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On Lack of Uniqueness in Heat Conduction through a Solid to Solid Contact

If two solids are pressed together and made to exchange heat by conduction, one possible steady state corresponds to the solids remaining in contact along the entire interface. It is shown that this state is not unique, and that it is possible to construct solutions involving localized separation if heat flows in one of the two possible directions.

Introduction

The question about the possible lack of uniqueness when two contacting solids exchange heat by conduction was raised by Barber [1]. It is also implicit in the results presented by Dundurs and Panek [2, p. 734]. The object of this article is to demonstrate actual nonuniqueness in steady-state heat conduction by means of a specific example.

Consider for this purpose two semi-infinite solids that are pressed together and made to conduct heat through the contact interface. One possible steady state is that of the solids remaining in contact along the whole interface which results in simple linear temperature distributions normal to the interface. We show, however, that another steady-state solution involving a localized separation zone at the interface and satisfying the customary boundary conditions, as well as the appropriate inequalities, can be constructed for heat flowing across the interface in one of the two possible directions. Moreover, the solutions for full contact and localized separation satisfy the same far-field boundary conditions. We also give a solution for a periodic array of separation zones which appears to be of fundamental importance.

The existence of competing steady-state solutions immediately creates new questions about history dependence, stability, and the rate at which possible disturbances grow if the process of heat conduction between two solids is viewed developing in time. There is also the matter of whether the lack of uniqueness demonstrated in this article is due to the strongly idealized boundary conditions (no resistance to heat flow in zones of solid to solid contact, no heat transfer between the solids in the separation zones) and the fact that the contacting bodies are treated as semi-infinite solids. Except for a few comments made at the end, which anticipate some of the answers, we must leave these issues unresolved at the present time.

The Problem and Its Formulation

Consider two semi-infinite solids that are forced together by the applied pressure, p, and carry the remotely established uniform heat flux, q^∞, in the direction normal to the interface. As mentioned before, one steady-state solution for the thermal and elastic fields corresponds to the solids remaining in contact along the whole interface. We seek here the steady-state solutions, satisfying identical boundary conditions, but which involve localized separation between the solids. The problem is posed in the framework of linear thermoelasticity [3], assuming plane strain conditions. For simplicity, the interface is taken as frictionless.

The coordinate system is placed in relation to the two bodies as shown in Fig. 1; subscripts or superscripts 1 and 2 are used to distinguish the field quantities and physical constants of the two materials. The thermal conductivity is denoted by k, the coefficient of thermal expansion by α, the shear modulus by μ, and Poisson's ratio by ν. The quantity

$$\delta = \frac{\alpha(1+\nu)}{k} \tag{1}$$

is called the distortivity of a material [2].

Assuming that the two materials are homogeneous and isotropic, the steady-state temperature distribution is a plane harmonic function, viz.,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2}$$

in the absence of internal heat generation [4]. Written in the customary indicial notation for the sake of compactness, the pertinent equations for the elastic fields, under the assumption of plane strain [3, Sec. 4.2], consist of the equilibrium conditions

$$\partial_j \sigma_{ji} = 0 \tag{3}$$

on the stress components, the relation between the total strain and displacement

$$\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{4}$$

and Hooke's law

$$\epsilon_{ij} = \frac{1}{2\mu}(\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}) + \alpha(1+\nu)T\delta_{ij} \tag{5}$$

Herein, $i, j, k = 1, 2$; $(x_1 = x, x_2 = y)$; $\partial_i = \partial/\partial x_i$; and repeated indices imply summation over the values 1 and 2. The first term on the right side of (5) constitutes the elastic strain, and the second term is the strain due to thermal expansion.

