the metric geometry of P^3 is elliptic, or equivalently, spherical. A point in the Image Space is called an image point of spherical displacement. With the spherical model of the Image Space, two antipodal image points, $\mathbf{Q}$ and $-\mathbf{Q}$ are considered to be identical that represent the same spherical displacement.

In kinematics, the word "displacement" is used in a rather special way. It implies that we have no interest in how a motion actually proceeds: we consider only the object position before and after the motion. In the case of a spherical displacement, this concept of displacement does not capture the "sense" of a rotation. By this I mean that starting from the same initial position, the same final position can be achieved with two rotational motions with opposite sense of rotation: the forward rotation has the rotation angle θ and is about the axis $\mathbf{s}$ and the backward rotation has the rotational angle $2\pi - \theta$ and is about the axis $-\mathbf{s}$, which is geometrically the same as $\mathbf{s}$ but oppositely directed. Another way to look at is that the instantaneous angular velocities of the two rotational motions are oppositely directed. The Euler parameters associated with the two geometrically equivalent but oppositely oriented rotations are given by $\mathbf{Q}$ and $-\mathbf{Q}$, respectively. This distinction of the sense of rotation is important in the development of freeform parametric motions such as rational Bézier motions as has been shown in Ge and Ravani (1994a, 1994b). In this paper, this distinction facilitates the unambiguous definition of a motion segment, which is the basis for applying the notion of convexity to kinematic geometry.

Thus, we associate each spherical displacement with a sense of rotation. This is done algebraically by requireing that the weight w in the quaternion coordinates $\mathbf{Q}$ to be positive, i.e., we consider $\mathbf{Q}$ as *signed homogeneous coordinates* of a projective three space. This allows to consider two antipodal points $\mathbf{Q}$ and $-\mathbf{Q}$ to be distinct as opposed to identical. In this way, we can attach a sense of direction or orientation to a geometric feature such as a line-segment or a plane in the projective space. For the lack of a better term, we refer to the resulting projective space as "oriented". This notion of oriented space is not entirely new. For example, a directed line-segment is known as a "spear" in classical geometry text. It is emphasized here because it provides a consistent computational framework for handling geometric computations in the Image Space. However, the notion of oriented projective space advocated here should not to be confused with the notion of "orientability" in topology, which means that there are no orientation reversing paths on a manifold.

2.2 Quaternions and Spherical Displacements

Quaternion is an elegant tool in spherical kinematics. Let $i, j, k, 1$ denote the quaternion units with properties such as $i^2 = -1$ and $ij = k$. Then the Euler parameters can be used to define a quaternion, $\mathbf{Q} = Q_1 i + Q_2 j + q_3 k + Q_4$. Let $\mathbf{P}$ and $\tilde{\mathbf{P}}$ denote, respectively, the homogeneous coordinates of a point of a rigid body before and after a rotation $\mathbf{Q}$. Then the rotational transformation of the point coordinates is given by the following quaternion equation:

$$\tilde{\mathbf{P}} = \mathbf{Q}\mathbf{P}\mathbf{Q}^* \tag{8}$$

where "$*$" denotes the conjugate of a quaternion, i.e., $\mathbf{Q}^* = -Q_1 i - Q_2 j - Q_3 k + Q_4$. It is clear from the quadratic equation that $\mathbf{Q}$ and $-\mathbf{Q}$ would result in the same point coordinate transformation.

Quaternion algebra is also used for composing two successive rotations. Let $\mathbf{Q}_0, \mathbf{Q}_1$ denote two rotations. The composition of the two rotations is given by the quaternion prodcut $\mathbf{Q}_0\mathbf{Q}_1$.

2.3 Line-Segments in the Oriented Image Space

We study the projective convexity associated with sets of spherical displacements in the oriented Image Space, which is an oriented projective three-space P^3. The projective convexity in P^3 is essentially the same as the affine definition of convexity. A set is said to be convex if it contains every line-segment whose end points lie on the set. The key here is to clarify the meaning of "line-segment", especially in the case of antipodal points.

