Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 678407, 4 pages
http://dx.doi.org/10.1155/2014/678407

Research Article

On Isometric Extension in the Space $s_n(H)$

Xiaohong Fu

Department of Mathematics, Jiaying University, Meizhou 514015, China

Correspondence should be addressed to Xiaohong Fu; gdfuxh@sina.com

Received 7 June 2014; Revised 23 September 2014; Accepted 7 October 2014; Published 21 October 2014

Academic Editor: Naseer Shahzad

We study the problem of isometric extension on a sphere of the space $s_n(H)$. We give an affirmative answer to Tingley's problem in the space $s_n(H)$.

1. Introduction

Let E and F be metric linear spaces. A mapping $V : E \to F$ is called an isometry if $d_F(Vx, Vy) = d_E(x, y)$ for all $x, y \in E$. The classical Mazur-Ulam theorem in [1] describes the relation between isometry and linearity and states that every onto isometry V between two normed spaces with $V(0) = 0$ is linear. In 1987, Tingley [2] posed the problem of extending an isometry between unit spheres as follows.

Let E and F be two real Banach spaces. Suppose that V_0 is a surjective isometry between the two unit spheres $S(E)$ and $S(F)$. Is V_0 necessarily a restriction of a linear or affine transformation to $S(E)$?

It is very difficult to answer this question, even in two-dimensional cases. In the same paper, Tingley proved that if E and F are finite-dimensional Banach spaces and $V_0 : S(E) \to S(F)$ is a surjective isometry, then $V_0(x) = -V_0(-x)$ for all $x \in S(E)$. In [3], Ding gave an affirmative answer to Tingley's problem, when E and F are Hilbert spaces. Kadets and Martín in [4] proved that any surjective isometry between unit spheres of finite-dimensional polyhedral Banach spaces has a linear isometric extension on the whole space. In the case when E and F are some metric vector spaces, the corresponding extension problem was investigated in [5, 6]. See also [7–14] for some related results.

We introduce a new space $s(H)$ which consists of all H-valued sequences, where H is a Hilbert space, and, for each element $x = \{x(k)\}$, the F-norm of x is defined by $\|x\| = \sum_{k=1}^{\infty}(1/2^k)(\|x(k)\|/(1 + \|x(k)\|))$. Let $s_n(H)$ denote the set of all elements of the form $x = (x(1), \dots, x(n))$ with $\|x\| = \sum_{k=1}^{n}(1/2^k)(\|x(k)\|/(1 + \|x(k)\|))$, where $x(i)$ $(i = 1, \dots, n)$ is an element in the Hilbert space H.

In this paper, we study the problem of isometric extension on a sphere $S_{r_0}(s_n(H))$ with radius r_0 and center 0 in $s_n(H)$. We prove that if V_0 is an isometric mapping from $S_{r_0}(s_n(H))$ onto itself, then it can be extended to an isometry on the whole space $s_n(H)$.

Here is a notation used throughout this paper:

$$e_{x(k)} = \underbrace{(0, \dots, x(k), \dots, 0)}_{k\text{th place}} \in s_n(H), \qquad (1)$$

where $\|x(k)\| = 1$. Particularly, when $\|x(k)\| = 0$, we define $e_{x(k)/\|x(k)\|} = (0, \dots, 0)$.

2. Main Results and Proofs

In this section, we give our main results. For this purpose, we need some lemmas that will be used in the proofs of our main results. We begin with the following result.

Lemma 1. *If* $x, y \in s(H)$, *then*

$$\|x - y\| = \|x\| + \|y\| \Longleftrightarrow \operatorname{supp} x \cap \operatorname{supp} y = \emptyset, \qquad (2)$$

where $\operatorname{supp} x = \{n : x(n) \neq 0, n \in \mathbf{N}\}$.

Proof. The sufficiency is trivial. In the following, we prove the necessity.

