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Quasi-Static Extension of a Tensile Crack Contained in a Viscoelastic-Plastic Solid

Final stretch criterion of failure is applied to the problem of quasi-static extension of a crack embedded in an elastic-plastic or viscoelastic-plastic matrix. The slow growth under subcritical conditions in a rate-sensitive Tresca solid is shown to be a superposition of creep rupture and McClintock's ductile growth. This type of growth occurs at subcritical magnitude of the imposed K-factor and can be accounted for only through a recognition of inelastic properties of solids. In the subcritical range there is no unique value for K_c independent of geometrical configuration and flaw size. Not only the produced states of stress and strain are dependent on the loading path, but also the material resistance to fracture turns out to be a function of the history of loading that precedes catastrophic failure. A nonlinear integro-differential equation of motion is derived for a crack progressing through a viscoelastic medium with some limited ability to plastic flow. Examples of numerical integration are given incorporating both monotonic and cyclic loading programs.

Introduction

An extension of McClintock's and Rice's theory of stable crack growth is proposed for the tensile mode of fracture and small-scale yielding condition. Time-dependent phenomena are incorporated so that the combined effect of plastic and viscous deformation may be taken into account. In the first original paper published on this subject in 1958 by McClintock [5][1] it has been noted that some creeping in biaxially stressed aluminum foils tested for subcritical crack growth, was indeed observed. McClintock's comment was that because of time effects present, variations in testing rate might have contributed to the scatter.

Our major objective here is to give a theoretical background which would allow one to describe McClintock's slow growth and a time-dependent failure occurring at subcritical intensity level (creeping crack) by one governing equation. The secondary objective is to show feasibility of the Dugdale model, equipped with a new failure criterion, to account for the stable propagation which is likely to precede fracture in elastic-plastic or in viscoelastic-plastic solids. Comparison of results of this work with those obtained for mode III by McClintock and Rice through the use of the incremental plasticity theories and von Mises yield condition appears to be encouraging. The criterion proposed here is named "final stretch criterion." It is postulated that the amount of deformation which occurs within the process zone during the time interval just prior to decohesion of this zone is a material constant. In contrast to the COD condition the final stretch criterion is path-dependent and thus it appears to withstand Rice's criticism [8] of an earlier work on this subject by Cherepanov [1] and Wnuk [10]. This new approach assumes nothing about the instantaneous tip displacement and the length of the associated plastic zone. In fact these two entities turn out to be functions of time to be known only after a history of loading is prescribed and a governing integro-differential equation is solved.

Elastic-Plastic Matrix

In the first part of this paper we consider an elastic-plastic solid in which yielding obeys the Tresca condition. Attention is confined to the plane-stress tensile mode of deformation and small-scale yielding range. Therefore a line plastic zone results in front of the crack. Normal component of the displacement within this zone is given by

$$u_y(x) = \frac{4Y}{\pi E}\left\{\sqrt{R(R-x+l)} - \frac{x-l}{2}\log\frac{\sqrt{R}+\sqrt{R-x+l}}{\sqrt{R}-\sqrt{R-x+l}}\right\} \quad (1)$$

$$u_y(tip) = u_y(l) = 4YR/\pi E$$

[1] Numbers in brackets designate References at end of paper.

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The origin of the (x, y) coordinates is the crack center with the crack on $y = 0$; Y denotes the yield stress, E is Young modulus, l is the crack half-length, and R is the length of plastic zone. Equation (1) assumes l is large in comparison to R, and that consideration need be given only for the right end of the crack. Of course R is a known function of the stress-intensity factor K, which for the considered range of loads is a unique parameter correlating fracture test data

$$R = \pi K^2/8Y^2 \tag{2}$$

It is convenient to rewrite the expression (1) in terms of coordinates (x_1, y_1) which are anchored at the crack tip and move along with the crack front, Fig. 1(*a*). It is evident that

$$x_1 = x - l, \qquad y_1 = y$$

If we consider a fixed material point, say P, lying ahead of the crack, and imagine that the crack moves from left to right, then of course both x_1, l and R will depend on time τ. Therefore equation (1) may be rewritten as

$$u_P[x_1(\tau), R(\tau)] = \frac{4Y}{\pi E}\left\{\sqrt{R(\tau)(R(\tau) - x_1(\tau))} - \frac{x_1(\tau)}{2}\log\frac{\sqrt{R(\tau)} + \sqrt{R(\tau) - x_1(\tau)}}{\sqrt{R(\tau)} - \sqrt{R(\tau) - x_1(\tau)}}\right\} \tag{3}$$

Henceforth for simplicity we shall drop the subscript "y" on the displacement.

Consider now the history of deformation at the point P. An infinitesimal material element located at this point enters the plastic zone at a certain instant $\tau = t_0$, undergoes stretching while the yielded zone passes by and collapses at $\tau = t$. Thus the total amount of deformation produced at the point $P(x(P) =$ fixed) is

$$u_P(t_0, t) = \int_{t_0}^{t} \dot{u}_P[x_1(\tau), R(\tau)]d\tau = u_P[x_1(t), R(t)] - u_P[x_1(t_0), R(t_0)] \tag{4}$$

where $x_1(t) = 0$, $x_1(t_0) = R(t_0)$. Since at the initial instant $(t_0)u_P$ is zero and at the final instant (t) u_P is equal to the tip displacement, one simply recovers half the COD as the total stretch $u_P(t_0, t)$

$$u_P(t_0, t) = \text{COD}/2 = (4Y/\pi E)R(t) \tag{5}$$

A common thing to do next is to require that this quantity equals a material constant. Although such a criterion may be correct for predicting the final instability point, we should emphasize that it is not a proper criterion within the subcritical stage of growth, simply because it is not supported by experiment, i.e., COD $\neq$ constant, compare Vincent [11]. The actual COD observed by Vincent depended perceptibly on the amount of stable crack growth, thus lending support to the concept of an "R-curve" or a "$\mathcal{G}$-curve," or, in other words, of a variable toughness which depends on the current crack length and the history of loading. The function $R = R(t)$ is, however, not given. To measure it first and then insert into equation (2) would be merely a semiempirical approach. We believe that the theory alone should be capable to supply an equation which would generate an R-curve.

