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The theory of chaotic attractors, Brian R. Hunt, Judy A. Kennedy, Tien-Yien Li, and Helena E. Nusse (Editors), Springer-Verlag, New York, 2004, vi+514 pp., US$69.95, ISBN 0-387-40349-3

James Yorke won the 2003 Japan Prize for his work in the field of chaos theory. This book was compiled by four of his best-known collaborators in honor of his 60th birthday and contains papers by various authors on chaos and chaotic attractors. The papers are organized around the topics of "natural" invariant measure and fractal dimension of the associated chaotic attractors. A few of the papers are written by Yorke and collaborators, but the editors say they have chosen the collection primarily for the papers' historical importance and their accessibility to students.

The book begins with the famous paper "Deterministic Nonperiodic Flow" by Edward Lorenz. Anyone who doubts the importance of interdisciplinary institutes or of "linking" figures like Yorke should read Gleick's [4] account of Yorke's role in the dissemination of this paper to the mathematics community. The shocking aspect of the story is that the paper had been in print a full 9 years before most mathematicians were aware of it. Lorenz had found a three-dimensional ODE (a reduction of a model for Rayleigh-Benard instability in hydrodynamics) containing a limit set A that appeared to have the following properties:

(1) The orbit of almost every initial condition in a neighborhood of A has its limit points in A (hence, it is an attractor).

(2) The limit set is not periodic or asymptotically periodic.

(3) The orbits show sensitive dependence on initial conditions. (Hence, by (2) and (3), the limit set A is chaotic.)

(4) The limit set A contains a dense orbit (hence, it is indecomposable).

Although several properties of a chaotic attractor are listed here, there is no universally accepted definition. Some prefer a more mathematical (technical) statement; others want conditions that could be verified by experimentalists. In a paper which appears in this volume, Milnor gives a definition of "attractor" in which condition (1) is replaced with the requirement that the basin of attraction should have positive measure. He also requires that no smaller closed subset has the same basin, up to measure zero. This definition has withstood the test of time, and a definition of "chaotic attractor" based on it appears in [2]. Milnor's paper also provides an insightful survey of the subject and many important examples.

The editors present an excellent selection of key works in the development of the statistical or measure-theoretic approach to chaotic attractors. The important work of Yorke was early in this development. To get a true feeling for his many contributions to nonlinear dynamics, a book would have to touch on almost all aspects of (low-dimensional) chaos, where he has seen new, sometimes startling, connections and in doing so has opened avenues of investigation and introduced innovative techniques.

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Two early papers illustrate his versatility. In order to explore the complexities of Lorenz's work, Yorke and others studied the simplification of the model to a 1-D map. In the famous 1975 paper "Period-Three Implies Chaos", Li and Yorke address the question of how complicated nonperiodic orbits can be. One important aspect of the paper lay in its early definition of sensitive dependence for an interval map. They prove that the existence of a period–three point for a continuous map of the interval implies (1) sensitive dependence on some subset of the interval, and (2) the existence of periodic orbits of every minimum period. The second part is actually a special case of a result of Sharkovsky [10] previously published in the USSR. Unfortunately, due to the state of communications at the time, the result was not known in the West. In addition, like many Yorke papers, this one was notable for introducing a new word to our lexicon: it was the first time the term "chaos" was used.

During the same period, Yorke was also interested in the measure–theoretic approach to understanding chaos. In an often-cited paper in this volume, Lasota and Yorke proved that a piecewise-smooth expanding map T of the interval admits an absolutely continuous invariant measure. The importance here is in connecting the expanding map with an attractor which is not a periodic orbit. (Since the measure is invariant under T, it is carried on an attracting limit set; and since it is absolutely continuous, the limit set is not a periodic orbit.) Their techniques include use of the Perron-Frobenius operator for T to study the evolution of densities under iteration by T. Then they introduce the idea of operating on functions of bounded variation and establish convergence for these functions, which are dense in the space of integrable functions. This method is now commonly used for constructing invariant measures. These two papers (of 1973 and 1975) are quintessential Yorke: easily stated theorems (with verifiable hypotheses) that opened areas to many students, scientists, and other researchers who might not have otherwise approached the subject.