Suppose that $x = \{x(n)\}$ and $y = \{y(n)\}$ are elements in $s(H)$ and that $\|x - y\| = \|x\| + \|y\|$. Then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x(n) - y(n)\|}{1 + \|x(n) - y(n)\|} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x(n)\|}{1 + \|x(n)\|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|y(n)\|}{1 + \|y(n)\|}. \tag{3}$$

In view of (3), it is sufficient to show that

$$\frac{\|x(n) - y(n)\|}{1 + \|x(n) - y(n)\|} \le \frac{\|x(n)\|}{1 + \|x(n)\|} + \frac{\|y(n)\|}{1 + \|y(n)\|}, \tag{4}$$

and the equality holds if and only if $\|x(n)\|\|y(n)\| = 0$.

Indeed, since $x/(1+x)$ is strictly increasing on $[0, \infty)$, this lemma is proven. □

Lemma 2. *Let $S_{r_0}(s(H))$ be a sphere with radius r_0 and center 0 in $s(H)$. Suppose that $V_0 : S_{r_0}(s(H)) \to S_{r_0}(s(H))$ is a surjective isometry; then* $(\operatorname{supp} x) \cap (\operatorname{supp} y) = \emptyset$ *if and only if* $(\operatorname{supp} V_0 x) \cap (\operatorname{supp} V_0 y) = \emptyset$.

Proof. Necessity. Take any two disjoint elements x and y in $S_{r_0}(s(H))$. Let $V_0(x) = \{x'(n)\}$ and $V_0(y) = \{y'(n)\}$.

Since V_0 is an isometry, we have by Lemma 1 and (4) that

$$\begin{aligned} 2r_0 &= \|x\| + \|y\| = \|x - y\| \\ &= \|V_0 x - V_0 y\| \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x'(n) - y'(n)\|}{1 + \|x'(n) - y'(n)\|} \\ &\le \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x'(n)\|}{1 + \|x'(n)\|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|y'(n)\|}{1 + \|y'(n)\|} \\ &= 2r_0. \end{aligned} \tag{5}$$

Thus,

$$\|V_0 x - V_0 y\| = \|V_0 x\| + \|V_0 y\|. \tag{6}$$

According to Lemma 1 again, we obtain

$$(\operatorname{supp} V_0 x) \cap (\operatorname{supp} V_0 y) = \emptyset. \tag{7}$$

The proof of sufficiency is similar to that of necessity because V_0^{-1} is also an isometry from $S_{r_0}(s(H))$ onto itself. □

Remark 3. The space $s(H)$ in Lemmas 1 and 2 can be replaced by the space $s_n(H)$.

Lemma 4. *Let $S_{r_0}(s_n(H))$ be a sphere with radius r_0 in the space $s_n(H)$, where $r_0 < 1/2^n$. Suppose that $V_0 : S_{r_0}(s_n(H)) \to S_{r_0}(s_n(H))$ is an isometry, $\lambda_k = 2^k r_0/(1 - 2^k r_0)$ ($k \in \mathbf{N}$, $1 \le k \le n$), and $e_{x(k)} \in s_n(H)$ ($\|x(k)\| = 1$). Then there exists $x'(k) \in H$ ($\|x'(k)\| = 1$) such that $V_0(\lambda_k e_{x(k)}) = \lambda_k e_{x'(k)}$ and $V_0(-\lambda_k e_{x(k)}) = -\lambda_k e_{x'(k)}$.*

Proof. We prove first that, for any k ($1 \le k \le n$), there exist l ($1 \le l \le n$) and $x'(l)$ ($\|x'(l)\| = 1$) such that $V_0(\lambda_k e_{x(k)}) = \lambda_l e_{x'(l)}$ (notice that the assumption of λ_k implies $\lambda_k e_{x'(k)} \in S_{r_0}(s_n(H))$). To this end, suppose on the contrary that $V_0(\lambda_{k_0} e_{x(k_0)}) = \sum_{k=1}^{n} \eta_k e_{x'(k)}$ and $\eta_{k_1} \ne 0$, $\eta_{k_2} \ne 0$. In view of Lemma 2, we have

$$[\operatorname{supp} V_0(\lambda_{k_0} e_{x(k_0)})] \cap [\operatorname{supp} V_0(\lambda_k e_{x(k)})] = \emptyset, \quad \forall k \ne k_0. \tag{8}$$

Hence, by the "pigeon nest principle" there must exist k_{i_0} ($1 \le k_{i_0} \le n$) such that $V_0(\lambda_{k_{i_0}} e_{x(k_{i_0})}) = 0$, which leads to a contradiction.