To achieve this end, we postulate that the prior-to-fracture work done at a fixed material point P, while the process zone of microstructural dimension Δ passes through it, is a material property, i.e.,

$$\int_{t-\delta t}^{t} S_P[x_1(\tau)]\dot{u}_P[x_1(\tau), R(\tau)]d\tau = \text{constant} \tag{6}$$

Here $S_P[x_1(\tau)]$ is the stress restraining separation of crack faces, δt is the time used by the crack front to traverse its own process

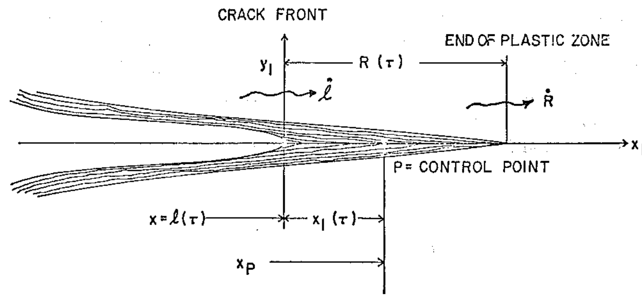

Fig. 1(*a*) Front of an advancing crack and associated yielded (or crazed) zone; *P* is control point

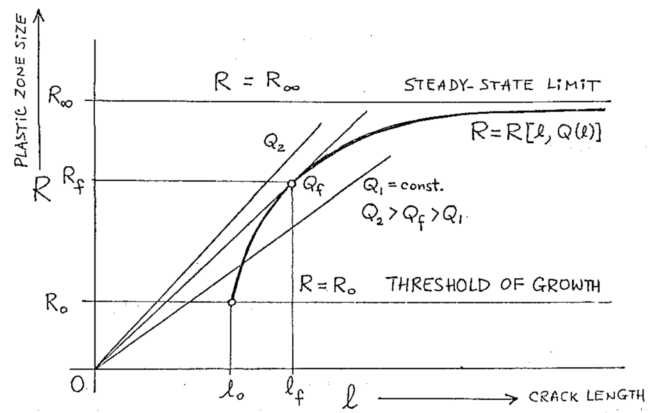

Fig. 1(*b*) The *R*-curve is shown by the heavy line. The propagation threshold R_0, the steady-state limit R_∞, and the fracture point (l_f, R_f) are marked. The latter can be identified with the point of tangential contact of the universal *R*-curve and a line showing variation of *R* with *l*, known for a prescribed geometry, and graphed at a constant loading parameter *Q*. Horizontal translation of the *R*-curve, due to variation of the initial crack size, results in different amounts of slow growth prior to instability.

zone. Note that the lower limit of the integral in equation (6) corresponds to the instant just prior to failure at point P, while the upper limit denotes the time at which P collapses. The length Δ is on the order of a characteristic microstructural size, compare McClintock and Irwin [6]. Since Δ is small versus crack size (it is exactly zero for a perfectly brittle continuum), one may apply the quasi-steady approximation of Glennie and Willis [2], and divide the unsteady motion of an accelerating crack into a number of constant speed segments, each of which occupies the time interval

$$\delta t = \Delta/\dot{l} \tag{7}$$

It is not difficult to relate the constant on the right-hand side of equation (6) to the work required to initiate the crack extension, say $\mathcal{G}_{\text{threshold}} = \mathcal{G}_0 = 2Yu_0$, where u_0 isi half of the initiation COD. With this assumption and with a constant restraining stress Y, our criterion for failure reads

$$\int_{t-\delta t}^{t} \dot{u}_P[x_1(\tau), R(\tau)]d\tau = u_0 \tag{8}$$

Physical meaning of the integral in equation (8) is readily seen: it is the amount of deformation produced just prior to failure, say[2]

$$\hat{u}_P(t - \delta t, t) = \int_{t-\delta t}^{t} \dot{u}_P[x_1(\tau), R(\tau)]d\tau \tag{9}$$

For this reason we suggest the name "final stretch" criterion, as

[2] A "hat" symbol is employed to denote "change of."

opposed to the COD criterion. These two are equivalent only in the limit when $\delta R \to 0$, that is, when R approaches its steady-state value R_∞. Here R_∞ is the maximum value R can reach before the unstable fracture sets on, and R_0 is the initiation plastic zone size, as shown in Fig. 1(b).

Expression (9) can be now expanded as follows:

$$\dot{u}_P(t - \delta t, t) = u_P(t) - u_P(t - \delta t)$$
$$= \frac{4Y}{\pi E}\left\{R(t) - R(t - \delta t)\left[\sqrt{\frac{R - \Delta}{R}} - \frac{\Delta}{2R}\log\frac{\sqrt{R} + \sqrt{R - \Delta}}{\sqrt{R} - \sqrt{R - \Delta}}\right]_{\tau = t - \delta t}\right\} \quad (10)$$

Since

$$R(t) = R(t - \delta t) + \dot{R}\delta t$$
$$= R(t - \delta t) + \frac{dR}{dl}\cdot\Delta \quad (11)$$

we arrive at

$$\dot{u}_P(t - \delta t, t) = \frac{4Y}{\pi E}\left\{R - \sqrt{R(R - \Delta)} + \Delta\frac{dR}{dl} + \frac{\Delta}{2}\log\frac{\sqrt{R} + \sqrt{R - \Delta}}{\sqrt{R} - \sqrt{R - \Delta}}\right\}_{t - \delta t} \quad (12)$$

Imposing the condition that this increment be equal to the constant $u_0 = (4Y/\pi E)\cdot R_0$, we obtain a differential equation which governs the quasi-static extension of a crack within the subcritical range of stress intensity

$$R - \sqrt{R(R - \Delta)} + \Delta\frac{dR}{dl} + \frac{\Delta}{2}\log\frac{\sqrt{R} + \sqrt{R - \Delta}}{\sqrt{R} - \sqrt{R - \Delta}} = R_0 \quad (13)$$

In a dimensionless form we have

$$\frac{dr}{d\Lambda} = r_0 - r + \sqrt{r(r - 1)} + \log\frac{\sqrt{r} - \sqrt{r - 1}}{\sqrt{r} + \sqrt{r - 1}} \quad (14)$$

where

$$r = R/\Delta, \qquad \Lambda = l/\Delta$$

The solution satisfying (14) gives the "universal" R-curve for an elastic-plastic solid. Since R is a very large entity compared to Δ (except perhaps for a low ductility solid, when $R_\infty \simeq R_0 \simeq \Delta$) one may consider a somewhat simplified form of equation (14):

$$\frac{dR}{dl} = \frac{R_0}{\Delta} - \frac{1}{2}\log(4R/\Delta) \quad (15)$$

This equation can be integrated in a closed form. Omitting the algebraic details, we have

$$l - l_0 = (1/8)\exp(2R_0/\Delta)\left\{\mathrm{ei}\left[\log\left(\frac{4R_0}{\Delta}\right) - \frac{2R_0}{\Delta}\right] - \mathrm{ei}\left[\log\left(\frac{4R}{\Delta}\right) - \frac{2R_0}{\Delta}\right]\right\} \quad (15a)$$

Function $R = R(l - l_0)$ is given implicitly by equation (15a); the symbol "ei" denoting the integro-exponential function

$$\mathrm{ei}(x) = \int_{-\infty}^{x}\frac{e^t}{t}\,dt \quad (15b)$$

It is of interest to see that the initial slope of the R-curve given by equation (14) closely compares with that predicted for mode III

