Topology and Chaos

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Abstract. We discuss some basic topological techniques used in the study of chaotic dynamical systems. This paper is partially motivated by a talk given by the second author at the 7th international summer school and conference *Chaos 2008: Let's Face Chaos Through Nonlinear Dynamics* (CAMTP, University of Maribor, Slovenia, 29 June - 13 July 2008).

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INTRODUCTION

Certain dynamical systems have invariant sets and induced maps on these invariant sets than are viewed as chaotic. After a review of the mathematical definition of chaos, we investigate how topological techniques naturally arise in this setting. We discuss first the basic example of the squaring map from the unit circle to itself. We then use this example to discuss a more complicated example, that of the dyadic solenoid. We show how topological inverse limits can be used to study the dynamics of the solenoid. Another more complicated example is also discussed, that of the Whitehead continuum. Finally, we list some additional examples of the interaction between topology and chaos theory.

A basic reference for the topological terms we use is Munkres' book on topology [16]. Two books that provide an introduction to dynamical systems are Devaney's text and Falconer's text [7, 8]. A very good reference for the topology of chaos is Gilmore's and Lefranc's text [12].

DYNAMICAL SYSTEMS AND CHAOS

Chapter 13 in Falconer [8] gives an introduction to dynamical systems. We summarize the terminology from this chapter. For D a subset of R^n and $f : D \to D$ a continuous function, the iterates $f, f^2, f^3, \ldots f^k, \ldots$ form a dynamical system on D where $f^k = f \circ f \circ \ldots \circ f$ is the composition of f with itself k times. We use the word map to refer to a continuous function and use the symbol $\cong$ to represent homeomorphism. A closed subset A of D is an attractor for this system if $f(A) = A$ and if for each point $p \in D$, the distance between the iterates $f^k(p)$ and A converges to 0. The orbit of a point p is the set consisting of the iterates $f^k(p)$. The point p is a periodic point if $f^k(p) = p$ for some k.

For an attractor A as above, the restriction of f to A, $f|_A$ is chaotic if the following three conditions hold:

1. The orbit of some point p in A is dense in A.

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2. The periodic points of f in A are dense in A.
3. The map f is sensitive to initial conditions on A. That is, there is a $\delta > 0$ so that for every $p \in A$, and for every neighborhood U of p, there is a point $x \in U$ and an iterate $f^k(x)$ of x so that the distance between $f^k(p)$ and $f^k(x)$ is greater than δ.

For compact A, the first condition can be replaced by topological transitivity, that is, for each pair of neighborhoods U and V there is an iterate f^k such that $f^k(V) \cap U \neq \emptyset$. See section 1.8 of Devaney [7].

CHAOTIC EXAMPLES

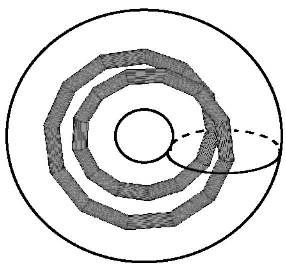
A Chaotic Map on the Circle

The proof of the results in this section can be found in Devaney [7]. Let S^1 be the unit circle centered at the origin in the plane. Viewing points in S^1 as complex numbers, let $f : S^1 \to S^1$ be the map $f(z) = z^2$. If the point $p \in S^1$ has rectangular coordinates $(\cos(\theta), \sin(\theta))$, then $f(p)$ had rectangular coordinates $(\cos(2\theta), \sin(2\theta))$.

The map f is chaotic on S^1. Arcs on the circle are stretched to twice their size under f. This can be used to show the topological transitivity of f and the sensitivity to initial conditions. To see that periodic points are dense, note that the points $\left(\cos(\frac{2\pi k}{2^n - 1}), \sin(\frac{2\pi k}{2^n - 1})\right)$ are periodic of order n.

The Dyadic Solenoid and Inverse Limits

A more complicated example related to the previous one is that of the dyadic solenoid. More detailed descriptions are available in both Falconer [8] and Devaney [7]. The inverse limit connection described below is explained in Schori [17]. Let T be a solid torus in R^3 and let f be a homeomorphism from R^3 to itself that takes T to a solid torus $f(T)$ in the interior of T that wraps around T in the longitudinal direction twice. If T is parameterized by a pair of angles (θ, ϕ), we may also require that f takes the meridional disc at angle θ into the meridional disc at angle 2θ, all angles measured mod 2π. Pictured below are T and $f(T)$.



Since f is a homeomorphism, the pair $(T, f(T))$ is homeomorphic to the the pair $(f(T), f^2(T))$ and thus $f^2(T)$ is interior to $f(T)$ and wraps twice around $f(T)$. Let

$T_n = f^n(T)$. We then have a sequence of tori $T \supset T_1 \supset T_2 \cdots$ with T_n in the interior of T_{n-1} and wrapping twice around T_{n-1} in the longitudinal direction.

The Dyadic Solenoid S is defined to be $\cap_{i=0}^{\infty} T_i$. S is an attractor for f restricted to T. So $f(S) = S$. It is also true that f is chaotic on S. Complete details are in Falconer [8] and Devaney [7]. To exhibit periodic points, we give an alternate description of S as a topological inverse limit.

Let a system of spaces $A_1, A_2, A_3, \ldots$ and maps $f_i : A_{i+1} \to A_i$ be given. The inverse limit of this system, denoted

$$\lim_{\leftarrow}(A_i, f_i)$$

is the subset of the topological product spaces $\prod_{i=1}^{\infty} A_i$ consisting of points $(a_1, a_2, a_3, \ldots)$ with $f_i(a_{i+1}) = a_i$ for all i.