The boundary conditions to be imposed in the *contact zones* are

$$T_1(x,0) = T_2(x,0) \tag{6}$$

$$k_1 \frac{\partial T_1(x,0)}{\partial y} = k_2 \frac{\partial T_2(x,0)}{\partial y} \tag{7}$$

$$u_y^{(1)}(x,0) = u_y^{(2)}(x,0) \tag{8}$$

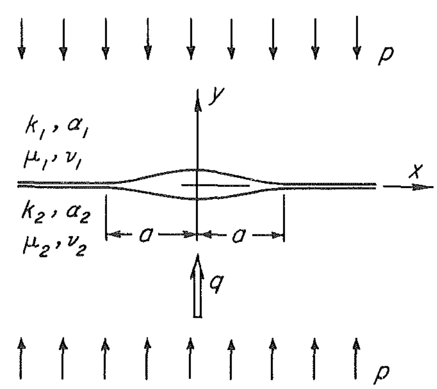

Fig. 1 Contacting solids with a single separation zone

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$$\sigma_{xy}{}^{(1)}(x, 0) = \sigma_{xy}{}^{(2)}(x, 0) = 0 \quad (9)$$

$$\sigma_{yy}{}^{(1)}(x, 0) = \sigma_{yy}{}^{(2)}(x, 0) \leq 0 \quad (10)$$

Equation (6) reflects the idealization that the interface offers no resistance to heat flow in the regions of solid to solid contact. The next boundary condition is simply a statement that no heat is generated at the interface. The boundary condition (8) on the normal components of displacements insures that the bodies are in contact. Equations (9) and (10) express Newton's third law, the assumption of no friction, and the condition that contacting bodies can only exert pressure on each other and that the surface tractions cannot be tensile.

The boundary conditions used in the *separation zones* are

$$\frac{\partial T_1(x, 0)}{\partial y} = \frac{\partial T_2(x, 0)}{\partial y} = 0 \quad (11)$$

$$\sigma_{xy}{}^{(1)}(x, 0) = \sigma_{xy}{}^{(2)}(x, 0) = 0 \quad (12)$$

$$\sigma_{yy}{}^{(1)}(x, 0) = \sigma_{yy}{}^{(2)}(x, 0) = 0 \quad (13)$$

$$u_y{}^{(1)}(x, 0) - u_y{}^{(2)}(x, 0) \geq 0 \quad (14)$$

Equation (11) expresses the assumption that no heat is transmitted across the gaps in the separation zones. The next two boundary conditions insure that the surfaces of the bodies are free of traction in the separation zones. Condition (14) reflects the requirement that the gap between the bodies cannot be negative. It is important to note that the inequalities (10) and (14) make the problem nonlinear.

The boundary conditions at infinity will be discussed after the solutions are constructed.

The field equations (2–5) and the boundary conditions (6–10) are satisfied for full contact by the following temperature, heat flux, displacement and stress distributions:

$$T = -\frac{q^\infty}{k} y \quad (15)$$

$$q_x = 0, q_y = q^\infty \quad (16)$$

$$u_x = \frac{\nu p}{2\mu} x, \; u_y = -\frac{(1-\nu)p}{2\mu} y - \frac{q^\infty \delta}{2(1-\nu)} y^2 \quad (17)$$

$$\sigma_{xx} = \frac{2\mu q^\infty \delta}{1-\nu} y, \; \sigma_{xy} = 0, \; \sigma_{yy} = -p \quad (18)$$

where subscripts 1 or 2 must be attached to k, δ, μ and ν, depending on the region considered. The arbitrary datum for temperature has been adjusted so that the contact interface is at zero temperature. The linearly distributed σ_{xx} or bending stress must be applied in order to prevent the bodies from globally warping away from each other. The behavior of the field quantities at infinity is clear from the explicit formulas.

Single Separation Zone

Next, we construct a steady-state solution in which the solids are allowed to separate over an interface segment of length $2a$. The separation zone is expected to disturb the temperature and stresses only locally. We assume that no heat is transmitted across the gap, and there is no contact resistance outside the separation zone, so that the boundary conditions (11) and (6) apply in $|x| < a$ and $|x| > a$, respectively.

