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EXOTIC CIRCLES OF A REMARKABLE GROUP OF PIECEWISE GENERALIZED (NON LINEAR) CIRCLE HOMEOMORPHISMS

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Abstract

Let G be a subgroup of $\text{Homeo}_+(S^1)$. An exotic circle of G is a subgroup of G which is conjugate to $SO(2)$ in $\text{Homeo}_+(S^1)$ but not conjugate to $SO(2)$ in G. The existence of exotic circles shows that the subgroup G is far from being a Lie group. Let $r \geq 1$ be an integer, $r = +\infty$ or $r = \omega$. In this paper, we prove that the subgroup $\mathcal{P}^r(S^1)$ of $\text{Homeo}_+(S^1)$ consisting of piecewise class $P\ C^r$ homeomorphisms of the circle has no exotic circles. However, we show that there exist exotic circles of a particular subgroup (denoted $\mathcal{P}^r_1(S^1)$) of $\mathcal{P}^r(S^1)$ and we determine the conjugacy classes of all exotic circles in $\mathcal{P}^r_1(S^1)$. In particular, for the group $PL_+(S^1)$ consisting of piecewise linear homeomorphisms we give a simple proof of Minakawa's Theorems in [7], [6].

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1. Introduction

Let $\text{Homeo}_+(S^1)$ denote the group of orientation-preserving homeomorphisms of the circle and $SO(2)$ denote the group of rotations of S^1. Let G be a subgroup of $\text{Homeo}_+(S^1)$. A topological circle of G is a subgroup of G which is conjugate to $SO(2)$ in $\text{Homeo}_+(S^1)$. An *exotic circle* of G is a topological circle of G which is not conjugate to $SO(2)$ in G. The existence of exotic circles shows that the topological subgroup G is very far from being a Lie group (cf. [6], [8]). The following Corollary is a consequence of Theorem 4 of Montgomery and Zipping (cf. [8], Theorem 4, p. 212):

Corollary 1.1. *(cf. [6]) For every integer $r \geq 1$, $r = \infty$, or $r = \omega$, $Diff_+^r(S^1)$ has no exotic circles.*

To consider general groups of piecewise circle homeomorphisms, we prove the following more precise result. Let $Diff_+^{1+BV}(S^1)$ denote the group of C^1-diffeomorphisms which derivative of bounded variation on S^1. Then:

Corollary 1.2. *$Diff_+^{1+BV}(S^1)$ has no exotic circle.*

The proof uses the following classical result.

Theorem 1.3. *([10]) If g is a measurable function defined on the interval $(0,1)$, and if, for every $\tau \in (0,1)$, $g(t+\tau) - g(t)$ is of bounded variation on the interval $(0, 1-\tau)$ then g is of bounded variation on $(0,1)$.*

Proof of Corollary 1.2. Let $S = h \circ SO(2) \circ h^{-1}$ be a topological circle of $Diff_+^{1+BV}(S^1)$ where $h \in Homeo_+(S^1)$. We let $f = h \circ R_\alpha \circ h^{-1}$, $\alpha \in S^1$. By Corollary 1.1, $h \in Diff_+^1(S^1)$. Hence, $Dh > 0$ and $(Df \circ h)Dh = Dh \circ R_\alpha$. So, $\log Dh \circ R_\alpha - \log Dh = \log Df \circ h$. We let $g = \log Dh$. We identify f, g, and h to their lifts on $[0,1]$. So, g is a measurable function on $[0,1]$ and satisfies $g(x+\alpha) - g(x) = \log Df \circ h$. Since Df is of bounded variation on $[0,1]$, and $h \in Homeo_+(S^1)$, by Theorem 1.3, g is of bounded variation on $[0,1]$. Therefore, Dh is of bounded variation and $h \in Diff_+^{1+BV}(S^1)$. $\square$

Let $PL_+(S^1)$ denote the subgroup of $\text{Homeo}_+(S^1)$ consisting of piecewise linear homeomorphisms. Minakawa [6],[7] showed that $PL_+(S^1)$ has exotic circles and obtained the conjugacy classes of all exotic circles of $PL_+(S^1)$:

Minakawa's Theorem ([6],[7]). Let $\sigma \in \mathbb{R}_+^\star > 0$, $\sigma \neq 1$ and denote by h_σ the homeomorphism of S^1 defined by

$$h_\sigma(x) = \frac{\sigma^x - 1}{\sigma - 1}, \ x \in [0, 1[.$$

Then the topological circles $S_\sigma = h_\sigma \circ SO(2) \circ h_\sigma^{-1}$ are exotic circles of $PL_+(S^1)$ and every exotic circle of $PL_+(S^1)$ is conjugate in $PL_+(S^1)$ to one of the S_σ.

In this paper, we consider the general case: piecewise class $P\ C^r$ ($r \geq 1$, $r = +\infty$ or $r = \omega$) homeomorphisms of the circle with break point singularities, that is maps f that are C^r except at some singular points in which the successive derivatives until the order r on the left and on the right exist. These piecewise classes $P\ C^r$ homeomorphisms of the circle form a group noted $\mathcal{P}^r(S^1)$ which contains $PL_+(S^1)$ (cf. [1]). The aim of this paper is to show that $\mathcal{P}^r(S^1)$ has no exotic circles, and that, there exist exotic circles of a subgroup (denoted $\mathcal{P}_1^r(S^1)$) of $\mathcal{P}^r(S^1)$. Moreover, we determine the conjugacy classes of all exotic circles in $\mathcal{P}_1^r(S^1)$. In the case of $PL_+(S^1)$, we give a simple proof of the classification of all exotic circles of $PL_+(S^1)$ up to PL conjugacy obtained by Minakawa in [7], [6].

2. Class $P\ C^r$ homeomorphisms of the circle

Denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and $p : \mathbb{R} \longrightarrow S^1$ the canonical projection. Let f be an orientation preserving homeomorphism of S^1. The homeomorphism f admits a lift $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ that is an increasing homeomorphism of $\mathbb{R}$ such that $p \circ \tilde{f} = f \circ p$. Conversely, the projection of such a homeomorphism of $\mathbb{R}$ is an orientation preserving homeomorphism of S^1. Let $x \in S^1$. We call *orbit* of x by f the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$,

Historically, the dynamic study of circle homeomorphisms was initiated by H. Poincaré ([9], 1886). He introduced the rotation number of a homeomorphism f of S^1 as $\rho(f) = \lim\limits_{n \longrightarrow +\infty} \frac{\tilde{f}^n(x) - x}{n} (mod\, 1)$.
Poincaré showed that this limit exists and does not depends on x and the lift $\tilde{f}$ of f.
We say that f is semi-conjugate to the rotation $R_\rho(f)$ if there exists an orientation preserving surjective continuous map $h : S^1 \longrightarrow S^1$ of degree one such that $h \circ f = R_\rho(f) \circ h$.

Poincaré's theorem. Let f be a homeomorphism of S^1 with rotation number $\rho(f)$ irrational. Then f is semi-conjugate to the rotation $R_\rho(f)$.

A natural question is whether the semi-conjugation h could be improved to be a conjugation, that is h to be an homeomorphism. In this case, we say that f is topologically conjugate to the rotation $R_{\rho(f)}$. In this direction, A. Denjoy ([2]) proved the following:

Denjoy's theorem([2]). Every C^2-diffeomorphism f of S^1 with irrational rotation number $\rho(f)$ is topologically conjugate to the rotation $R_{\rho(f)}$.

Other classes of circle homeomorphisms commonly referred to as the *class P homeomorphisms* are known to satisfy the conclusion of Denjoy's theorem (see [4]; [3], chapter VI).

Definition 2.1. Let f be an orientation preserving homeomorphism of the circle. The homeomorphism f is called of *class P* if it is derivable except in finitely or countable points called break points of f in which f admits left and right derivatives (denoted, respectively, by Df_- and Df_+) and such that the derivative $Df : S^1 \longrightarrow \mathbb{R}_+^\star$ has the following properties:

- there exist two constants $0 < a < b < +\infty$ such that:

$a < Df(x) < b$, for every x where Df exists, $a < Df_+(c) < b$, and $a < Df_-(c) < b$ at the break point c.