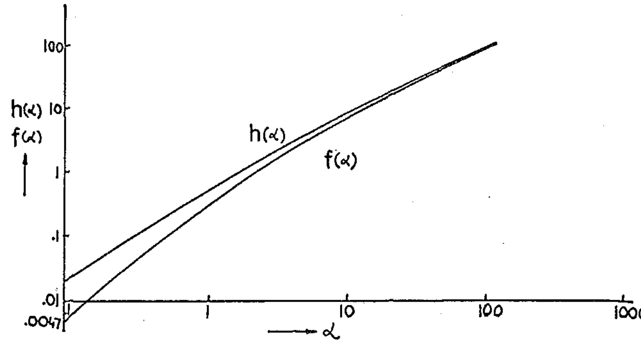

Fig. 2 Initial slope of the R-curve versus ductility parameter α: $h(\alpha)$ denotes the present solution, $f(\alpha)$ is Rice's solution

from the incremental theory of plasticity, cf. Rice, [7]. From equation (14) we have the initial slope

$$\left(\frac{dR}{dl}\right)_0 = \sqrt{\alpha(1 + \alpha)} + \log\frac{\sqrt{1 + \alpha} - \sqrt{\alpha}}{\sqrt{1 + \alpha} + \sqrt{\alpha}} \quad (16)$$

where α is the ductility parameter defined as a ratio of the plastic component of the strain at fracture to the yield strain ($\alpha = \epsilon_f^P/\epsilon_Y$). Rice has $(dR/dl)_0 = r_0 - 1 - \log(r_0)$ or $(\alpha - \log(1 + \alpha))$. Both results are plotted in Fig. 2. It can be shown that for ductility parameter $\alpha \to 0$ and for $\alpha \to \infty$ both solutions converge.

Noteworthy is also the fact that under certain assumptions the final stretch criterion can be derived from McClintock's criterion of critical plastic strain achieved at a fixed microstructural distance ahead of the crack front. For this purpose the strains $\epsilon(x_1)$ within Dugdale's plastic zone are assigned as follows:

$$\epsilon(x_1) = \epsilon_Y + \frac{\epsilon_f{}^p\Delta}{u_0}\{-|\mathrm{grad}\ u(x_1)|\}, \qquad \frac{R}{\Delta} \gg 1 \quad (17)$$

or

$$\epsilon(x_1) = \epsilon_Y + \frac{\pi\alpha}{4(1 + \alpha)}\left\{-\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial R}\frac{dR}{dl}\right\}, \qquad \frac{R}{\Delta} \gg 1 \quad (17a)$$

Here $\epsilon_f{}^p$ is the plastic strain at fracture, ϵ_Y is the yield strain, and α is the ratio of these two. The displacement $u = u(x_1, R(x_1))$ is given by equation (1), in where x_1 is treated as a timelike parameter. Applying McClintock's criterion, which means setting $\epsilon(x_1)$ at $x_1 = \Delta$ equal ϵ_f, gives then

$$\frac{1}{2}\log\left(\frac{4R}{\Delta}\right) + \frac{dR}{dl} = \frac{R_0}{\Delta} \quad (18)$$

which is identical with equation (15). To allow for the change from the Δ units to the R_∞ units, or from the micro-to-macro units, and to satisfy the boundary condition at the steady-state limit $R = R_\infty$, one has to incorporate the following equality involving the threshold opening stretch, namely,

$$u_0 = \frac{4Y}{\pi E}\cdot\frac{\Delta}{2}\cdot\log(4R_\infty/\Delta) \quad (19)$$

This transforms both equations (15) and (18) into a simple common form

$$\frac{dR}{dl} = \frac{1}{2}\log(R_\infty/R) \quad (20)$$

Fig. 3(a) shows a few graphs which resulted from integration of equation (20). Two distinct initial conditions $R_0 = 0.1R_\infty$ and $R_0 = 0.5R_\infty$ were chosen to illustrate the effect of the propagation threshold on the shape of the universal curve. For comparison the broken lines depict curves resulting from the "constant fracture energy" concept, cf., Cherepanov [1] and Wnuk [10].[3] Note that in both cases the R function depends only on the difference $(l - l_0)$, regardless of the initial crack size. The amount of slow growth which occurs prior to unstable propagation, however, will be affected by l_0 as pointed out by Rice [7].

To illustrate the effect of ductility on the slow growth let us integrate equation (20) at various levels of a ductility parameter R_∞/Δ. To be more specific we choose a configuration of a plane crack contained in an infinite sheet for which equation (20) reduces to

$$\frac{dQ}{dl} = \frac{\log(2R_\infty/lQ^2) - Q^2}{2lQ} \tag{20a}$$

Here Q denotes the loading parameter, $Q = \pi\sigma/2Y$, and σ is the nominal applied stress. The foregoing equation defines load versus crack length relation, $Q = Q(l)$, valid within the subcritical range for a perfectly elastic-plastic solid (compare Rice's and McClintock's analysis for mode III).

For the four chosen ductility ratios R_∞/Δ we obtain numbers shown in Table 1; see also Fig. 3(b).

Note that not only the location of the final instability points varies, but also the Q versus l curve changes its shape when ductility increases. For a very ductile solid the "plateau" or the flat portion of this curve extends over a considerable section of the subcritical history. This means that a large amount of growth occurs at an almost constant load, just below the failure load Q_f. Such an effect is yet better visible in a viscoelastic-plastic solid, considered in the next section.

Table 1

R_∞/Δ	l_f/Δ	Q_f
10	131	0.365
100	198	0.756
1000	306	1.218
10,000	451	1.665[a]

[a] Gross fracture occurs at load slightly above the yield stress.

Viscoelastic-Plastic Matrix

In many ductile solids tested at the temperature which activates ability to creep and to relax the initially imposed stresses, the slope of the R-curve *varies* with the rate of loading. This can be assigned to

1 Rate-dependent yield point which enters into failure criterion (6) which in turn governs equation of motion of a progressing crack.

2 Viscoelastic behavior of the bulk of the solid in which the crack moves.

Let us consider the second cause in more detail. Imagine the crack embedded in a viscoelastic-plastic solid. The matrix behaves as a linear viscoelastic medium for all points at which stresses are below the yield point, while in the high stress regions it yields according to the Tresca plasticity condition. Therefore material will yield within narrow bands emanating from the tips (such a behavior is observed in a number of polymers and known as a "crazing" process). The constitutive equations of the matrix containing the crack are then

$$\begin{aligned} s_{ij}(t) &= \int_{-\infty}^{t} G_1(t-\tau)\,\frac{\partial e_{ij}(\tau)}{\partial\tau}\,d\tau \\ s(t) &= \int_{-\infty}^{t} G_2(t-\tau)\,\frac{\partial e(\tau)}{\partial\tau}\,d\tau \end{aligned} \tag{21}$$

where $G_1(t)$ and $G_2(t)$ are the relaxation moduli for shear and hydrostatic stress states, respectively. Yielding within the high stress regions obeys the Tresca condition, thus the viscoelastic displacement which gives the shape of the plastic zone can be computed from Graham [3] and Wnuk and Knauss [9] as follows:

$$u(x,t) = u^0(x,t) + \int_{t_0}^{t} \dot{\Psi}(t-\tau)u^0(x,\tau)d\tau \tag{22}$$

Here $u^0(x, t)$ denotes the elastic counterpart of the considered viscoelastic problem, and $\Psi(t)$ is defined as the ratio[4] of the creep compliance and its value at $t = 0$. Dot denotes differentiation with respect to the argument shown in parentheses.