Given two oriented image points $\mathbf{Q}_i$ ($i = 0, 1$) in P^3, any point $\mathbf{Q}$ on the line-segment Q_0Q_1 is given by the following linear combination:

$$\mathbf{Q}(\alpha_0, \alpha_1) = \alpha_0\mathbf{Q}_0 + \alpha_1\mathbf{Q}_1 \tag{9}$$

where we restrict the sum $\mathbf{Q}(\alpha_0, \alpha_1)$ to be nonzero and we require the real coefficients α_i to be non-negative. The "nonzero sum" restriction implies that $(\alpha_0, \alpha_1) \neq (0, 0)$ when the two points are not antipodal and that the ratio $\alpha_1 : \alpha_0$ is not inverse proportional to the weight weight $w_0 : w_1$ associated with $\mathbf{Q}_i$ when the two points are antipodal.

Now consider a spherical model of P^3. If the two points, $\mathbf{Q}_0$ and $\mathbf{Q}_1$, are distinct and not antipodal, then Q_0Q_1 is the shorter of the two great circular arcs connecting them. If the two points are identical, then we get a single point; if the two points are antipodal, we get the two antipodal points.

With Eq.(9), we can also talk about open line-segment as opposed to closed line-segment as defined above by removing the end points. This is done by requiring that the coefficients α_i to be strictly positive. An open line-segment of two identical points is an empty set. An open line-segment of two antipodal points is also an empty set.

To investigate the kinematic meaning of the line-segment, we assume, without loss of generality, that initially the moving and fixed frames are coincident so that $\mathbf{Q}_0 = (0, 0, 0, 1)$. In addition, we let $\mathbf{Q}_1 = (\tan(\theta_1/2)\mathbf{s}_1, 1)$. In this way, Eq. (9) becomes,

$$\mathbf{Q} = (\alpha_1 \tan(\theta_1/2)\mathbf{s}_1, \alpha_0 + \alpha_1),$$

which is equivalent to

$$\mathbf{Q} = (\tan(\beta/2)\mathbf{s}_1, 1), \tag{10}$$

where

$$\tan(\beta/2) = \frac{\alpha_1 \tan(\theta_1/2)}{\alpha_0 + \alpha_1}. \tag{11}$$

It is clear from (10) and (11) that as we vary the values of (α_0, α_1), the axis of rotation remains fixed and that the angle of rotation β varies in the range $0 \leq \beta \leq \theta_1$. Therefore, a line-segment in P^3 corresponds to a motion segment, called *convex hull of the two spherical displacements*, which contains all spherical displacements that belong to a pure rotation from the starting position (or displacement) to the end position (or displacement)[1].

[1] In this paper, we use the term "position" and "displacement" inerchangeably.

When the two oriented points are identical, the convex hull reduces to a single displacement; when the two oriented points are antipodal, the convex hull consists of two displacements that geometrically identical but with opposite sense of orientation. The presence of the non-point in between the two antipodal points may be interpreted as that the two oppositely oriented displacements are related not by a Euclidean displacement in E^3 but by a reflection in E^4 that changes the sense of direction for the vector normal to the hyperplane E^3 but leaves hyperplane E^3 itself invariant.

2.4 Convex Sets of Spherical Displacements

Since this paper deals only with kinematic geometry of E^3, from now on, when defining a set of oriented displacements, we exclude those that are geometrically equivalent but oppositely oriented. This leads to the following definition:

Definition 1. A set Z of oriented spherical displacements is convex if it is not empty, has no geometrically equivalent but oppositely oriented displacements, and for any pair $Q_0, Q_1 \in Z$, the convex hull of Q_0 and Q_1 is also in Z.

Let a set of oriented spherical displacements be represented by a set of points $\mathbf{Q}_i$ ($i = 0, 1, 2, \ldots, n$) in P^3. These points form a simplex S in P^3 with $\mathbf{Q}_i$ as their vertices. Similar to the affine definition of convex combination, we define the following as a convex combination of the set of points in P^3:

$$\mathbf{Q} = \sum_{i=0}^{n} \alpha_i \mathbf{Q}_i \tag{12}$$

where $\alpha_i \geq 0$ for all i but $(\alpha_0, \alpha_1, \ldots, \alpha_n) \neq (0, 0, \ldots, 0)$. The point $\mathbf{Q}$ lies inside of the simplex S. All convex combinations of the set of points $\mathbf{Q}_i$ span a convex set called the convex hull of the point set. Kinematically, every point of the convex hull in P^3 defines a spherical displacement that belong to the convex hull of the set of oriented spherical displacements represented by $\mathbf{Q}_i$ ($i = 0, 1, 2, \ldots, n$).

We note that Woo (1994) studied the problem of convex hull on a sphere for visiability analysis.