In the late 60's and 70's, the field of dynamical systems in the U.S. was dominated by Stephen Smale, together with his collaborators and students. The effort at that point was to understand nonlinear systems with sufficient hyperbolicity, called "Axiom A" systems. For these systems, either differential equations or iterated diffeomorphisms, the attractors are hyperbolic; i.e., there is a continuous splitting of the tangent bundle over points of the attractor, and the derivative of the map restricted to each of these subbundles is either uniformly contracting or uniformly expanding. One of Smale's important contributions is the example of the "horseshoe" [12], which remains the basic building block of chaotic diffeomorphisms. None of Smale's papers appear in this volume (probably because they do not have direct bearing on the aspects of attractor theory presented); however, his work is referenced frequently in the early papers.

Aided by numerical experiments made possible by the advent of personal computers, researchers began to discover and prove properties of attractors. In an early example, Hénon showed a (computed) chaotic attractor for an iterated planar diffeomorphism. The underlying map is quadratic, and a large part of the example's importance lies in its transparency and ease of computation. For example, researchers could now plot stable and unstable manifolds. (The "unstable manifold" is the set of points whose backward iterates converge to a fixed point. There is an analogous definition for periodic points. For a fixed point saddle in the plane, the unstable manifold is an embedded one-dimensional curve.) Unstable manifolds are of particular interest here, as they are seen to carry the invariant measures for chaotic attractors. Yorke, who has always worked with many scientific collaborators, explored the phenomenology of chaotic sets: see, for example, papers on crises (sudden jumps in attractors) and transient chaos [5], the creation of period-doubling cascades of attractors in the development of chaotic attractors [13] and the concurrent annihilation of others [7], and the existence of “riddled” basins [1], which satisfied Milnor’s positive measure condition on basins in a new and unexpected way.

Still, it took many years to prove the existence of a chaotic attractor for nonhyperbolic systems, such as Hénon’s. What was needed was an understanding of a “natural” invariant measure, carried by the attractor. This is the major emphasis of the papers included in this book. Given a transformation T, define the measure of a set to be the fraction of iterates of a typical orbit that fall within the set. When this fraction is the same for the orbit of almost every (in Lebesgue measure) initial point in the basin of attraction, then the measure is called “natural”. This definition is reminiscent of the Birkhoff Ergodic Theorem, which states that for an ergodic measure μ, invariant under the map T, the space average equals the time average (the asymptotic average value of the iterates), for μ almost every point. The difference here is that for a natural measure, the condition holds for almost every point in Lebesgue measure.

There are two approaches to obtaining a natural measure which are stressed in this book. The first is that employed by Yorke and Lasota which uses the Perron-Frobenius operator. The other approach is primarily due to Sinai, Ruelle, and Bowen. Sinai’s work on Gibbs measures and equilibrium states in ergodic theory appeared in the late 60’s and early 70’s [11]. In a paper included in this volume, Bowen and Ruelle proved that systems with hyperbolic attractors have natural invariant measures, which then began to be called SRB measures. Thereafter, theorems proving the existence of natural measures with similar techniques have named the measures with various permutations of these letters and occasionally together with other names. The main technique for constructing an SRB measure is to take Lebesgue measure on a small segment of a local unstable manifold and average it over forward iterates to obtain an invariant measure on the entire unstable manifold.