Consider now a moving crack whose front approaches the control point P, Fig. 1(a). The amount of deformation gen-

[3] The basic physical assumption made by Cherepanov [1] is that the total work done in separating two surfaces during an incremental growth is a material constant. This statement expressed in terms of elastic field entities and with the assumption that the Dugdale model applies, reads

$$4\int_{l}^{l+R} Y\delta u[x, Q(l), l]dx = 2\mathcal{G}_c\delta l \tag{a}$$

or

$$2Y\int_{l}^{l+R}\left[\left(\frac{\partial u}{\partial l}\right)_Q + \left(\frac{\partial u}{\partial Q}\right)_l \frac{dQ}{dl}\right]dx = \mathcal{G}_c \tag{b}$$

or

$$2Y\left[\frac{\partial}{\partial l} + \frac{dQ}{dl}\frac{\partial}{\partial Q}\right]\int_{l}^{l+R} u[x, Q(l), l]dx + 2Yu(tip) = \mathcal{G}_c \tag{c}$$

The foregoing relations describe a slowly moving crack within the subcritical range of the applied load Q. An extension of the Cherepanov theory for the viscoelastic solids was proposed by Wnuk [10].

Let us briefly summarize here the essential results pertinent to the small-scale yielding range and a rate insensitive solid. The integrals involved in equation (c) can be then evaluated as follows: $(R/l \ll 1)$

$$\begin{aligned} \int_{l}^{l+R}\frac{\partial u}{\partial l}dx &= -\int_{l}^{l+R}\frac{\partial u}{\partial x}dx = u(l) - u(l+R) = u(l) \\ &= (4Y/\pi E)R \\ \int_{l}^{l+R}\frac{\partial u}{\partial Q}dx &= \frac{\partial}{\partial Q}\int_{l}^{l+R} u dx = \frac{\partial}{\partial Q}(4Y/\pi E)R^2/3 \\ &= (4Y/\pi E)(2/3)R(\partial R/\partial Q) \end{aligned} \tag{d}$$

Combining these results with equation (c) and replacing $\mathcal{G}_c$ by $2Y(4Y/\pi E)R_\infty$, we arrive at

$$R + \frac{2}{3}R\frac{\partial R}{\partial Q}\frac{dQ}{dl} = R_\infty \tag{e}$$

The derivative dQ/dl can now be eliminated from the obvious relation

$$dR/dl = \partial R/\partial l + (\partial R/\partial Q)dQ/dl \tag{f}$$

Thus the differential equation (e) takes on the form

$$\frac{dR}{dl} = \frac{3}{2}(R_\infty - R)/R + \partial R/\partial l \tag{g}$$

or

$$\frac{dR}{dl} = \frac{3}{2}(R_\infty - R)/R \tag{h}$$

if one considers the last term in equation (g) negligible for the small-scale yielding range (this term can be shown to be proportional to $(Q/Y)^2$). Equation (h) was used for graphing the R curves shown in Fig. 3(a) by the dotted lines, while equation (g) transformed into the (Q, l) plane for a crack contained in an infinite plate, was employed to generate the curves marked by the dotted lines in Fig. 3(b). Equation (g) reads then $dQ/dl = (3/2)(2R_\infty - lQ^2)/Q^3l^2$.

[4] It is so only for a unequivocal value of the Poisson ratio. Otherwise, if ν is also a function of time, the kernel Ψ is given as follows:

$$\Psi(t) = \Omega(t)/\Omega(0)$$

where

$$\Omega(t) = \mathcal{L}^{-1}\left[\frac{2(2G_1^*(s) + G_2^*(s))}{s^2(G_1^*(s) + 2G_2^*(s)G_1^*(s))};\quad s \to t\right]$$

and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

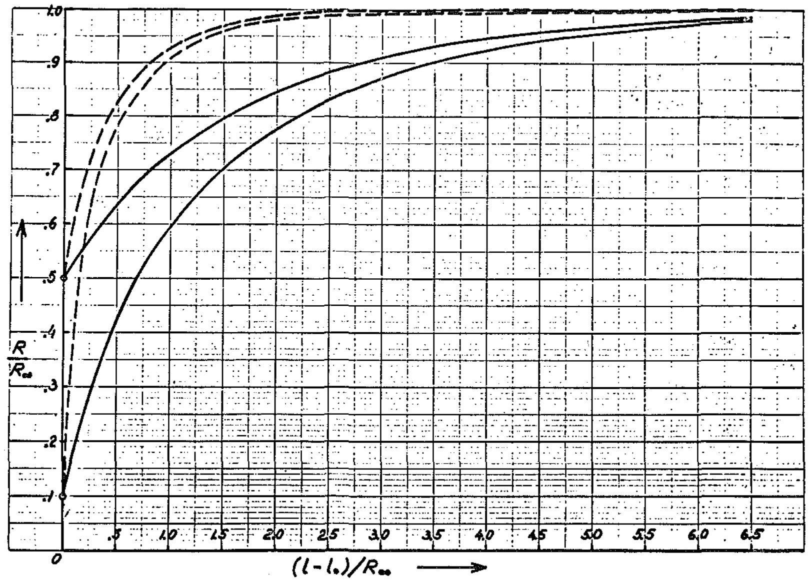

Fig. 3(a) R-curves resulting from the final stretch criterion of failure (continuous lines) and from the Cherepanov theory (broken lines)

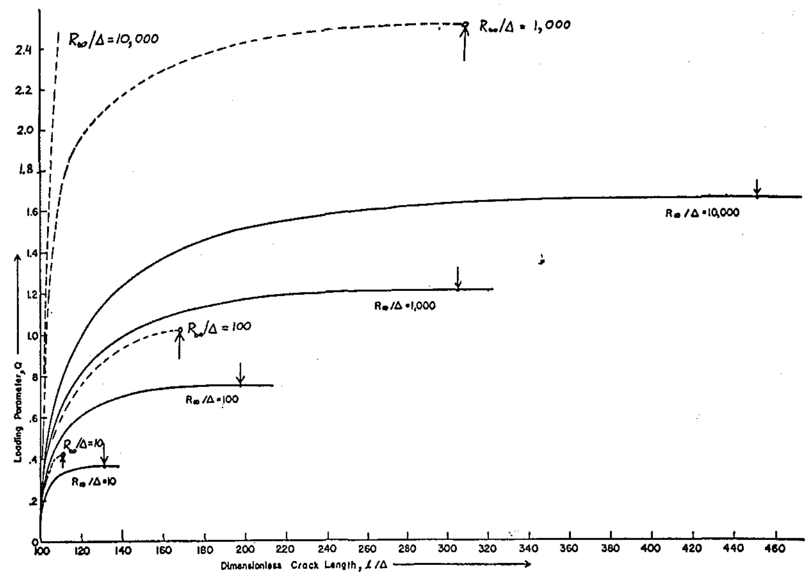

Fig. 3(b) Slow growth curves at various ductility levels R/Δ for a tensile crack contained in an infinite plate. Continuous lines result from the equation of motion based on the final stretch criterion, while the broken lines follow from Cherepanov's theory. Arrows indicate points of transition to unstable propagation.