There is no need to start at the level of the field equations and the boundary conditions other than (11) and (13), because we can take advantage of a Green's function for exterior thermoelastic contact that consists of a thermoelastic field (heat vortex) and a purely elastic field (edge dislocation at a freely slipping interface) [5]. The full expressions for the field quantities associated with the Green's function are given in [5] and of immediate interest are only the relationships at the interface.

An isolated heat vortex of strength ω acting at the point $(\xi, 0)$ leads to the following quantities at the interface:

temperature discontinuity across the interface

$$\tau(x) = T_2(x, 0) - T_1(x, 0) = \omega H(x - \xi) \quad (19)$$

heat flux through the interface

$$q_y(x, 0) = -\frac{\omega}{\pi} \frac{k_1 k_2}{k_1 + k_2} \frac{1}{x - \xi} \quad (20)$$

gap between the solids

$$g(x) = u_y{}^{(1)}(x, 0) - u_y{}^{(2)}(x, 0) = 0 \quad (21)$$

and the normal tractions

$$\sigma_{yy}(x, 0) = 2\omega M(\delta_1 - \delta_2) \frac{k_1 k_2}{k_1 + k_2} \mathrm{H}(x - \xi) \quad (22)$$

where H( ) is the Heaviside step function, and

$$M = \frac{\mu_1 \mu_2}{2[\mu_1(1-\nu_2) + \mu_2(1-\nu_1)]} \quad (23)$$

The edge dislocation with the Burgers vector b_y leads to no thermal quantities, but gives

$$g(x) = -b_y \mathrm{H}(x - \xi) \quad (24)$$

$$\sigma_{yy}(x, 0) = \frac{2 b_y M}{\pi} \frac{1}{x - \xi} \quad (25)$$

The desired solution for the single separation zone can be constructed by correcting the fields given by (15–18) for the full contact between the solids. In order to cancel the heat flux through the separation zone, we distribute heat vortices with the density $\Omega(x)$ over the interval $-a < x < a$. On the basis of (20), this leads to the relation

$$q^\infty - \frac{1}{\pi} \frac{k_1 k_2}{k_1 + k_2} \int_{-a}^{a} \frac{\Omega(\xi) d\xi}{x - \xi} = 0, \; |x| < a \quad (26)$$

From (19), it also follows that

$$\tau(x) = \int_{-\infty}^{x} \Omega(\xi) d\xi \quad (27)$$

and consequently

$$\Omega(x) = \frac{d\tau(x)}{dx} \quad (28)$$

If the temperature jump is to vanish outside the inteval $-a < x < a$, We must have

$$\int_{-a}^{a} \Omega(\xi) d\xi = 0 \quad (29)$$

Nomenclature

a = half length of separation zone
b_y = Burgers vector of an isolated dislocation
B_y = intensity of distributed dislocations
g = gap between the solids
h = half length of period
k = thermal conductivity
M = bimaterial constant (see equation (23))
p = applied pressure
q^∞ = heat flux at infinity
q_x, q_y = components of heat flux
T = temperature above the datum at which contact is established
u_i = displacement vector
x, y = cartesian coordinates
α = coefficient of thermal expansion
δ_{ij} = Kronecker delta
$\delta = \alpha(1+\nu)/k$ = distortivity
ϵ_{ij} = strain tensor
μ = shear modulus
ν = Poisson's ratio
σ_{ij} = stress tensor
τ = temperature jump across the interface
ω = strength of an isolated heat vortex
Ω = intensity of distributed heat vortices

Equation (26) is a Cauchy singular integral equation, and its solution with the constraint (29) is known [6]:

$$\Omega(x) = -\frac{q^\infty(k_1+k_2)}{k_1k_2}\frac{x}{(a^2-x^2)^{1/2}}, \; |x| < a \tag{30}$$

It follows then using (22) that the distributed heat vortices give the following interface tractions:

$$\sigma_{yy}(x,0) = \begin{cases} 2Mq^\infty(\delta_1-\delta_2)(a^2-x^2)^{1/2}, & |x|<a \\ 0, & a<|x| \end{cases} \tag{31}$$