2.5 Triangular Segments

The boundaries of a convex hull of a point set in P^3 include vertices, line-segments, and triangular segments. We know that a vertex represents a spherical displacement, a line-segment represents a rotational motion segment connecting one spherical displacement to another. What about a triangular segment?

Let $\mathbf{Q}_i$ ($i = 0, 1, 2$) be homogeneous vectors of the image points of three spherical displacements Q_i. A point on a triangular segment is given by

$$\mathbf{Q} = \alpha_0\mathbf{Q}_0 + \alpha_1\mathbf{Q}_1 + \alpha_2\mathbf{Q}_2. \tag{13}$$

The pole of the plane containing the triangle is given by

$$\mathbf{R} =^* (\mathbf{Q}_0 \wedge \mathbf{Q}_1 \wedge \mathbf{Q}_2), \tag{14}$$

where "$\wedge$" denotes the wedge product of vectors and "*" denotes the dual operator in multi-linear algebra (see Flanders, 1989). The result of the wedge product of the three vectors $\mathbf{Q}_i$ $(i = 0,1,2)$ is the trivector $\mathbf{Q}_0 \wedge \mathbf{Q}_1 \wedge \mathbf{Q}_2$ whose coordinates are given by the determinants of the four minors of the 3×4 array formed by $\mathbf{Q}_i$ $(i = 0,1,2)$. The dual operator converts the resulting tri-vector $\mathbf{Q}_0 \wedge \mathbf{Q}_1 \wedge \mathbf{Q}_2$ into a conventional vector.

Regardless of the choice for α_i, we have

$$\mathbf{Q} \cdot \mathbf{R} = Q_1R_1 + Q_2R_2 + Q_3R_3 + Q_4R_4 = 0. \tag{15}$$

Since the metric geometry of the Image Space is elliptic, it follows from (15) that the angular distance between any point $\mathbf{Q}$ on the triangular segment to the polar point $\mathbf{R}$ is always $\pi/2$. This means that the spherical displacements defined by $\mathbf{Q}$ and $\mathbf{R}$, respectively, are related by a half turn. In other words, all spherical displacements belonging to the triangular segment are line-symmetric displacements with respect to the symmetric position defined by $\mathbf{R}$. Let $\mathbf{l}_i$ $(i = 0,1,2)$ denote the three-dimensional vectors representing lines of symmetry associated with three spherical displacements. They can be obtained from the quaternion product

$$\mathbf{l}_i = \mathbf{Q}_i\mathbf{R}^*, \tag{16}$$

where $\mathbf{R}^*$ is the conjugate of $\mathbf{R}$. Thus the line of symmetry associated with a spherical displacement $\mathbf{Q}$ as defined by (13) is on the line congruence

$$\mathbf{l} = \alpha_0\mathbf{l}_0 + \alpha_1\mathbf{l}_1 + \alpha_2\mathbf{l}_2, \tag{17}$$

which is a triangular cone defined by three lines of symmetry. We conclude that the convex hull of three spherical displacements Q_i $(i = 0,1,2)$ is a two-parameter line-symmetric spherical motion bounded by three one-parameter pure rotations from Q_0 to Q_1, Q_1 to Q_2, and Q_2 to Q_0, respectively.

2.6 Projective Representation of Spherical Bézier Motions

As an application of the projective convexity, we now present a projective representation of spherical Bézier motions by recursive application of the linear combination (9). A comprehensive treatment of Bézier curves in E^3 can be found in Farin (1993).

Projective De Castelau Algorithm: Let $\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_n \in P^3$ denote a set of oriented image points that represent a set of spherical displacements. Let $\alpha_i \geq 0$ $(i = 0,1)$ but $(\alpha_0, \alpha_1) \neq (0,0)$. Set

$$\mathbf{b}_i^r(\alpha_0,\alpha_1) = \alpha_0\mathbf{b}_i^{r-1} + \alpha_1\mathbf{b}_{i+1}^{r-1} \quad r = 1,\cdots,n, \quad i = 0,\cdots,n-r. \tag{18}$$

Please note we follow the convention commonly used in CAGD text, i.e., the superscript is an index, not an exponent. When $r = 0$, $\mathbf{b}_i^0 = \mathbf{b}_i$. When $r = n$, $\mathbf{b}_0^n(\alpha_0,\alpha_1)$ traces out a Bézier curve in P^3 (denoted by $\mathbf{b}^n$) as (α_0,α_1) varies. Every image point on the Bézier curve corresponds to a spherical displacement that belong to a Bézier spherical motion in E^3. The points $\mathbf{b}_i$ are called *Bézier points* or *control points* of the Bézier curve. The spherical displacements that they represent are called *Bézier spherical displacements*. It is clear that the Bézier spherical motion must lie in the convex hull of the Bézier spherical displacements.