- $\log Df$ has bounded variation on S^1

Denote by

- $\sigma_f(c) := \frac{Df_-(c)}{Df_+(c)}$ called the *f-jump in c*.
- $C(f)$ the set of break points of f.
- $\pi_s(f)$ the product of f-jumps in the break points of f : $\pi_s(f) = \prod_{c\in C(f)} \sigma_f(c)$.
- $V = Var \log Df$ the total variation of $\log Df$. We have

$$V := \sum_{j=0}^{p} Var_{[c_j, c_{j+1}]} \log(Df) + |\log(\sigma_f(c_j)| < +\infty$$

where $c_0, c_1, c_2, ..., c_p$ are the break points of f with $c_{p+1} := c_0$. In this case, V is the total variation of $\log Df$, $\log Df_-$, $\log Df_+$.

Among the simplest examples of class P homeomorphisms, there are:

- C^2-diffeomorphisms,

- Piecewise linear (PL) homeomorphisms. An orientation preserving circle homeomorphism f is called *PL* if f is derivable except in many finitely break points $(c_i)_{1\leq i\leq p}$ of S^1 such that the derivative Df is constant on each $]c_i,\ c_{i+1}[$.

Definition 2.2. We say that f has the *(D)-property* (cf. [5], [7]) if the product of f-jumps on each orbit is equal to 1 i.e. $\pi_s(f)(c) = \prod_{x\in C(f)\cap O_f(c)} \sigma_f(x) = 1$.

In particular, if f has the (D)-property then $\pi_s(f) = 1$. Conversely, if $\pi_s(f) = 1$ and if all break points belong to a same orbit then f has the (D)-property.

If f is a (PL) homeomorphism, always we have $\pi_s(f) = 1$. Therefore, a PL homeomorphism f satisfies the (D)-property if all its break points are on the same orbit.

Proposition 2.3. *Let f, g be two circle orientation preserving C^1-homeomorphisms. Then $\pi_s(g\circ f) = \pi_s(g)\pi_s(f)$.*

Proof. Let $c \in S^1$. We have $\sigma_{g\circ f}(c) = \sigma_g(f(c))\sigma_f(c)$. So,

$$\pi_s(g\circ f) = \prod_{c\in C(g\circ f)} \sigma_{g\circ f}(c) = \prod_{c\in C(g\circ f)} \sigma_g(f(c))\sigma_f(c).$$

Since $C(g\circ f) \subset C(f) \cup f^{-1}(C(g))$ and $\sigma_{g\circ f}(c) = 1$ for every $c \in S^1\backslash C(g\circ f)$,

$$\pi_s(g\circ f) = \prod_{c\in C(f)} \sigma_g(f(c))\sigma_f(c) \prod_{c\in f^{-1}(C(g))\backslash C(f)} \sigma_g(f(c))\sigma_f(c)$$

$$= \pi_s(f) \prod_{c\in C(f)} \sigma_g(f(c)) \prod_{c\in f^{-1}(C(g))\setminus C(f)} \sigma_g(f(c))$$

$$= \pi_s(f) \prod_{c\in f^{-1}(C(g))} \sigma_g(f(c)) = \pi_s(f)\pi_s(g).$$

□

Corollary 2.4. *(Invariance of π_s by piecewise C^1-conjugation). Let f, g be two circle orientation preserving C^1-homeomorphisms. If f and g are bi-piecewise C^1 conjugated then $\pi_s(f) = \pi_s(g)$.*

Definition 2.5. Let $r \geq 1$ be an integer, $r = +\infty$, or $r = \omega$. A class P circle homeomorphism is called of piecewise class $P\ C^r$ if f is C^r except in a finitely many points called *singular points* and in which the successive derivatives of f until the order r on the left and on the right exist.

Denote by

- $S(f)$ the set of singular points of f.