Next, we prove that if $V_0(\lambda_k e_{x(k)}) = \lambda_l e_{x'(l)}$, $V_0(-\lambda_k e_{x(k)}) = \lambda_p e_{x''(p)}$, then $l = p$.

Indeed, if $l \ne p$, we have

$$\begin{aligned} &\|V_0(\lambda_k e_{x(k)}) - V_0(-\lambda_k e_{x(k)})\| \\ &\quad = \|2\lambda_k e_{x(k)}\| = \frac{1}{2^k} \frac{2\lambda_k}{1 + 2\lambda_k} \ne 2r_0, \\ &\|V_0(\lambda_k e_{x(k)}) - V_0(-\lambda_k e_{x(k)})\| \\ &\quad = \|\lambda_l e_{x'(l)} - \lambda_p e_{x''(p)}\| = 2r_0, \end{aligned} \tag{9}$$

and this contradiction implies $l = p$.

Finally, we assert that, for any k ($1 \le k \le n$), there exists $x'(k)$ such that $V_0(\lambda_k e_{x(k)}) = \lambda_k e_{x'(k)}$ and that $V_0(-\lambda_k e_{x(k)}) = -\lambda_k e_{x'(k)}$.

Indeed, if $V_0(\lambda_k e_{x(k)}) = \lambda_l e_{x'(l)}$, by the result in the last step, we have $V_0(-\lambda_k e_{x(k)}) = \lambda_l e_{x''(l)}$; thus

$$\begin{aligned} &\|V_0(\lambda_k e_{x(k)}) - V_0(-\lambda_k e_{x(k)})\| = \|2\lambda_k e_{x(k)}\| = \frac{1}{2^k} \frac{2\lambda_k}{1 + 2\lambda_k}, \\ &\|V_0(\lambda_k e_{x(k)}) - V_0(-\lambda_k e_{x(k)})\| \\ &\quad = \|\lambda_l e_{x'(l)} - \lambda_l e_{x''(l)}\| = \frac{1}{2^l} \frac{\lambda_l \|x'(l) - x''(l)\|}{1 + \lambda_l \|x'(l) - x''(l)\|}. \end{aligned} \tag{10}$$

Therefore,

$$\frac{1}{2^k} \frac{2\lambda_k}{1 + 2\lambda_k} = \frac{1}{2^l} \frac{\lambda_l \|x'(l) - x''(l)\|}{1 + \lambda_l \|x'(l) - x''(l)\|} \le \frac{1}{2^l} \frac{2\lambda_l}{1 + 2\lambda_l}. \tag{11}$$

It follows that

$$\frac{2r_0}{1 + 2^k r_0} \le \frac{2r_0}{1 + 2^l r_0}, \tag{12}$$

and so $k \ge l$. Applying the "pigeon nest principle" again, we have $k = l$. Thus $\|x'(l) - x''(l)\| = 2$. Since $x'(l)$ and $x''(l)$ are elements in Hilbert space H, we have $x'(l) = -x''(l)$. □

Lemma 5. *Suppose that x_1 and y_1 are elements in the Hilbert space H, λ and μ are some nonzero real numbers, and $\|\lambda x_1 \pm \mu y_1\| = \|\lambda x_2 \pm \mu y_2\|$, $\|x_1\| = \|x_2\|$, and $\|y_1\| = \|y_2\|$. Then $\|x_1 - y_1\| = \|x_2 - y_2\|$.*

Proof. It is easy to prove this lemma by the parallelogram law. □

Now we are in a position to state the main result and proof in this paper.