We describe two particular inverse systems each with inverse limit the Dyadic Solenoid. For the first, each A_i is S^1 and the map $f_i : S^1 \to S^1$ is the squaring map of the previous section. For the second, each A_i is $T_i \subset R^3$ and the map $g_i : T_{i+1} \to T_i$ is inclusion. Standard topological techniques show that

$$\lim_{\leftarrow}(T_i, g_i) \cong S$$

Consider the following diagram:

$$\begin{array}{ccccccccccccc}
S^1 & \xleftarrow[f_1]{} & S^1 & \xleftarrow[f_2]{} & S^1 & \longleftarrow \cdots\cdots \longleftarrow & S^1 & \xleftarrow[f_n]{} & S^1 & \longleftarrow \cdots\cdots & & & Y \\
\uparrow{\scriptstyle h_0} & & \uparrow{\scriptstyle h_1} & & \uparrow{\scriptstyle h_2} & & \uparrow & & \uparrow{\scriptstyle h_n} & & & & \uparrow{\scriptstyle h} \\
T_0 & \xleftarrow[g_1]{} & T_1 & \xleftarrow[g_2]{} & T_2 & \longleftarrow \cdots\cdots \longleftarrow & T_{n-1} & \xleftarrow[g_n]{} & T_n & \longleftarrow \cdots\cdots & & & S
\end{array}$$

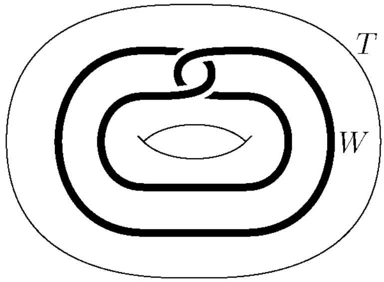
The space Y is the inverse limit of the top row, and consists of points $(a_0, a_1, a_2, \cdots)$ with each $a_i \in S^1$ and with $f_i(a_i) = a_i^2 = a_{i-1}$. The maps $h_i : T_i \to S^1$ are projection onto the longitudinal coordinate. The description of the T_i shows that the diagram commutes, i.e. $f_i \circ h_i = h_{i-1} \circ g_i$ for each i. The maps h_i then induce a map from the inverse limit S to the inverse limit Y. One then checks that the map h is one to one and onto, so that h is a homeomorphism.

The map f on S described earlier can thus be represented by a map s (for squaring) on Y that takes $(a_0, a_1, a_2, \cdots)$ to $(a_0^2, a_0, a_1, a_2, \cdots)$. The dynamics of f can be investigated by examining the dynamics of this equivalent map s defined on Y.

To exhibit periodic points of the map s on Y, take sequences of the form $(a_n^{2^n}, a_n^{2^{n-1}}, \ldots a_n^2, a_n, \ldots)$ where the first $n+1$ coordinates are repeated and where a_n had order n as a periodic map of the squaring map on the circle.

A Chaotic Embedding of the Whitehead Continuum

We describe another example related to the previous one. We again take T be a solid torus in R^3 and now g be a homeomorphism from R^3 to itself that takes T to a solid torus $g(T) = W$ as in the diagram below.



Since g is a homeomorphism, the pair $(T, g(T))$ is homeomorphic to the the pair $(g(T), g^2(T))$ and thus $g^2(T)$ is interior to $g(T)$ and is placed in $g(T)$ just as W is placed in T. Let $T_n = g^n(T)$. We then have a sequence of tori $T \supset T_1 \supset T_2 \cdots$ with T_n in the interior of T_{n-1}.

The Whitehead Continuum $\widetilde{W}$ is defined to be $\cap_{i=0}^{\infty} T_i$. S is an attractor for g restricted to T. So $g(\widetilde{W}) = \widetilde{W}$. However this is where this section diverges from the previous section. The map g as described is not necessarily chaotic. Jubran [14], in a preprint shows that the standard construction fails to have topological transitivity, and so the map g is not chaotic.

Garity, Jubran, and Schori [9, 10] then show that it is possible to modify the construction, still yielding the Whitehead continuum, in such a way as to produce a chaotic embedding. They use more advanced topological techniques from Decomposition Theory (see Daverman's text [6]) to achieve this. The techniques in their paper apply to many spaces constructed as cell-like subsets of R^3, a broad class of spaces that have importance in topological applications.

OTHER RESULTS

(i) It is rather easy to construct plenty of global attractors by using continuous maps $f : I \to I$, where I is the unit interval [0,1]. Using such interval mappings, Barge and Martin [4] proved that every inverse limit space of an interval mapping (called Snake-like Continuum by Bing [5]) can be realized as a global attractor for a homeomorphism of the plane R^2.
(ii) It is well known that the Dyadic Solenoid is an attractor of a homeomorphism of a three-dimensional manifold [17]. On the other hand, Günther [13] showed that the Generalized Solenoid, obtained from a sequence of pairwise relatively prime integers, cannot be an attractor of any self-map of a topological manifold.
(iii) Using forcing of periodic points in orientation-reversing twist maps of the plane (for example, the Hénon maps), van den Berg et al. [18] showed that the fourth iterate of an orientation-reversing twist map can be written as the composition of four orientation-preserving twist maps. Then they reformulated the problem in terms of parabolic flows, which form the natural dynamics on a certain space of braid diagrams. For example, they classify period-4 points in terms of their corresponding braid diagrams.
(iv) An efficient method for estimating the topological complexity of isolated invariant sets of flows is to use the Conley Index, i.e. the homotopy type of the topological quotient

obtained from a special isolating neighbourhood by identifying to a point the region on the boundary of the neighbourhood through which the flow exits. Ghrist et al. [11] developed such a Conley-type approach for certain flows on spaces of discretized braids. They also discussed applications to the detection of periodic orbits for certain second order Lagrangian systems and they proved a forcing theorem showing the existence of infinitely many braid classes of periodic orbits.

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