The final task is to cancel the normal tractions in the separation zone and allow for the gap. This is done by introducing a distribution $B_y(x)$ of edge dislocations in the interval $-a < x < a$. Thus, from (25) and (31),

$$-p + 2Mq^\infty(\delta_1-\delta_2)(a^2-x^2)^{1/2} + \frac{2M}{\pi}\int_{-a}^{a}\frac{B_y(\xi)d\xi}{x-\xi} = 0, \; |x| < a \tag{32}$$

Using (24), we also get

$$g(x) = -\int_{-\infty}^{x} B(\xi)d\xi \tag{33}$$

and

$$B_y(x) = -\frac{dg(x)}{dx} \tag{34}$$

Since the gap between the solids must vanish in the contact zones

$$\int_{-a}^{a} B(\xi)d\xi = 0 \tag{35}$$

Due to the symmetry of the problem, $B_y(x)$ must be odd in x, and (35) is automatically satisfied, but this will be verified a posteriori. Moreover, $B_y(x)$ must be bounded at $x = \pm a$ [7], which also insures that the interface tractions vanish at the ends of the separation zone.

A bounded solution of a Cauchy singular integral equation is possible only if the right side of the equation satisfies a so-called consistency condition [6]. The consistency condition for (32) is

$$\int_{-a}^{a}\{-p + 2Mq^\infty(\delta_1-\delta_2)(a^2-\xi^2)^{1/2}\}(a^2-\xi^2)^{-1/2}d\xi = 0 \tag{36}$$

Carrying out the elementary integrations, (36) reduces to

$$-\pi p + 4Mq^\infty(\delta_1-\delta_2)a = 0 \tag{37}$$

Equation (37) determines the length of the separation zone. It also shows that a solution is possible (viz., $a > 0$) only when $q^\infty(\delta_1-\delta_2) > 0$, or when heat flows into the material with the larger distortivity ($q^\infty > 0$ corresponds to heat flowing in the direction of increasing y).

The solution of (32) is [6]

$$B_y(x) = -\frac{1}{2\pi M}(a^2-x^2)^{1/2}\int_{-a}^{a}\frac{2Mq^\infty(\delta_1-\delta_2)(a^2-\xi^2)^{1/2}-p}{(a^2-\xi^2)^{1/2}(\xi-x)}d\xi = \frac{1}{\pi}q^\infty(\delta_1-\delta_2)(a^2-x^2)^{1/2}\log\left|\frac{a+x}{a-x}\right|, \; |x|<a \tag{38}$$

It is seen that $B_y(x)$ is odd in x, as expected, and that (35) is satisfied. The gap between the solids can be obtained from (33) and (38) by numerical integration. Using (25), the normal tractions between the solids in the contact zones are seen to be

$$\sigma_{yy}(x,0) = -p + \frac{2M}{\pi}\int_{-a}^{a}\frac{B_y(\xi)d\xi}{x-\xi}, \; a<|x| \tag{39}$$

Substituting (38) into (39), and carrying out the integrations [8], we obtain

$$\sigma_{yy}(x,0) = -\frac{p}{2a}(\mathrm{sgn}x)(x^2-a^2)^{1/2}\log\left|\frac{x+a}{x-a}\right|, \; a<|x| \tag{40}$$

The normal tractions are seen to be compressive for $q^\infty(\delta_1-\delta_2) > 0$.

The shape of the gap and the distribution of normal tractions are shown in Figs. 2 and 3.

It follows from (29) and (35) that the far field behavior of the disturbance caused by the separation zone is the same as that of a doublet of heat vortices and a doublet of dislocations. From the expressions given in [5], we can readily deduce that a doublet of heat vortices gives a temperature change and stresses that are of order $1/R$ as $R \to \infty$, where R is the distance from the doublet. A doublet of dislocations gives stresses that decay as $1/R^2$. It is clear, therefore, that the solution with a separation zone satisfies the same boundary conditions at infinity as the temperature and stress fields for full contact.