Theorem 6. *Let $S_{r_0}(s_n(H))$ be a sphere with radius r_0 in the space $s_n(H)$, where $r_0 < 1/2^n$. Suppose that $V_0 : S_{r_0}(s_n(H)) \to S_{r_0}(s_n(H))$ is a surjective isometry. Then V_0 can be extended to an isometry on the whole space $s_n(H)$.*

Proof. Let $x = \{x(i)\}_{i=1}^n \in S_{r_0}(s_n(H))$. For i and j are points in supp x such that $i \neq j$, it follows from Lemma 2 that

$$\operatorname{supp} V_0 \left(\lambda_i e_{x(i)/\|x(i)\|}\right) \bigcap \operatorname{supp} V_0 \left(\lambda_j e_{x(j)/\|x(j)\|}\right) = \emptyset. \quad (13)$$

Since V_0 is surjective, there is an element $z = \{z(i)\}_{i=1}^n \in S_{r_0}(s_n(H))$ such that

$$V_0(z) = \sum_{i=1}^n \frac{\|x(i)\|}{\lambda_i} V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right). \quad (14)$$

If $\|x(i)\| = 0$, then $V_0(\lambda_i e_{x(i)/\|x(i)\|}) \overset{\text{def}}{=} 0$ and $e_{x(i)/\|x(i)\|} \overset{\text{def}}{=} 0$. In the following, we explain why the right-hand side of (14) is norm r_0.

By Lemma 4, we can see that, for any i $(1 \le i \le n)$, there exists $x'(i)$ such that $\|x(i)\| = \|x'(i)\|$ and

$$V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right) = \lambda_i e_{x'(i)/\|x'(i)\|}. \quad (15)$$

So

$$\begin{aligned} &\left\|\sum_{i=1}^n \frac{\|x(i)\|}{\lambda_i} V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right)\right\| \\ &\quad = \left\|\sum_{i=1}^n \frac{\|x(i)\|}{\lambda_i} \lambda_i e_{x'(i)/\|x'(i)\|}\right\| \\ &\quad = \left\|\sum_{i=1}^n e_{x'(i)}\right\| \\ &\quad = \sum_{i=1}^n \frac{1}{2^i} \frac{\|x'(i)\|}{1 + \|x'(i)\|} = r_0. \end{aligned} \quad (16)$$

Since V_0 is an isometry, we have

$$\begin{aligned} &\left\|z - \lambda_i e_{x(i)/\|x(i)\|}\right\| \\ &\quad = \left\|V_0(z) - V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right)\right\| \\ &\quad = \left\|\sum_{i=1}^n \frac{\|x(i)\|}{\lambda_i} V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right) - V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right)\right\| \\ &\quad = \left\|\sum_{j\neq i} \frac{\|x(j)\|}{\lambda_j} V_0\left(\lambda_j e_{x(j)/\|x(j)\|}\right) + \left(\frac{\|x(i)\|}{\lambda_i} - 1\right) V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right)\right\| \\ &\quad = \left\|\sum_{j\neq i} \frac{\|x(j)\|}{\lambda_j} \lambda_j e_{x'(j)/\|x'(j)\|} + \left(\frac{\|x(i)\|}{\lambda_i} - 1\right) \lambda_i e_{x'(i)/\|x'(i)\|}\right\| \\ &\quad = r_0 - \frac{1}{2^i} \frac{\|x'(i)\|}{1 + \|x'(i)\|} + \frac{1}{2^i} \frac{\lambda_i - \|x(i)\|}{1 + \lambda_i - \|x(i)\|} \\ &\quad = r_0 - \frac{1}{2^i} \frac{\|x(i)\|}{1 + \|x(i)\|} + \frac{1}{2^i} \frac{\lambda_i - \|x(i)\|}{1 + \lambda_i - \|x(i)\|}. \end{aligned} \quad (17)$$

On the other hand

$$\begin{aligned} &\left\|z - \lambda_i e_{x(i)/\|x(i)\|}\right\| \\ &\quad = \sum_{j\neq i} \frac{1}{2^j} \frac{\|z(j)\|}{1 + \|z(j)\|} + \left\|\|z(i)\| e_{z(i)/\|z(i)\|} - \lambda_i e_{x(i)/\|x(i)\|}\right\| \\ &\quad \ge r_0 - \frac{1}{2^i} \frac{\|z(i)\|}{1 + \|z(i)\|} + \frac{1}{2^i} \frac{\lambda_i - \|z(i)\|}{1 + \lambda_i - \|z(i)\|}. \end{aligned} \quad (18)$$