- $\mathcal{P}^r(S^1)$ the set of class $P\ C^r$ circle homeomorphisms ($r \geq 1$ integer, $r = +\infty$, or $r = \omega$). One can check that $\mathcal{P}^r(S^1)$ is a group.

Notice that if $r = 1$, $S(f) = C(f)$.

The set $S(f)$ of singular points is partitioned into finite subsets $S_i(f)$ which are supported by disjoints orbits:

$$S(f) = \coprod_{i=1}^{p} S_i(f)$$

where $S_i(f) = S(f) \cap O_f(c_i)$, $c_i \in S(f)$ and $O_f(c_i)_{1\leq i\leq p}$ are on distinct orbits.

Definition-Notation. The set $M_i(f) = \{x_i, f(x_i), .., f^{N(f,x_i)}(x_i)\}$ is called the *envelope* of $S_i(f)$ $(1 \leq i \leq p)$ where $N(f, x_i) \in \mathbb{N}$, x_i, $f^{N(f,x_i)}(x_i) \in S(f)$ and

$$S(f) \cap M_i(f) = S(f) \cap O_f(x_i) = S_i(f).$$

Definition 2.6. Let $r \geq 1$ be an integer. Let $f \in \mathcal{P}^r(S^1)$. We say that f has the *(D_r)-property* if $f^{N(f,x_i)+1}$ is C^r on x_i, for $i = 1, ..., p$.

Notice that if $N(f, x_i) = 0$ for some i then x_i is the unique singular point in its orbit and the (D_r)-property is not satisfied.

Remark 1. In the case $r = 1$, the (D_r)-property is equivalent to the *(D)-property.* For every $i = 1, ..., p$,

$$\prod_{d \in M_i(f)} \sigma_f(d) = 1 = \prod_{d \in S_i(f)} \sigma_f(d).$$

Indeed, $f^{N(f,x_i)+1}$ is C^1 on x_i, $i = 1, ..., p$ means that

$$\sigma_{f^{N(f,x_i)+1}}(x_i) = 1 = \prod_{c \in S_i(f)} \sigma_f(c) = \prod_{j=0}^{N(f,x_i)} \sigma_f(f^j(x_i)),$$

in other words: f satisfies the (D)-property.

Proposition 2.7. *([1], Corollary 2.8). Let* f, $g \in \mathcal{P}^r(S^1)$ *(*$r \geq 1$ *be a real,* $r = +\infty$ *or* $r = \omega$*) with irrational rotation numbers that are rationally independent. If* $f \circ g = g \circ f$ *then* f *and* g *have* (D_r)*-property.*

Theorem 2.8. *([1], Theorem 2.1) Let* $r \geq 1$ *be a real,* $r = +\infty$ *or* $r = \omega$ *and* $f \in \mathcal{P}^r(S^1)$ *with irrational rotation number. Then the following properties are equivalent:*

i) f *is conjugated in* $\mathcal{P}^r(S^1)$ *to a* C^r*-diffeomorphism,*

ii) f *has the* (D_r)*-property,*

iii) f *is conjugated to a* C^r*-diffeomorphism by a piecewise polynômial homeomorphism* $K \in \mathcal{P}^r(S^1)$

Proposition 2.9. *([1], Lemma 5.1) Let* $f \in Diff_+^r(S^1)$ *with irrational rotation number and let* $g \in \mathcal{P}^r(S^1)$*. If* $f \circ g = g \circ f$ *then* $g \in Diff_+^r(S^1)$*.*

Our main results are the following:

Theorem 2.10. *Let* $r \geq 1$ *be an integer,* $r = +\infty$ *or* $r = \omega$*. Then* $\mathcal{P}^r(S^1)$ *has no exotic circles.*

Let $\mathcal{P}_1^r(S^1)$ denote the subgroup of $\mathcal{P}^r(S^1)$ consisting of class $P\,C^r$ circle homeomorphisms f with $\pi_s(f) = 1$. Then:

Theorem 2.11. *Let* $r \geq 1$ *be an integer,* $r = +\infty$ *or* $r = \omega$, $\sigma \in \mathbb{R}_+^\star > 0$, $\sigma \neq 1$ *and let* $h_\sigma \in \mathcal{P}^r(S^1)$ *with one break point* c *such that* $\sigma_{h_\sigma}(c) = \sigma$*. Then:*

i) $S_\sigma = h_\sigma \circ SO(2) \circ h_\sigma^{-1} \subset \mathcal{P}_1^r(S^1)$ *is an exotic circle of* $\mathcal{P}_1^r(S^1)$*.*

ii) Two exotic circles $S_1 = h_1 \circ SO(2) \circ h_1^{-1}$, $S_2 = h_2 \circ SO(2) \circ h_2^{-1}$ *of* $\mathcal{P}_1^r(S^1)$ *are conjugated in* $\mathcal{P}_1^r(S^1)$ *if and only if* $\pi_s(h_1) = \pi_s(h_2)$*.*

iii) Every exotic circle of $\mathcal{P}_1^r(S^1)$ *is conjugate in* $\mathcal{P}_1^r(S^1)$ *to one of the* S_σ*.*

3. No exotic circle of $\mathcal{P}^r(S^1)$

Lemma 3.1. *Let* $S = h \circ SO(2) \circ h^{-1}$ *be a topological circle of* $\mathcal{P}^r(S^1)$, $h \in Homeo_+(S^1)$. *Then every element of* S *with irrational rotation number has the* (D_r)*-property.*

Proof. Let $f \in S$ with irrational rotation number $\alpha \in S^1$, that is $f = h \circ R_\alpha \circ h^{-1} \in \mathcal{P}^r(S^1)$. Let $g = h \circ R_\beta \circ h^{-1}$ with β irrational such that α, β are rationally independent. Since $f \circ g = g \circ f$, by Proposition 2.7, f and g have the (D_r)-property. □

Proof of Theorem 2.10.

Let $S = h \circ SO(2) \circ h^{-1} \subset \mathcal{P}^r(S^1)$ where $h \in Homeo_+(S^1)$. Take $f \in S$ with irrational rotation number $\beta \in S^1$. By Lemma 3.1, f has the (D_r)-property. Hence, by Theorem 2.8, there exists a polynômial homeomorphism $K \in \mathcal{P}^r(S^1)$ such that $F = K \circ f \circ K^{-1} \in Diff_+^r(S^1)$. Now for every $g = h \circ R_\alpha \circ h^{-1} \in S$, $G = K \circ g \circ K^{-1} = (K \circ h) \circ R_\alpha \circ (K \circ h)^{-1} \in \mathcal{P}^r(S^1)$. Since $G \circ F = F \circ G$, by Proposition 2.9, $G \in Diff_+^r(S^1)$. It follows by Corollary 1.1, that $K \circ h = u \in Diff_+^r(S^1)$. Hence, if $r \geq 2$, $r = +\infty$ or $r = \omega$, $h = K^{-1} \circ u \in \mathcal{P}^r(S^1)$.

If $r = 1$ then $G \in \mathcal{P}^1(S^1) \cap Diff_+^1(S^1)$, so, $G \in Diff_+^{1+BV}(S^1)$. By Corollary 1.2, we have $K \circ h = u \in Diff_+^{1+BV}(S^1)$. Hence $h = K^{-1} \circ u \in \mathcal{P}^1(S^1)$. This completes the proof. □

4. Existence of exotic circles of $\mathcal{P}_1^r(S^1)$

In this section, $r \geq 1$ is an integer, $r = +\infty$ or $r = \omega$. Let us consider the set

$$\mathcal{P}_1^r(S^1) = \{f \in \mathcal{P}^r(S^1) : \pi_s(f) = 1\}.$$

Lemma 4.1. $\mathcal{P}_1^r(S^1)$ *is a subgroup of* $\mathcal{P}^r(S^1)$.