erated at the point P during a small interval δt, preceding failure at P, is

$$\hat{u}_P(t - \delta t, t) = \int_{t-\delta t}^{t} \dot{u}(x_P, \tau) d\tau = u(x_P, t) - u(x_P, t - \delta t) \tag{23}$$

This is in fact the "final stretch" which we shall use to describe crack growth in the subcritical range. Let us compute

$$\begin{aligned} \hat{u}_P(t - \delta t, t) &= u^0(x_P, t) + \int_{t_0}^{t} \dot{\Psi}(t - \tau) u^0(x_P, \tau) d\tau \\ &\quad - \hat{u}^0(x_P, t - \delta t) - \int_{t_0}^{t-\delta t} \dot{\Psi}(t - \delta t - \tau) u^0(x_P, \tau) d\tau \\ &= \hat{u}^0(t - \delta t, t) + \int_{t_0}^{t-\delta t} [\dot{\psi}(t - \tau) \\ &\quad - \dot{\Psi}(t - \delta t - \tau)] u^0(x_P, \tau) d\tau + \int_{t-\delta t}^{t} \dot{\Psi}(t - \tau) u^0(x_P, \tau) d\tau \end{aligned} \tag{24}$$

The first term on the right-hand side of the foregoing expression is the purely elastic-plastic increment of deformation at point P, and it has been evaluated in the preceding section. The second term may be neglected if one agrees that the change in $\dot{\Psi}$ due to the shift of argument by δt is small. In other words we assume that the function Ψ does not vary rapidly and that for the interval involved it may be represented by just the first two terms of the Maclaurin expansion

$$\begin{aligned} \Psi(\delta t) &\simeq \Psi(0) + \dot{\Psi}(0)\delta t \\ \delta t &= \Delta/\dot{l},\ \dot{\Psi}(\delta t) \simeq \dot{\Psi}(0) = B \end{aligned} \tag{25}$$

It means that although the kernel $\dot{\Psi}$ may vary arbitrarily within the time interval (t_0, t), it is assumed that it is approximately constant inside each δt section.

Let us consider the third term in a more detailed way. Inserting $u^0(x_P, \tau)$ from equation (1) and $\dot{\Psi}$ from equation (25) we have

$$\int_{t-\delta t}^{t} \dot{\Psi}(t-\tau)u^0(x_P, \tau)d\tau = \frac{4Y}{\pi E} B \times \int_{t-\delta t}^{t} \left\{ \sqrt{R(\tau)[R(\tau) - l(t) + l(\tau)]} - \frac{l(t) - l(\tau)}{2} \times \log \frac{\sqrt{R(\tau)} + \sqrt{R(\tau) - l(t) + l(\tau)}}{\sqrt{R(\tau)} - \sqrt{R(\tau) - l(t) + l(\tau)}} \right\} d\tau \quad (26)$$

Note that the coordinate x_P is fixed and equal $l(t)$. All functions appearing in the integrand of expression (26), although unknown, can be represented by the following Taylor expansions:

$$\begin{aligned} l(t) &= l(t - \delta t) + \dot{l}\cdot\delta t \\ l(\tau) &= l(t - \delta t) + (\tau - t + \delta t)\dot{l} \\ R(\tau) &= R(t - \delta t) + (\tau - t + \delta t)\dot{R} \\ R(t) &= R(t - \delta t) + \dot{R}\cdot\delta t \end{aligned} \quad (27)$$

where both $\dot{l}$ and $\dot{R}$ are considered constant within the δt interval. Inserting (27) into (26) gives

$$B\left(\frac{4Y}{\pi E}\right) R(t - \delta t) \int_{t-\delta t}^{t} \left\{ \sqrt{\rho(\tau)\left[\rho(\tau) - \frac{(t-\tau)\dot{l}}{R(t-\delta t)}\right]} - \frac{(t-\tau)\dot{l}}{2R(t-\delta t)} \log \frac{\sqrt{\rho(\tau)} + \sqrt{\rho(\tau) - (t-\tau)\dot{l}/R(t-\delta t)}}{\sqrt{\rho(\tau)} - \sqrt{\rho(\tau) - (t-\tau)\dot{l}/R(t-\delta t)}} \right\} d\tau$$

$$\rho(\tau) = R(\tau)/R(t - \delta t) \quad (28)$$

It is plausible to write the foregoing expression in a more compact form

$$\left(\frac{4Y}{\pi E}\right) B \frac{R^2(t-\delta t)}{\dot{l}} \int_0^{\epsilon} G[\rho(s), s]ds \quad (29)$$

where

$$s = (t - \tau)\dot{l}/R(t - \delta t), \qquad \rho(s) = 1 + (\epsilon - s)dR/dl$$

$$\epsilon = \Delta/R(t - \delta t), \qquad G[\rho(s), s] = \sqrt{\rho(\rho - s)} - \frac{s}{2} \log \frac{\sqrt{\rho} + \sqrt{\rho - s}}{\sqrt{\rho} - \sqrt{\rho - s}} \quad (30)$$

If we restrict the attention to the case of $R/\Delta \gg 1$ or $\epsilon \to 0$, it is not difficult to give a following estimate of the integral appearing in the expression (29):

$$\int_0^{\epsilon} G[\rho(s), s]ds = G[\rho(0), 0]\cdot\epsilon - \delta G\cdot\epsilon$$

$$\delta G = G[\rho(0), 0] - G[\rho(\epsilon), \epsilon]$$

$$G[\rho(0), 0] = 1 + (dR/dl)\epsilon + \ldots$$

$$G[\rho(\epsilon), \epsilon] = \left\{ \sqrt{1-\epsilon} - \frac{\epsilon}{2} \log \frac{1 + \sqrt{1-\epsilon}}{1 - \sqrt{1-\epsilon}} \right\}_{\epsilon \to 0} = 1 - \frac{\epsilon}{2} \log (4/\epsilon) + \ldots \quad (31)$$

The term $\delta G\cdot\epsilon$ is of higher order and therefore will be neglected. Thus the third term of the right-hand side of expression (24), which represents the viscous component of deformation produced at point P, can be cast into this form

$$\frac{4Y}{\pi E} B \frac{R^2(t-\delta t)}{\dot{l}} G[\rho(0), 0] \frac{\Delta}{R(t-\delta t)} = \delta\Psi\cdot u_{tip}{}^0(t - \delta t) \quad (32)$$

Combining this with $\hat{u}^0$ given in the preceding section by equation (12), we obtain the final formula for the total amount of deformation generated at P due to the combined effect of yielding and creep. It reads

$$\hat{u}_P(t - \delta t, t) = \hat{u}^0(t - \delta t, t) + \delta\Psi\cdot u_{tip}{}^0(t - \delta t) = \frac{4Y}{\pi E}\left\{ \frac{\Delta}{2} \log\left(\frac{4R}{\Delta}\right) + \Delta \frac{dR}{dl} + \delta\Psi\cdot R \right\}_{t-\delta t} \quad (33)$$

or

$$\hat{u}_P = \frac{4Y}{\pi E}\left\{ \frac{\Delta}{2} \log\left(\frac{4R}{\Delta}\right) + \Delta[1 + CR(\partial R/\partial Q)^{-1}] \frac{dR}{dl} - \Delta C(\partial R/\partial Q)^{-1} R \frac{\partial R}{\partial l} \right\} \quad (34)$$

Here $(dR/dl - \partial R/\partial l)C\Delta(\partial R/\partial Q)^{-1}$ has been substituted for $\delta\Psi$. Both $\partial R/\partial l$ and $\partial R/\partial Q$ are known functions of plastic zone size R, crack length l, and loading parameter Q, if geometry is prescribed. The constant $C = B/\dot{Q}$, where B denotes the initial slope of the normalized creep compliance, $B = \dot{\Psi}(0)$. It is regarded here as a measure of material rate sensitivity.