Now let us impose an additional requirement that $\alpha_0 + \alpha_1 = 1$. This implies that we treat the Bézier points as points in affine geometry. After letting $\alpha_1 = t$ and $\alpha_0 = 1-t$, Then Eq.(18) becomes the following well-known linear interpolation:

$$\mathbf{b}_i^r(t) = (1-t)\mathbf{b}_i^{r-1}(t) + t\mathbf{b}_{i+1}^{r-1}(t). \tag{19}$$

In view of (8), every point of a rigid body under the rotational motion defined by (19) traces out a circular arc in quadratic rational form. The Bézier curve $\hat{\mathbf{b}}^n(t)$ in this case defines a rational Bézier spherical motion of degree $2n$, for every point of the moving body traces out a rational curve of degree $2n$. More details about rational Bézier spherical motions can be found in Ge and Larochelle (1999).

As opposed to treat Bézier image points as points in affine geometry, we can treat them as points in elliptic geometry. This means that, for unit vectors $\mathbf{b}_i^{r-1}$ and $\mathbf{b}_{i+1}^{r-1}$, the coefficients α_0 and α_1 should be selected such that $\mathbf{b}_i^r(\alpha_0,\alpha_1)$ is also a unit vector. In this case, the coefficients can be shown to be

$$\alpha_0 = \frac{\sin((1-t)\phi)}{\sin\phi}, \quad \alpha_1 = \frac{\sin(t\phi)}{\sin\phi}, \tag{20}$$

where ϕ is the angular distance from $\mathbf{b}_i^{r-1}$ to $\mathbf{b}_{i+1}^{r-1}$, which is half of the rotation angle between the two spherical displacements that they represent. Shoemake (1985) was the first to apply "spherical linear interpolation" of the form (20) to defining Bézier spherical motions. The resulting Bézier spherical motion is not a rational motion but it has also the convex hull property, i.e., all spherical displacements belonging to the Bézier motion are within the convex hull of the Bézier spherical displacements.

3 Projective Convexity in Spatial Kinematic Geometry
3.1 The Oriented Image Space of Spatial Kinematics

It is well known that a spatial displacement in Euclidean three-space E^3 is equivalent to a screw displacement about a fixed axis. Let $\hat{\mathbf{s}}$ denote a unit dual vector representing the screw axis and let $\hat{\theta} = \theta + \varepsilon h$ denote the dual angle representing a rotation of angle θ about and a translation of distance h along the screw axis. They can be used to define the so-called dual Euler parameters:

$$\hat{\mathbf{q}} = (\hat{\mathbf{s}}\sin(\hat{\theta}/2), \cos(\hat{\theta}/2). \tag{21}$$

The dual Euler parameters can be separated into a real part $\mathbf{q}$, which is the normalized Eulers parameters of rotation, and a dual part, $\mathbf{q}^0$, i.e., $\hat{\mathbf{q}} = \mathbf{q} + \varepsilon\mathbf{q}^0$. The real and dual parts satisfy the Plückerian condition $\mathbf{q}\cdot\mathbf{q}^0 = 0$. It follows that $\hat{\mathbf{q}}\cdot\hat{\mathbf{q}} = \mathbf{q}\cdot\mathbf{q} + 2\mathbf{q}\cdot\mathbf{q}^0 = 1$, which means that $\hat{\mathbf{q}}$ are normalized dual Euler parameters. Furthermore, if a quaternion representation is used, then $\hat{\mathbf{q}}$ is a unit dual quaternion. It can be shown that the dual part, which is associated with the translational component of the spatial displacement, can be alternatively defined by the following quaternion product

$$\mathbf{q}^0 = (1/2)\mathbf{d}\mathbf{q}, \tag{22}$$

where $\mathbf{d}$ is a vector quaternion associated with the translation vector.