Given (17) and (18) and the fact that $f(x) = -(x/(1 + x)) + (\lambda_i - x)/(1 + \lambda_i - x)$ is decreasing on $[0, +\infty)$, we have

$$\|x(i)\| \le \|z(i)\|. \quad (19)$$

By (14) and Lemma 4, we get

$$\begin{aligned} &\sum_{i=1}^n \frac{1}{2^i} \frac{\|z(i)\|}{1 + \|z(i)\|} \\ &\quad = \|z\| = \|V_0 z\| \\ &\quad = \left\|\sum_{i=1}^n \frac{\|x(i)\|}{\lambda_i} V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right)\right\| \\ &\quad = \sum_{i=1}^n \frac{1}{2^i} \frac{\|x(i)\|}{1 + \|x(i)\|}. \end{aligned} \quad (20)$$

Combining (19) and (20) implies

$$\|x(i)\| = \|z(i)\|. \quad (21)$$

It follows from (17), (18), and (21) that

$$\left\|\|z(i)\| e_{z(i)/\|z(i)\|} - \lambda_i e_{x(i)/\|x(i)\|}\right\| = \frac{1}{2^i} \frac{\lambda_i - \|z(i)\|}{1 + \lambda_i - \|z(i)\|} \quad (22)$$

and consequently

$$x(i) = z(i). \quad (23)$$

That is,

$$V_0 x = \sum_{i=1}^n \frac{\|x(i)\|}{\lambda_i} V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right). \quad (24)$$

We now define a mapping on the space $s_n(H)$ as follows:

$$V\left(\{x(i)\}_{i=1}^{i=n}\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{\|x(i)\|}{\lambda_i} V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right) \tag{25}$$

for all $\{x(i)\}_{i=1}^{i=n} \in s_n(H)$. If $\|x(i)\| = 0$, then $V_0(\lambda_i e_{x(i)/\|x(i)\|}) \stackrel{\text{def}}{=} 0$.

Suppose that $\{x(i)\}_{i=1}^{i=n}$ and $\{y(i)\}_{i=1}^{i=n}$ are elements in $s_n(H)$. By Lemma 4, (24), and (25), we can assume

$$\begin{aligned} V\left(\{x(i)\}_{i=1}^{i=n}\right) &= \left\{x'(i)\right\}_{i=1}^{i=n}, \\ V\left(\{y(i)\}_{i=1}^{i=n}\right) &= \left\{y'(i)\right\}_{i=1}^{i=n}, \end{aligned} \tag{26}$$

where $\|x(i)\| = \|x'(i)\|$ and $\|y(i)\| = \|y'(i)\|$.

To prove

$$\left\|V\left(\{x(i)\}_{i=1}^{i=n}\right) - V\left(\{y(i)\}_{i=1}^{i=n}\right)\right\| = \left\|\{x(i)\}_{i=1}^{i=n} - \{y(i)\}_{i=1}^{i=n}\right\|, \tag{27}$$

we proceed as follows.

Since V_0 is an isometry, it follows from Lemma 4 that

$$\begin{aligned} &\left\|V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right) \pm V_0\left(\lambda_i e_{y(i)/\|y(i)\|}\right)\right\| \\ &= \left\|\lambda_i e_{x(i)/\|x(i)\|} \pm \lambda_i e_{y(i)/\|y(i)\|}\right\| \\ &= \frac{1}{2^i} \frac{\lambda_i \left\|x(i)/\|x(i)\| \pm y(i)/\|y(i)\|\right\|}{1 + \lambda_i \left\|x(i)/\|x(i)\| \pm y(i)/\|y(i)\|\right\|}. \end{aligned} \tag{28}$$

On the other hand

$$\begin{aligned} &\left\|V_0\left(\lambda_i e_{x(i)/\|x(i)\|}\right) \pm V_0\left(\lambda_i e_{y(i)/\|y(i)\|}\right)\right\| \\ &= \left\|\lambda_i e_{x'(i)/\|x'(i)\|} \pm \lambda_i e_{y'(i)/\|y'(i)\|}\right\| \\ &= \frac{1}{2^i} \frac{\lambda_i \left\|x'(i)/\left\|x'(i)\right\| \pm y'(i)/\left\|y'(i)\right\|\right\|}{1 + \lambda_i \left\|x'(i)/\left\|x'(i)\right\| \pm y'(i)/\left\|y'(i)\right\|\right\|}. \end{aligned} \tag{29}$$