Periodic Array of Separation Zones

Another solution of fundamental importance is that involving a periodic array of separation zones. It can be constructed by means similar to those used in the previous section. As before, the initially unknown lengths of the separation zones are denoted by $2a$. The length of a period is taken to be $2h$.

The heat conduction part of the problem for a periodic array of insulated segments can first be written as

$$q^\infty - \frac{1}{\pi}\frac{k_1k_2}{k_1+k_2}\int_{-\infty}^{\infty}\frac{\Omega(\xi)d\xi}{x-\xi} = 0 \tag{41}$$

Taking advantage of the periodicity [9], we obtain

$$q^\infty + \frac{1}{2h}\frac{k_1k_2}{k_1+k_2}\int_{-a}^{a}\Omega(\xi)\cot\frac{\pi(\xi-x)}{2h}d\xi = 0, \; |x|<a \tag{42}$$

In addition, we must require that

$$\int_{-a}^{a}\Omega(\xi)d\xi = 0 \tag{43}$$

Equation (42) can be transformed to a Cauchy singular integral equation for the new unknown function

$$\Phi(u) = \frac{\Omega(u)}{1+c^2u^2} \tag{44}$$

by the following change of variables

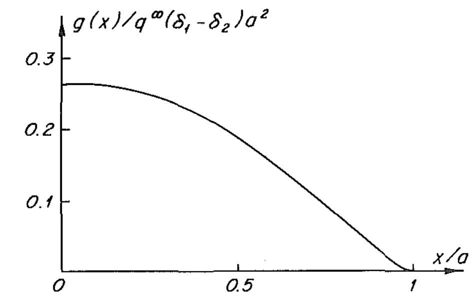

Fig. 2 Shape of the gap for a single separation zone

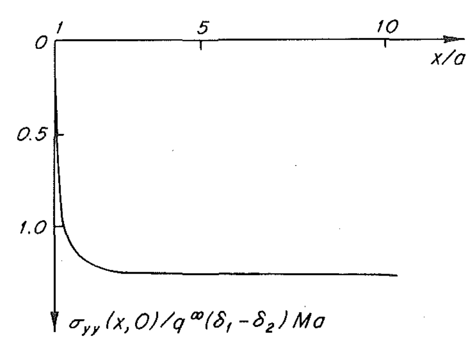

Fig. 3 Contact pressure for a single separation zone

$$\tan\frac{\pi\xi}{2h} = cu,\ \tan\frac{\pi x}{2h} = cs,\ c = \tan\frac{\pi a}{2h} \tag{45}$$

Condition (43) then becomes

$$\int_{-1}^{1}\Phi(u)du = 0 \tag{46}$$

Omitting the details which can be found in reference [9], the solution is

$$\Omega(s) = -\frac{q^\infty(k_1+k_2)(1+c^2)^{1/2}}{k_1k_2}\frac{s}{(1-s^2)^{1/2}},\ |s|<1 \tag{47}$$

The interface tractions corresponding to the periodic array of heat vortices can be obtained with the aid of (22). After some integrations, the result is

$$\sigma_{yy}(x,0) = \begin{cases}\frac{2Mh}{\pi}q^\infty(\delta_1-\delta_2)I(x), & |x|<a\\ 0, & a<|x|<h\end{cases} \tag{48}$$

where

$$I(x) = \log\left|\frac{\cos\frac{\pi x}{2h}+\left(\cos^2\frac{\pi x}{2h}-\cos^2\frac{\pi a}{2h}\right)^{1/2}}{\cos\frac{\pi x}{2h}-\left(\cos^2\frac{\pi x}{2h}-\cos^2\frac{\pi a}{2h}\right)^{1/2}}\right| \tag{49}$$