Let $\hat{w} = w(1 + \varepsilon\sigma)$ denote a dual number weight. Then a general four-dimensional dual vector $\hat{\mathbf{Q}}$ is related to the unit dual vector $\hat{\mathbf{q}}$ by

$$\hat{\mathbf{Q}} = \hat{w}\hat{\mathbf{q}}. \tag{23}$$

The homogeneous vector (23) can be considered as defining a point in a projective dual three-space $\hat{P}^3$. This space is referred to as the Image Space of Spatial Kinematics by Ravani and Roth (1984). They have shown that a change of the moving and fixed reference frame in Euclidean three-space corresponds to dual-orthogonal transformation in $\hat{P}^3$ that leaves dual weight $\hat{w}$ or equivalently the quadratic form $\hat{Q}_1^2 + \hat{Q}_2^2 + \hat{Q}_3^2 + \hat{Q}_4$ invariant. Thus the metric geometry of $\hat{P}^3$ is elliptic.

Similar to the case of spherical displacements, we may say that a spatial displacement is equivalent to two oppositely oriented screw displacements given by $\hat{\mathbf{Q}}$ and $-\hat{\mathbf{Q}}$. The forward screw displacement, $\hat{\mathbf{Q}}$, is about the screw axis $\hat{\mathbf{s}}$ with dual angle $\hat{\theta}$ and the backward screw displacement, $-\hat{\mathbf{Q}}$, is about he screw axis $-\hat{\mathbf{s}}$ with dual angle $2(\pi - \hat{\theta})$. Thus, we associate each spatial displacement with a sense of orientation and consider the Image Space of spatial kinematics as an oriented projective dual three-space $\hat{P}^3$ by considering two antipodal points $\hat{\mathbf{Q}}$ and $-\hat{\mathbf{Q}}$ to be distinct as opposed to identical. In order to maintain the sign of $\hat{\mathbf{Q}}$, we also require that the real weight w to be positive.

It is noted here again that in this paper, the term "orientation" is used to indicate the sense of direction for a displacement or a geometric feature. It is not to be confused with the notion of orientation in topology.

3.2 Quaternions and Spatial Displacements

Let $\mathbf{P}$, $\tilde{\mathbf{P}}$ denote quaternions whose components are homogeneous point coordinates. Then point coordinate transformation can be expressed in terms of quaternion algebra as:

$$\tilde{\mathbf{P}} = \mathbf{QPQ}^* + \mathbf{Q}^0\mathbf{Q}^* - \mathbf{Q}(\mathbf{Q}^0)^* \tag{24}$$

where "$*$" denotes the conjugate of a quaternion.

Let $\hat{\mathbf{l}}$ and $\tilde{\hat{\mathbf{l}}}$ denote dual vectors of a line before and after a spatial displacement. Then the displacement of the line is given by

$$\tilde{\hat{\mathbf{l}}} = \hat{\mathbf{Q}}\hat{\mathbf{l}}\hat{\mathbf{Q}}^*. \tag{25}$$

We note that the above equations for point and line transformations are completely homogeneous in dual-quaternion components. Thus if $\hat{\mathbf{Q}}$ is defined as n^{th} degree polynomial functions of time t, then Eqs. (24) and (25) define a rational motion of degree $2n$ whose point and line trajectories are rational functions of time.

Similar to the spherical case, quaternion algebra is also used for composing two successive spatial displacements. Let $\hat{\mathbf{Q}}_0, \hat{\mathbf{Q}}_1$ denote two spatial displacements. The composition of the two spatial displacements is given by the quaternion prodcut $\hat{\mathbf{Q}}_0\hat{\mathbf{Q}}_1$.

3.3 Line-Segments in the Oriented Image Space

We study the projective convexity associated with sets of spatial displacements in the oriented Image Space of spatial kinematics, which is an oriented projective dual three-space $\hat{P}^3$. The projective convexity in $\hat{P}^3$ is essentially the same as that in the real projective three-space P^3.

First, we clarify the meaning of "line-segments" in $\hat{P}^3$. Given two oriented points $\hat{\mathbf{Q}}_i$ ($i = 0, 1$) in $\hat{P}^3$, we define the following linear combination by dualizing (9):

$$\hat{\mathbf{Q}}(\hat{\alpha}_0, \hat{\alpha}_1) = \hat{\alpha}_0\hat{\mathbf{Q}}_0 + \hat{\alpha}_1\hat{\mathbf{Q}}_1, \tag{26}$$

where $\hat{\alpha}_i = \alpha_i(1+\varepsilon\gamma_i^0)$ are dual-number coefficients such that $\alpha_i \geq 0$ and the real part of the sum is nonzero, i.e., $\alpha_0\mathbf{Q}_0 + \alpha_1\mathbf{Q}_1 \neq 0$.