If $\|x(i)\| = 0$, then $x(i)/\|x(i)\| \stackrel{\text{def}}{=} 0$ and $x'(i)/\|x'(i)\| \stackrel{\text{def}}{=} 0$. It follows from (28) and (29) that

$$\left\|\frac{x(i)}{\|x(i)\|} \pm \frac{y(i)}{\|y(i)\|}\right\| = \left\|\frac{x'(i)}{\left\|x'(i)\right\|} \pm \frac{y'(i)}{\left\|y'(i)\right\|}\right\|. \tag{30}$$

Notice that $\|x(i)\| = \|x'(i)\|$ and $\|y(i)\| = \|y'(i)\|$ and notice (30); it follows from Lemma 5 that

$$\|x(i) - y(i)\| = \left\|x'(i) - y'(i)\right\|. \tag{31}$$

Since

$$\begin{aligned} &\left\|V\left(\{x(i)\}_{i=1}^{i=n}\right) - V\left(\{y(i)\}_{i=1}^{i=n}\right)\right\| \\ &= \sum_{i=1}^{n} \frac{1}{2^i} \frac{\left\|x'(i) - y'(i)\right\|}{1 + \left\|x'(i) - y'(i)\right\|}, \end{aligned} \tag{32}$$

$$\left\|\{x(i)\}_{i=1}^{i=n} - \{y(i)\}_{i=1}^{i=n}\right\| = \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|x(i) - y(i)\|}{1 + \|x(i) - y(i)\|}.$$

Equations (31) and (32) assure that (27) holds. That is, we have obtained an isometry on the space $s_n(H)$ and it is the extension of V_0. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The helpful suggestions from the referees are appreciated. This work is supported by the Project of Department of Education of Guangdong Province (Grant no. 2013KJCX0170).

References

[1] S. Mazur and S. Ulam, "Sur les transformations isometriques d'espaces vectoriels normés," *Comptes Rendus de l'Académie des Sciences*, vol. 194, pp. 946–948, 1932.

[2] D. Tingley, "Isometries of the unit sphere," *Geometriae Dedicata*, vol. 22, no. 3, pp. 371–378, 1987.

[3] G. Ding, "The 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space," *Science in China. Series A. Mathematics*, vol. 45, no. 4, pp. 479–483, 2002.

[4] V. Kadets and M. Martín, "Extension of isometries between unit spheres of finite-dimensional polyhedral Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 441–447, 2012.

[5] G. An, "Isometries on unit sphere of l^{β_n}," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 1, pp. 249–254, 2005.

[6] X. Fu, "Isometries on the space s," *Acta Mathematica Scientia Series B*, vol. 26, no. 3, pp. 502–508, 2006.

[7] G.-G. Ding and J.-Z. Li, "Sharp corner points and isometric extension problem in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 405, no. 1, pp. 297–309, 2013.

[8] X. Fu and S. Li, "Some properties of $l^p(A, X)$ spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 562507, 8 pages, 2009.

[9] X. Fu, "The isometric extension of the into mapping from the unit sphere $S_1(E)$ to $S_1(l^\infty(\Gamma))$," *Acta Mathematica Sinica*, vol. 24, no. 9, pp. 1475–1482, 2008.

[10] X. Fu and S. Stević, "The problem of isometric extension in the unit sphere of the space $s_p(\alpha)$," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 3, pp. 733–738, 2011.

[11] X. Fu, "The problem of isometric extension in the unit sphere of the space $s_p(\alpha, H)$," *Banach Journal of Mathematical Analysis*, vol. 8, no. 1, pp. 179–189, 2014.

[12] T. M. Rassias, "Properties of isometric mappings," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 1, pp. 108–121, 1999.

[13] S. Stevic and S.-I. Ueki, "Isometries of a Bergman-Privalov-type space on the unit ball," *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 725860, 16 pages, 2009.

[14] D.-N. Tan, "Some new properties and isometries on the unit spheres of generalized James spaces J_p," *Journal of Mathematical Analysis and Applications*, vol. 393, no. 2, pp. 457–469, 2012.