Proof. Let us consider the map $\pi_s : \mathcal{P}^r(S^1) \longrightarrow \mathbb{R}^\star; f \longmapsto \pi_s(f)$. Since $\pi_s(g \circ f) = \pi_s(g)\pi_s(f)$ by Proposition 2.3, π_s is a group's homomorphism. Its kernel $Ker\ \pi_s = \mathcal{P}_1^r(S^1)$ is then a subgroup of $\mathcal{P}^r(S^1)$. □

Lemma 4.2. *Let* $\sigma \in \mathbb{R}_+^\star \setminus \{1\}$ *and* $h_\sigma \in \mathcal{P}^r(S^1)$ *with one break point* c *such that* $\sigma_{h_\sigma}(c) = \sigma$. *Then* $S_\sigma = h_\sigma \circ SO(2) \circ h_\sigma^{-1}$ *is an exotic circle of* $\mathcal{P}_1^r(S^1)$.

Proof. Letting $f = h_\sigma \circ R_\alpha \circ h_\sigma^{-1} \in S_\sigma$. Then $f \in \mathcal{P}^r(S^1)$ and has exactly two break points c_1 and $c_2 = f(c_1)$ and the product of f-jumps : $\pi_s(f) = \sigma_f(c_1)\sigma_f(c_2) = 1$, hence $f \in \mathcal{P}_1^r(\mathrm{S}^1)$. Therefore, $S_\sigma \subset \mathcal{P}_1^r(S^1)$. Since $\pi_s(h_\sigma) = \sigma_{h_\sigma}(c) = \sigma \neq 1$, $h_\sigma \notin \mathcal{P}_1^r(S^1)$. This completes the proof. □

Proof of Theorem 2.11. Assertion i) follows from Lemma 4.2.

Assertion ii): Let $S_1 = h_1 \circ SO(2) \circ h_1^{-1}$ and $S_2 = h_2 \circ SO(2) \circ h_2^{-1}$ be two exotic circles of $\mathcal{P}_1^r(S^1)$.

Suppose that $\pi_s(h_1) = \pi_s(h_2)$. Then $S_2 = L \circ S_1 \circ L^{-1}$ where $L = h_2 \circ h_1^{-1}$. Since S_1 and S_1 are topological circles of $\mathcal{P}^r(S^1)$, so, by Theorem 2.10, $L \in \mathcal{P}^r(S^1)$. Moreover, by Proposition 2.3,

$$\pi_s(L) = \frac{\pi_s(h_1)}{\pi_s(h_2)} = 1.$$

Hence, $L \in \mathcal{P}_1^r(S^1)$ and then S_1 and S_2 are conjugated in $\mathcal{P}_1^r(S^1)$.

Conversely, suppose that S_1 and S_2 are conjugated in $\mathcal{P}_1^r(S^1)$, that is $S_2 = L \circ S_1 \circ L^{-1}$ where $L \in \mathcal{P}_1^r(S^1)$. Let $\alpha \in S^1$ be irrational. We have

$$L \circ h_1 \circ R_\alpha \circ h_1^{-1} \circ L^{-1} = h_2 \circ R_\alpha \circ h_2^{-1},$$

hence,

$$h_1^{-1} \circ L^{-1} \circ h_2 \circ R_\alpha = R_\alpha \circ h_1^{-1} \circ L^{-1} \circ h_2.$$

Since α is irrational, $h_1^{-1} \circ L^{-1} \circ h_2$ must belong to $SO(2)$, so $h_1^{-1} \circ L^{-1} \circ h_2 = R_\beta$ for some $\beta \in S^1$. Thus, we have

$$L = h_2 \circ R_\beta^{-1} \circ h_1^{-1} = T \circ h_2 \circ h_1^{-1}$$

where $T = h_2 \circ R_\beta^{-1} \circ h_2^{-1} \in S_2$. Since $L, T \in \mathcal{P}_1^r(S^1)$, $h_2 \circ h_1^{-1} \in \mathcal{P}_1^r(S^1)$, so $\pi_s(h_2 \circ h_1^{-1}) = 1$, that is $\pi_s(h_1) = \pi_s(h_2)$.