Such a substitution results from the following consideration:

$$\delta\Psi = \Psi(\delta t) - \Psi(0) \simeq \dot{\Psi}(0)\delta t$$

Since $R = R(Q(l), l)$, we have

$$\frac{dR}{dl} = \frac{\partial R}{\partial l} + \frac{\partial R}{\partial Q}\frac{dQ}{dl}$$

which combined with the previous expression gives

$$\delta\Psi = \left(\frac{dR}{dl} - \frac{\partial R}{\partial l}\right)(\partial R/\partial Q)^{-1}C\Delta$$

Applying the final stretch criterion, that is requiring that $\hat{u}_P(t - \delta t, t) = u_0$, leads us to the following equation of motion:

$$\frac{dR}{dl} = \frac{R_0 - (\Delta/2)\log(4R/\Delta) + \Delta C(\partial R/\partial Q)^{-1}R\partial R/\partial l}{\Delta[1 + CR(\partial R/\partial Q)^{-1}]} \quad (35)$$

or

$$\frac{dR}{dl} = \frac{\frac{1}{2}\log(R_\infty/R) + CR(\partial R/\partial Q)^{-1}(\partial R/\partial l)}{1 + CR(\partial R/\partial Q)^{-1}} \quad (35a)$$

Two limit cases can be readily investigated. These are as follows:

1 Infinite rate of loading ($\dot{Q} \to \infty$) or zero material rate sensitivity ($\dot{\Psi} = 0$). In either case the entity C approaches zero, and we recover the previously studied differential equation which governs slow growth of McClintock's type, i.e., growth in a ductile but rate-insensitive solid. The initial slope of the R-curve, or the toughness-curve is then large, and eventually it levels off in the neighborhood of the instability point.

2 Zero rate of loading $dQ = 0$. Here C becomes infinite and the equation of motion (35) simplifies to just $dR/dl = \partial R/\partial l$. This relation, when translated into the (Q, l) plane, describes a straight horizontal line which originates at (Q_0, l_0). This is exactly the case of a creeping crack, whose length extends at a constant load. Such a problem is considered in more detail by Wnuk, [10].

For all intermediate values of C the motion proceeds at the slopes of the R-curves which vary between zero (at $\dot{Q} = 0$) and that given by equation (15) of the preceding chapter (at $\dot{Q} \to \infty$). Examples of integration of the equation of motion are given in the next two sections. They illustrate the effect of the rate-sensitivity of the material and the rate of loading $\dot{Q}$ on the shape of the R-curve. It will be seen that not only the slope of the curve is affected, but also pronounced changes in location of the ultimate instability point are observed.

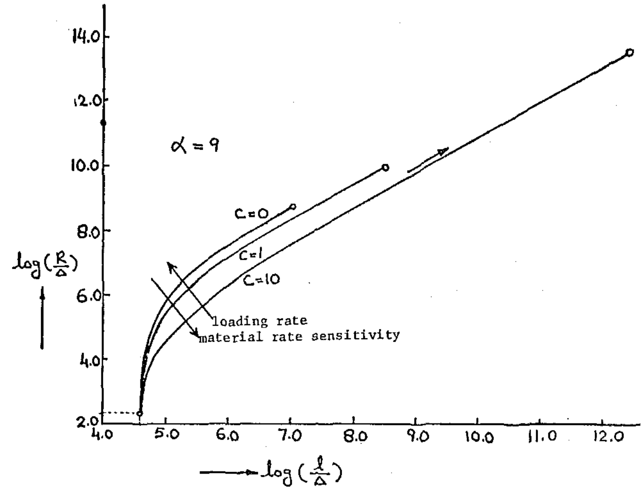

Fig. 4 Slow growth of a crack in a ductile rate-dependent solid. Ultimate instability is reached when the slope of the R-curve plotted on log-log scale equals unity (marked by circles). Initial crack size is 100 Δ.

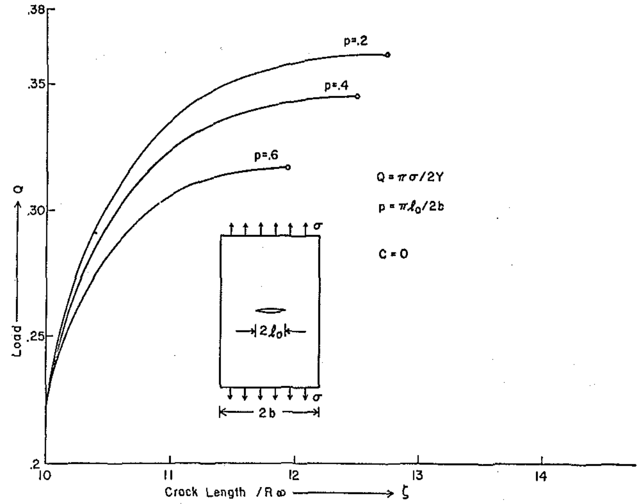

Fig. 5(a) $C = B/\dot{Q} = 0$

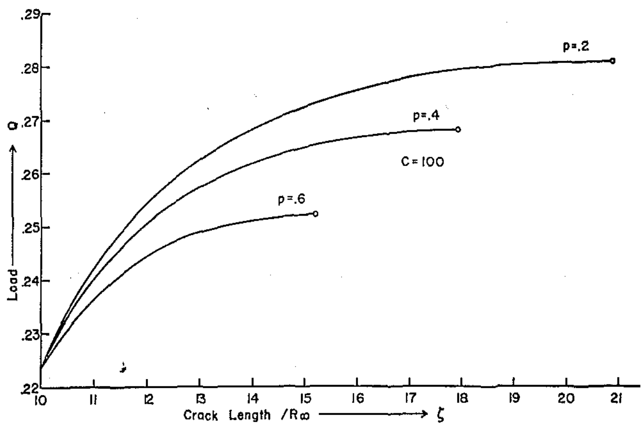

Fig. 5(b) $C = B/\dot{Q} = 100$

Fig. 5 Slow growth in a finite width panel containing a central crack; two different material rate-sensitivity (or rate of loading) levels are considered