The real part of Eq.(26) is the same as Eq.(9), which describes a rotational motion-segment; the dual part of (26) is given by

$$\mathbf{Q}^0(\hat{\alpha}_0, \hat{\alpha}_1) = \alpha_0(\mathbf{Q}_0^0 + \gamma_0\mathbf{Q}_0) + \alpha_1(\mathbf{Q}_1^0 + \gamma_1\mathbf{Q}_1), \tag{27}$$

which specifies whether and how the translational component of a spatial motion couples with the rotational component.

To investigate the kinematic meaning of the line-segment, we assume, without loss of generality, that $\mathbf{Q}_0 = (0,0,0,1)$ and let $\hat{\mathbf{Q}}_1 = (\tan(\hat{\theta_1}/2)\hat{\mathbf{s}}_1, 1)$. It can be shown that (26) leads to

$$\hat{\mathbf{Q}} = (\tan(\hat{\beta}/2)\hat{\mathbf{s}}_1, 1), \tag{28}$$

where

$$\tan(\hat{\beta}/2) = \frac{\hat{\alpha}_1 \tan(\hat{\theta_1}/2)}{\hat{\alpha}_0 + \hat{\alpha}_1}. \tag{29}$$

It can be seen that as $(\hat{\alpha}_0, \hat{\alpha}_1)$ varies, the dual angle $\hat{\beta}$ varies in the range $0 \leq \hat{\beta} \leq \hat{\theta_1}$ but the axis $\hat{\mathbf{s}}_1$ remains fixed. Therefore, a line-segment (26) defines the set of all displacements about a fixed axis such that the associated dual angles are within the range $[0, \hat{\theta}_1]$, where $\hat{\theta}_1$ is the dual angle for the relative screw displacement between two spatial positions represented by $\hat{\mathbf{Q}}_0$ and $\hat{\mathbf{Q}}_1$. Mechanically, this is the set of displacements allowed by a cylindrical joint.

For the linear combination (26), only the ratio of $\hat{\alpha}_i$ $(i = 0,1)$ is significant. With the help of dual-number algebra (see Bottema and Roth, 1979), the ratio is reduced to

$$\hat{\alpha}_1/\hat{\alpha_0} = (\alpha_1/\alpha_0)(1+\varepsilon(\gamma_1-\gamma_0)).$$

In other words, only two parameters, namely α_1/α_0 and $(\gamma_1-\gamma_0)$ are not redundant. Thus, we say (26) defines a *two-fold line-segment*. It defines a two-parameter set of displacements about a screw axis, which is referred to as a *two-fold convex hull* of two spatial displacements.

Another form of line-segment can be defined by restricting the non-redundant parameters to one. This results in a *unifold line-segment* which corresponds to a *unifold convex hull* of two displacements. The resulting motion is a one-parameter screw motion in the usual sense. The simplest way to obtain a unifold line-segment is to let $\gamma_i = 0$ $(i = 0,1)$, i.e., we use only real coefficients α_i in linear combination. Furthermore, if we require that $\alpha_0 + \alpha_1 = 1$, then the resulting screw motion has the property that every point of the moving body traces out a quadratic curve (see Ge and Ravani, 1994a; Li and Ge, 1999). It can be shown that the pitch of the screw motion is defined by a harmonic function. The motion has been referred to as a *vertical Darboux motion* (Bottema and Roth, 1979, pp. 321-322). Another approach is to define $\hat{\alpha}_i$ $(i = 0,1)$ based on the elliptic metric, i.e., requiring that $\hat{Q}_1^2 + \hat{Q}_2^2 + \hat{Q}_3^2 + \hat{Q}_4^2 = 1$. This would result in a unifold linear interpolation in the dual ellpitic three-space. The resulting screw motion has constant pitch (see Ge and Ravani, 1994b).

Analogous to the spherical case, when the two oriented image points are identical, the line-segment, whether unifold or two-fold, reduces to a point; when the two oriented points are antipodal, the line-segment consists of two antipodal points. When the real parts of the image points are indentical but their dual parts are not, then the line-segment is a special unifold line-segment representing a translation. When the dual parts of the image points are identical but their real parts are not, then the line-segment is another special unifold line-segment representing a pure rotation.