Both approaches were vigorously pursued. The stumbling block to extending the Lasota-Yorke approach was the lack of an appropriate notion of bounded variation in higher dimensions, although there are advances represented by papers in this book. Gradually, the SRB approach was extended to more general settings, as researchers succeeded in dealing with nonuniformly expanding maps. A breakthrough for nonhyperbolic maps came in 1981, when Jakobson [6] proved that the one-parameter family of quadratic maps of the unit interval, $f_r(x) = rx(1-x)$, $3 \le r \le 4$, has an absolutely continuous invariant measure for a positive measure set of r. The editors have included a paper of Rychlik with a less technical proof of this result. Another 10 years passed before this type of result was obtained for the Hénon map, the two-dimensional version of the quadratic family. Based on the remarkable paper of Benedicks and Carleson [3], who analyzed the dynamics of this system through changing rates of expansion and their changing directions, Benedicks and Young proved that there is a positive measure set of parameters for which the Hénon map admits a unique SRB measure.

When do the two approaches to natural measure coincide? The SRB measure shows how the trajectory from a typical initial point is distributed asymptotically, and the Lasota-Yorke approach finds the average distribution for trajectories of a large collection of initial conditions, not necessarily restricted to a single basin of attraction. If there are distinct attractors, the SRB construction (without some sort of averaging) yields one attractor and thus is not, strictly speaking, absolutely continuous, while the Lasota-Yorke measure is. In cases in which both approaches yield a natural measure and in which there is only one basin of attraction, the editors (in their own paper) show they are the same through a change of variables.

The stated aim of this book was to bring together a "coherent collection of readable, interesting, outstanding" papers for study. Clearly, a serious attempt was made to find accessible papers of quality by leaders in the field; however, this subject is not for the faint of heart. There are several exceptionally good survey articles by both mathematicians and scientists which would be good starting points for the novice. First, the classic 1985 survey *Ergodic Theory of Chaos and Strange Attractors* by Eckmann and Ruelle is a forty-page compendium on topics as difficult and diverse as Hausdorff dimension, Oseledec's multiplicative ergodic theorem, topological entropy, reconstruction of the dynamics from an experimental signal, Pesin theory, Axiom A, Hamiltonian systems, stochastic perturbations, et al. On the science side, there is an early (1981) article by Edward Ott, who has been Yorke's primary scientific collaborator for the past 20 years. One valuable aspect of this paper is his discussion of turbulence in the papers of Ruelle and Takens [9] and Lorenz. The editors end the volume with their paper on the similarities and differences of the two approaches to natural measure which surveys all aspects and includes several useful derivations.

Yorke's important role in science is evident in his many papers that present algorithms and sound computing techniques. Probably most frequently cited is the formula for computing the (fractal) dimension of an attractor, given by the Kaplan-Yorke Conjecture. Dimension is an important distinguishing property of the underlying fractal geometry of a chaotic set. When observing strange behavior, the scientist would like to be able to decide whether it is caused by noise or perhaps is multiply periodic rather than chaotic. The successful approximation of dimension is complicated by the fact that there are many approaches to defining dimension, some of which depend on a natural invariant measure and some of which do not. Many geometric definitions (not taking into account the frequency of iterates in different regions of the attractor) are simply impractical to compute in higher dimensions. In 1978, Kaplan and Yorke [8] suggested that the dimension of the natural measure might be calculated using Lyapunov exponents. (The Lyapunov exponent is a measure of the expansion rate of the linearized dynamics along a trajectory.) Their conjecture quickly became known as the "Kaplan-Yorke formula". A survey of this, the so-called Lyapunov dimension, together with geometric and other measure dependent dimensions is given by Farmer, Ott, and Yorke. As Grassberger and Procaccia offer in their paper, "If [the formula] were correct, [it] would obviously be very useful." Well, despite counterexamples, the Kaplan-Yorke formula has proved extremely useful. In fact, the two predominant methods used by scientists are the Lyapunov dimension and the correlation dimension, due to Grassberger and Procaccia.

In addition to assembling a quality collection of papers, the editors have written two introductions. The first is a discussion of Yorke's scientific work; the second is a survey of important contributions to the theory of natural measure, how these results developed one from another, and how the papers included in the book fit into this development. This second introduction is a masterful treatment and, together with their paper at the end of the book, provides a valuable guide.

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