Growth Under Monotone Loading

To exemplify the use of the foregoing equations let us choose a configuration of a central crack in an infinite plate under tensile stress σ. Then $R = \frac{1}{2}Q^2(l)l$, $Q = \pi\sigma/2Y$, and the governing equation of motion becomes

$$\frac{dR}{dl} = \frac{R_0 - \dfrac{\Delta}{2}\log\left(\dfrac{4R}{\Delta}\right) + \Delta CR^2/l\sqrt{2lR}}{\Delta + \Delta CR/\sqrt{2lR}} \tag{36}$$

Both R and l are then expressed in Δ units, and this leads to some astronomical numbers. To avoid such an inconvenience new variables are suggested, much more suitable for the numerical work. Setting

$$\begin{aligned} \log(l/\Delta) &= X \\ \log(R/\Delta) &= Y, \qquad Y = Y(X) \end{aligned} \tag{37}$$

we transform equation (36) into the form

$$\frac{dY}{dX} = \frac{A - Y/2 + (C/\sqrt{2})\exp\frac{3}{2}(Y - X)}{1 + (C/\sqrt{2})\exp\frac{1}{2}(Y - X)}\exp(X - Y) \tag{38}$$

This is the final form of the equation of motion used for numerical integration. The constant $A = (\alpha + 1) - \frac{1}{2}\log 4$. Fig. 4 gives some examples of the resulting integral curves. Now, the "universal" R-curve is seen to be strongly affected by the rate-sensitivity parameter C (which includes the material-sensitivity $\dot{\Psi}(0)$ and the rate of loading $\dot{Q}$). It can be also shown that the R-curve is no longer a function of the difference $(l - l_0)$ only, but rather it depends on the current length l.

It is seen from Fig. 4 that the extent of subcritical growth is pronouncely dependent on the rate-sensitivity. The final instability points for the three runs can be read out directly from the curves shown, since they coincide in each case with the points at which the slope dY/dX equals unity. They are

At $C = 0$	$l_f/\Delta = 1153$	$R_f/\Delta = 5710$
At $C = 1$,	$l_f/\Delta = 5064$	$R_f/\Delta = 22026$
At $C = 10$,	$l_f/\Delta = 268337$	$R_f/\Delta = 627814$

The index "f" stands for "failure."

For engineering applications it is convenient to rewrite the governing equation (35) in terms of the ratios R/R_∞ and l/R_∞, where R_∞ denotes the steady-state limit of the plastic zone size, say $R_\infty = \pi K_\infty^2/8Y^2$. The symbol K_∞ denotes the maximum plane-stress fracture toughness which would be attained in an ideal case, when the conditions of the test are such that the prior to failure growth is fully developed. Of course, the actual fracture toughness, i.e., the value of K at which the rapid motion begins, is bracketed by the initiation toughness K_0 and the maximum steady-state toughness K_∞.

If we define

$$Q = \pi\sigma/2Y \tag{39}$$

then for the central crack contained in an infinite plate we have

$$R = \tfrac{1}{2}Q^2 l \tag{40}$$

and for a central crack located in a panel of width $2b$ we have

$$R = \tfrac{1}{2}Q^2 l \sec\left(\frac{\pi l}{2b}\right) \tag{41}$$

The corresponding equations governing the subcritical growth take on the form

$$\frac{dR}{dl} = \frac{\frac{1}{2}\log(R_\infty/R) + CR^2/l\sqrt{2lR}}{1 + CR/\sqrt{2lR}} \tag{42a}$$

$$\frac{dQ}{dl} = \frac{\log(2R_\infty/Q^2 l) - Q^2}{lQ(2 + CQ)} \tag{42b}$$

for an infinite plate, and

$$\frac{dR}{dl} = \frac{\frac{1}{2}\log (R_\infty/R) + CR^2\{1 + (\pi l/2b)\tan (\pi l/2b)\}/l\sqrt{2lR \sec (\pi l/2b)}}{1 + CR/\sqrt{2lR \sec (\pi l/2b)}} \quad (43a)$$

$$\frac{dQ}{dl} = \frac{\log [2R_\infty/Q^2 l \sec (\pi l/2b)] - Q^2[1 + (\pi l/2b)\tan (\pi l/2b)] \sec (\pi l/2b)}{lQ(2 + CQ)\sec (\pi l/2b)} \quad (43b)$$

for a finite plate. The locus of terminal instability follows readily from equations (42*b*) and (43*b*) if dQ/dl is set equal to zero. Then for an infinite plate we obtain

$$l_f = R_\infty Q_f^{-2} \exp(-Q_f^2) \quad (44)$$

while for a finite width panel we have

$$\log [2R_\infty/Q_f^2 l_f \sec (\pi l_f/2b)] = Q_f^2[1 + (\pi l_f/2b)\tan (\pi l_f/2b)] \times \sec (\pi l_f/2b) \quad (45)$$

Of course equations (42) and (44) are "contained" in equations (43) and (45) and can be deduced from the latter by simply setting the panel half width $b = \infty$. Examples of the subcritical growth in a finite width panel and due to a monotonically increasing load are shown in Figs. 5(*a*, *b*).

Interestingly, the rate-sensitivity C does not enter explicitly in the relations (44) and (45). It is present here, though, in an implicit way, since both the critical load Q_f and the critical crack size l_f are pronouncely affected by the rate-sensitivity. This can be seen only after the integration of equations (42*b*) and (43*b*) is completed.

In general, we conclude then, the amount of slow growth before the final instability sets in is a function of

1 Ductility.
2 Rate sensitivity.
3 Rate of loading.
4 Initial crack size.
5 Geometry of the test.

These conclusions are illustrated by the graphs shown in Figs. 3–5. To establish such a dependence for a large-scale yielding range one would require a much more complete analytical investigation.

Growth Under Cyclic Loading

In this section, we restrict our attention to a crack contained in an infinite sheet.

Viewing fatigue crack extension as a sequence of slow growth curves (or steps) we integrate equation (42*b*) over a single cycle

$$(dl)_{\text{per cycle}} = 2l \int_{Q_{\min}}^{Q_{\max}} \frac{QdQ}{\log [2R_\infty/lQ^2] - Q^2} + Bl \int_0^T \frac{Q^2(t)dt}{\log [2R_\infty/lQ^2(t)] - Q^2(t)} \quad (46)$$

The current crack length l does not alter appreciably during just one cycle and therefore is considered constant in (46). Note also that the first integral on the right-hand side of equation (46) is extended over the ascending portion of the cycle only, while the second integration covers the complete cycle. This is so because of different physical interpretations of the two terms involved. The first one accounts for the McClintock type of slow growth which is ascribed totally to the plasticity effects and can occur only during the active process of loading. The second term arises from the viscoelastic behavior, and therefore it is not restricted to just the active loading. A more careful analysis of the governing equation (42*b*) reveals that some growth occurs at the sustained load or even at the decreasing load. Therefore, if a loading cycle is decomposed into an ascending ($0 \le t \le T/2$) and descending ($T/2 \le t \le T$) parts, the amount of growth generated in these two portions would be given by the equations

$$(dl)_{\text{ascending}} = l \int_{Q_{\min}}^{Q_{\max}} \frac{(2Q + BQ^2/\dot{Q})dQ}{\log [2R_\infty/lQ^2] - Q^2}$$
$$(dl)_{\text{descending}} = Bl \int_{Q_{\max}}^{Q_{\min}} \frac{(Q^2/\dot{Q})dQ}{\log [2R_\infty/lQ^2] - Q^2} \quad (47)$$

It is seen that the sum of these two gives indeed equation (46).