3.4 Convex Sets of Spatial Displacements

Excluding the antipodal points that represent are geometrically equivalent but oppositely oriented screw displacements, we have the following definition for the convex set of spatial displacements.

Definition 2. A set Z of oriented spatial displacements is two-fold (or unifold) convex if it is not empty, has no geometrically equivalent but oppositely oriented displacements, and for any pair $Q_0, Q_1 \in Z$, the two-fold (or unifold) convex hull of Q_0 and Q_1 is also in Z.

From now on , we restrict our study on unifold case only. Let a set of oriented spatial displacements be represented by a set of image points $\hat{\mathbf{Q}}_i$ $(i = 0,1,2,\ldots,n)$ in $\hat{P}^3$. These points form a simplex S in $\hat{P}^3$ with $\hat{\mathbf{Q}}_i$ as their vertices. A convex combination of the set of points is given by

$$\hat{\mathbf{Q}} = \sum_{i=0}^{n} \alpha_i \hat{\mathbf{Q}}_i \tag{30}$$

where $\alpha_i \geq 0$ for all i but $(\alpha_0, \alpha_1, \ldots, \alpha_n) \neq (0,0,\ldots,0)$. The point $\hat{\mathbf{Q}}$ lies inside of the simplex S. All convex combinations of the set of points $\hat{\mathbf{Q}}_i$ span a convex set, which is the convex hull of the point set. Kinematically, every point of the convex hull in $\hat{P}^3$ defines a spatial displacement that belong to the convex hull of the set of oriented spatial displacements represented by $\hat{\mathbf{Q}}_i$ $(i = 0,1,2,\ldots,n)$.

3.5 Triangular Segments

Now let us consider a unifold triangular segment in $\hat{P}^3$ defined by the image points $\hat{\mathbf{Q}}_i$ $(i = 0,1,2)$. A point on the triangular segment is given by

$$\hat{\mathbf{Q}} = \alpha_0\hat{\mathbf{Q}}_0 + \alpha_1\hat{\mathbf{Q}}_1 + \alpha_2\hat{\mathbf{Q}}_2. \tag{31}$$

The pole of the unifold plane containing the triangle is given by

$$\hat{\mathbf{R}} =^* (\hat{\mathbf{Q}}_0 \wedge \hat{\mathbf{Q}}_1 \wedge \hat{\mathbf{Q}}_2). \tag{32}$$

Regardless of the choice for α_i, we have $\hat{\mathbf{Q}} \cdot \hat{\mathbf{R}} = 0$, which means that all spatial displacements belonging to the unifold triangular segment are line-symmetric displacements with respect to the symmetric position defined by $\hat{\mathbf{R}}$. Let $\hat{\mathbf{l}}_i$ $(i = 0,1,2)$ denote the dual vectors representing lines of symmetry associated with three spatial displacements. They can be obtained from the quaternion product

$$\hat{\mathbf{l}}_i = \hat{\mathbf{Q}}_i\hat{\mathbf{R}}^*, \tag{33}$$

where $\hat{\mathbf{R}}^*$ is the conjugate of $\hat{\mathbf{R}}$. Thus the line of symmetry associated with a spatial displacement $\hat{\mathbf{Q}}$ as defined by (31) is on the line congruence

$$\hat{\mathbf{l}} = \alpha_0\hat{\mathbf{l}}_0 + \alpha_1\hat{\mathbf{l}}_1 + \alpha_2\hat{\mathbf{l}}_2. \tag{34}$$

The line congruence is bounded by three cylindroidal surface patches connecting the three lines $\hat{\mathbf{l}}_i$. Line geometric approach to define ruled surfaces and line congruences can be found in Ravani and Wang (1991) and Ge and Ravani (1998). We conclude that the convex hull of three spatial displacements Q_i $(i = 0,1,2)$ is a two-parameter line-symmetric spatial motion bounded by three one-parameter screw motions from Q_0 to Q_1, Q_1 to Q_2, and Q_2 to Q_0, respectively.

Conclusions and Discussions

In this paper, we have presented the concepts of projective convexity for sets of spherical and spatial displacements. These concepts have been developed in the image spaces of spherical and spatial kinematics. These image spaces are oriented projective spaces. Special attention is given to the orientation of a displacement so that the notion of "line-segments" can be unambiguously defined. We have also shown with two different approaches for motion interpolation, namely the rational approach and the spherical approach, can be unified in the language of projective geometry.

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