Next we introduce a distribution $B_y(x)$ of edge dislocations to cancel the normal tractions in the separation zones given by (48) and thus obtain the following singular integral equation:

$$-p+\frac{2Mh}{\pi}q^\infty(\delta_1-\delta_2)I(x) - \frac{M}{h}\int_{-a}^{a}B_y(\xi)\cot\frac{\pi(\xi-x)}{2h}d\xi = 0,\ |x|<a \tag{50}$$

with the constraint

$$\int_{-a}^{a}B_y(\xi)d\xi = 0 \tag{51}$$

Using the change of variables given by (45), defining a new unknown function

$$\Psi(u) = \frac{B_y(u)}{1+c^2u^2} \tag{52}$$

and proceeding as in the case of a single separation zone, we first obtain from the consistency condition that

$$\frac{p}{Mq^\infty(\delta_1-\delta_2)a} = \frac{2h}{\pi^2a}(1+c^2)^{1/2}R(c) \tag{53}$$

where

$$R(c) = \int_{-1}^{1}\frac{F(u,c)du}{(1-u^2)^{1/2}(1+c^2u^2)^{1/2}} \tag{54}$$

and

$$F(u,c) = \log\left|\frac{(1+c^2)^{1/2}+c(1-u^2)^{1/2}}{(1+c^2)^{1/2}-c(1-u^2)^{1/2}}\right| \tag{55}$$

The solution of the transformed integral equation (50) together with (53) yields

$$\frac{\Psi(s)}{q^\infty(\delta_1-\delta_2)a} = -\frac{h}{\pi^2a}(1-s^2)^{1/2}\left\{\frac{c^2R(c)s}{1+c^2s^2} + \int_{-1}^{1}\frac{F(u,c)du}{(1-u^2)^{1/2}(1+c^2u^2)(u-s)}\right\} \tag{56}$$

Equation (53) determines a for given p. Alternatively, it gives directly the pressure p required to produce a specified a. The integral in (54) can be evaluated by the Gauss-Jacobi quadrature [10] and the singular integral in (56) by the related quadrature developed by Erdogan and Gupta [11]. Finally, the normal tractions in the new variables are

$$\sigma_{yy}(s) = -p-\frac{2M}{\pi}(1+c^2u^2)\int_{-1}^{1}\frac{\Psi(u)du}{u-s} \tag{57}$$

Substituting (53) and (56) into (57) and performing some integrations, we obtain in the contact zones

$$\frac{\sigma_{yy}(s)}{Mq^\infty(\delta_1-\delta_2)a} = \frac{2h}{\pi^2a}(s^2-1)^{1/2}\left\{c^2R(c)|s| + (\mathrm{sgn}s)(1+c^2s^2)\int_{-1}^{1}\frac{F(u,c)du}{(1-u^2)^{1/2}(1+c^2u^2)(u-s)}\right\}\quad |s|>1 \tag{58}$$

The results are shown in Figs. 4–6 in the original variables. Figure 4 shows how the dimensionless parameter $p/Mq^\infty(\delta_1-\delta_2)a$ depends on a/h. As $a/h \rightarrow 0$, we recover the results for the single separation zone. Figure 5 shows the shapes of the gaps and Fig. 6, the contact pressure for various values of a/h.

If the rate of heat flow is to be the same for full contact and separation, the periodic array of gaps requires a larger far-field difference in temperatures. The additional temperature differential needed is readily extracted from the expressions given in [2], and it is

$$\Delta T = -\frac{2q^\infty h}{\pi}\left(\frac{1}{k_1}+\frac{1}{k_2}\right)\log\left[\sin\frac{\pi(h-a)}{2h}\right] \tag{59}$$

Conclusion

We have shown that two bodies that are pressed together and exchange heat by conduction may not necessarily remain in full contact and can separate locally if heat flows into the material with the larger distortivity. The questions raised in the Introduction cannot be answered in the present mathematical context due to its general complexity. However, insight into the phenomena involved can be gained by using materials that have a simpler response than that of the elastic solids conducting heat. A particularly simple model for this purpose was suggested to us by Aldo [12]. It consists of two blocks with dif-