The integrals in equation (46) principally can be carried out for any given loading regime $Q = Q(t)$. Interestingly, the first of the two integrals can be evaluated in terms of the maximum and the minimum load levels within the cycle, while the knowledge of the precise nature of the Q versus time variation is not necessary. However, to complete the integration in the second term of equation (46) one has to know the function $Q(t)$. Since this term results from the rate-dependency of the material, it will turn out upon completion of the integration, to be frequency-dependent. Unfortunately, the resulting formulas although cast into a closed form, contain a nonelementary integro-exponential function ei (R/R_∞), which was defined in the preceding section, cf., equation (15*b*). Even for the simplest case, i.e., the high cycle range with negligible rate-sensitivity ($Q^2 \to 0$, $B \to 0$), one arrives at the rate of fatigue crack growth expressed as follows:

$$\frac{dl}{dN} = 2 \int_{R_{\min}}^{R_{\max}} \frac{dR}{\log (R_\infty/R)} \quad (48)$$

which again reduces to the integro-exponential function.

Numerical integration of equation (46) presents no difficulty. Fig. 6 shows some examples of such approach, if the load cycle is given by a simple sawtooth-like function

$$Q(t) = \begin{cases} \dot{Q}t \text{ for } nT \le t \le (2n + 1)T/2 \\ -\dot{Q}t \text{ for } (2n + 1)T/2 \le t \le T(n + 1), \end{cases} \quad n = 0, 1, 2, \ldots$$

The examples shown resulted from application of the Runge-Kutta subroutine for numerical integration performed on IBM-360. The frequency of the load cycle was chosen as $\omega = 5 \sec^{-1}$ (or $\dot{Q} = \pi^{-1} \sec^{-1}$), while $Q_{\min}$ was 0.2, $Q_{\max}$ was 0.4, and the rate-sensitivity B assumed the values 0, 1, 10, and 100. It is obvious from the few cycles pictured in Fig. 6 that the fatigue-crack propagation is highly asymmetric within a single cycle. The more rate-sensitive is the solid (or the lower is the rate of loading), the greater contribution is obtained from the viscous portion of slow growth. In other words, the descending part of the crack extension cycle increases for larger values of the parameter $C(= B/\dot{Q})$.

Preliminary tests involving fatigue runs of varied amplitude of the applied K-factors have shown also that the increased rate-sensitivity of the material tends to upset the agreement with Miner's law of cumulative damage. Again, the discrepencies become appreciable for large values of C.

It appears that the future research on this subject should perhaps zero-in on extension of the theory developed here onto large-scale yielding range and possibly on incorporating the interaction terms between creep failure and slow growth phenomena. These terms, being of higher order of magnitude, were neglected here.

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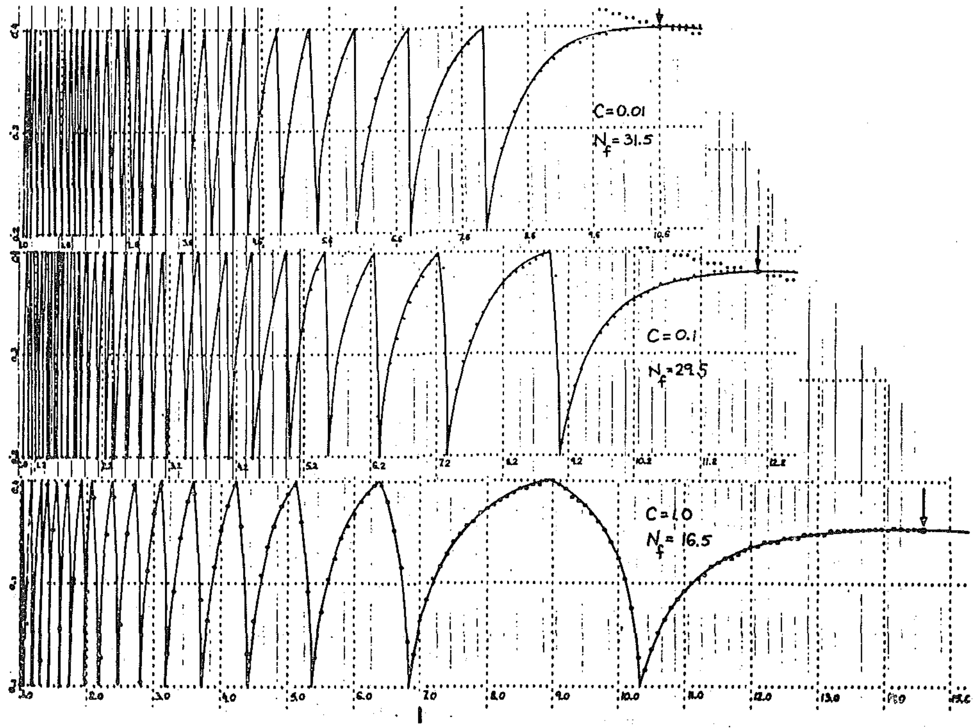

Fig. 6 Cyclic growth of a crack at three various rate-sensitivity levels

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References

1 Cherepanov, G. P., *Journal of Applied Mathematics and Mechanics*, Vol. 32, No. 6, pp. 1050–1058; translation of *Prikladnaya Matematika i Mekhanika*, Pergamon Press, 1968, Vol. 32, 6, pp. 1034–1042.

2 Glennie, E. B., and Willis, J. R., *Journal of Mechanics and Physics of Solids*, Vol. 19, 1971, pp. 11–30.

3 Graham, G. A. C., *Quarterly of Applied Mathematics*, Vol. 26, 1968, pp. 167–174.

4 Knauss, W. G., and Dietmann, H., *International Journal of Engineering Science*, Vol. 8, 1970, p. 643.

5 McClintock, F. A., Journal of Applied Mechanics, Vol. 25, Trans. ASME, Vol. 80, 1958, pp. 582–588.

6 McClintock, F. A., and Irwin, G. R., ASTM STP 381, 1964, pp. 84–113.

7 Rice, J. R., *Fracture*, an advanced treatise, ed., Liebowitz, H., Vol. 2, 1968, pp. 191–311.

8 Rice, J. R., Review No. 6185, *Applied Mechanics Reviews*, Vol. 23, No. 6, Aug. 1970, p. 917.

9 Wnuk, M. P., and Knauss, W. G., *International Journal of Solids and Structures*, Vol. 6, 1970, pp. 995–1009; also in report GALCIT SM 68-8, California Institute of Technology.

10 Wnuk, M. P., *International Journal of Fracture Mechanics*, Vol. 7, 1971, pp. 383–407.

11 Vincent, P. T., *Polymer*, Vol. 12, 1971, pp. 534–546.