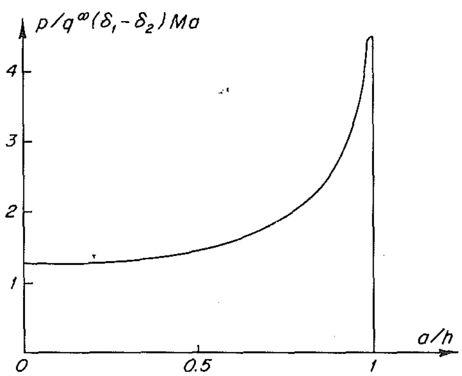

Fig. 4 Dependence of $p/q^\infty(\delta_1-\delta_2)Ma$ on a/h

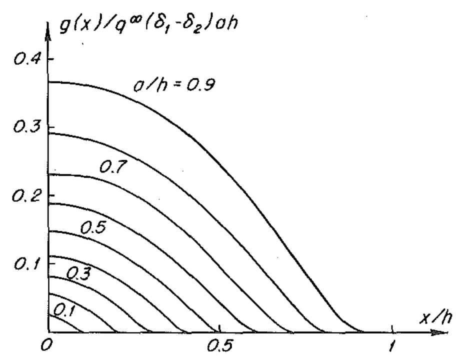

Fig. 5 Shapes of gaps for periodic arrays of separation zones for a/h from 0.1 to 0.9

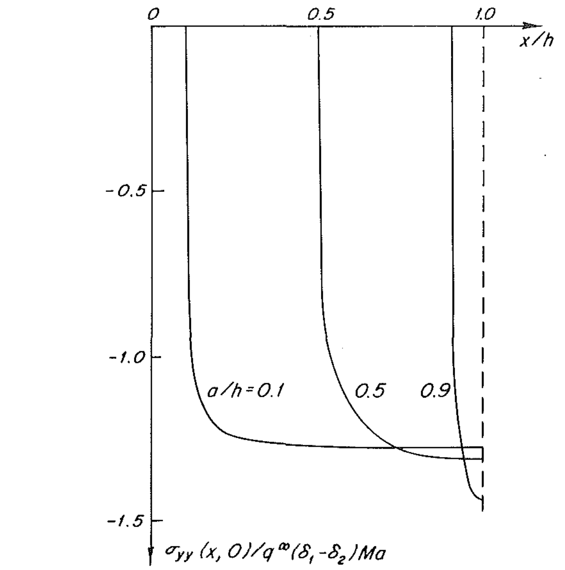

Fig. 6 Contact pressure for periodic arrays of separation zones with a/h = 0.1, 0.5 and 0.9

ferent properties. Each block is an assemblage of thin rods with insulated and frictionless sides, so that heat is conducted only in the direction of the rods and there are no shearing stresses when the blocks are deformed. The contact interface is normal to the rods. An investigation using the Aldo model has revealed the following behavior:

1 The steady state involving full contact is always possible. The state with partial separation becomes possible when the imposed temperature differential exceeds a certain critical value.

2 The state with partial separation corresponds to a lower level of total mechanical energy.

The details of this study are beyond the scope of the present article and will be published at a later time.

The Aldo model consisting of two finite blocks shows quite clearly that the nonuniqueness demonstrated here does not arise from the consideration of semi-infinite solids. Some very recent work by Barber [13], using a simplified model but a general and physically much more realistic contact condition for heat transfer between the solids, has shown that the lack of uniqueness is not caused by the idealized boundary conditions (6) and (11). Thus it appears that the lack of uniqueness for heat flowing into the material with the larger distortivity is not purely mathematical, that it has a physical basis, and that it signals possible instabilities.

It may be interesting to note in conclusion that there is also some difficulty with heat flowing between the contacting solids in the opposite direction, or into the material with the smaller distortivity. In such cases, the difficulty is exactly the opposite or connected with existence rather than uniqueness. It appears, however, that lack of existence is connected only with the highly idealized boundary conditions (6) and (11), and that existence can be achieved by appropriately modifying the conditions imposed at the contact interface [14, 15].

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