Assertion iii): Let $S = h \circ SO(2) \circ h^{-1}$ be an exotic circle of $\mathcal{P}_1^r(S^1)$. By Theorem 2.10, $h \in \mathcal{P}^r(S^1)$ but $h \notin \mathcal{P}_1^r(S^1)$. Hence $\pi_s(h) = \sigma \neq 1$. Since $\pi_s(h_\sigma) = \sigma_{h_\sigma}(c) = \sigma = \pi_s(h)$, S is conjugated in $\mathcal{P}_1^r(S^1)$ to S_σ by Assertion ii). This completes the proof. □

5. The PL case

In this section, we consider the group $PL_+(S^1)$ and we give a new proof of Minakawa classification of all exotic circles of $PL(S^1)$.

Lemma 5.1. *Let $h \in Homeo_+(S^1)$. Then $S = h \circ SO(2) \circ h^{-1}$ is an exotic circle of $PL_+(S^1)$ if and only if there exists $\lambda \in \mathbb{R}^\star$ and a subdivision $c_0, c_1, ..., c_{p-1}$ of S^1 such that*

$$h(x) = \frac{\alpha_i}{\lambda} e^{\lambda x} + \beta_i,\ x \in]c_{i-1}, c_i[$$

where $\alpha_i \in \mathbb{R}_+^\star, \beta_i \in \mathbb{R}$ are constants.

Proof. Suppose that S is an exotic circle of $PL_+(S^1)$. Since $PL_+(S^1) \subset \mathcal{P}^\infty(S^1)$ then by Theorem 2.10, $h \in \mathcal{P}^\infty(S^1)$. We let $f = h \circ R_\alpha \circ h^{-1}$ with $\alpha \in S^1$ irrational. The set $h^{-1}(S(f)) \cap R_\alpha^{-1}(S(f)) \cap S(h)$ is finite and partitioned S^1 into segments $[c_{i-1}, c_i]$, $1 \leq i \leq p$ $(c_p = c_0)$. So, $f(h(x)) = k_i$, for every $x \in [c_{i-1}, c_i[$. Differentiating the relation $f \circ h = h \circ R_\alpha$, we obtain successively $k_i Dh(x) = Dh(R_\alpha(x))$ and $k_i D^2h(x) = D^2h(R_\alpha(x))$ for every $x \in]c_{i-1}, c_i[$. Hence

$$\frac{D^2h(x)}{Dh(x)} = \frac{D^2h(R_\alpha(x))}{Dh(R_\alpha(x))}.$$

Letting

$$\varphi(x) = \{ \begin{array}{c} \frac{D^2h(x)}{Dh(x)} \ \ if \ \ x \in S^1 \backslash \{c_0, ..., c_{p-1}\} \\ \frac{D^2h_+(c_i)}{Dh_+(c_i)} \ if \ \ x = c_i \end{array}$$

then we have $\varphi \circ R_\alpha = \varphi$ on S^1. Since $\varphi \in L^2(S^1)$ and R_α is ergodic with respect to the Haar measure m (α is irrational), φ is constant m a.e.; that is there exists a subset E in S^1 with $m(E) = 0$ such that $\varphi(x) = \lambda$ for every $x \in S^1 \backslash E$. Since $h \notin PL_+(S^1)$, $\lambda \neq 0$. We have $\frac{D^2h(x)}{Dh(x)} = \lambda$ for every $x \in]c_{i-1}, c_i[\backslash E$. Since $\frac{D^2h}{Dh}$ is continuous on $]c_{i-1}, c_i[$ and $]c_{i-1}, c_i[\backslash E$ is dense in $]c_{i-1}, c_i[$, $\frac{D^2h}{Dh} = \lambda$ on $]c_{i-1}, c_i[$ for every i. The resolution of the differential equation $D^2h(x) = \lambda Dh(x)$, $x \in]c_{i-1}, c_i[$ implies that there exist two constants $\alpha_i \in \mathbb{R}^\star_+$, $\beta_i \in \mathbb{R}$ such that

$$h(x) = \frac{\alpha_i}{\lambda} e^{\lambda x} + \beta_i, \ x \in]c_{i-1}, c_i[.$$

Conversely, we let $h(x) = \frac{\alpha_i}{\lambda} e^{\lambda x} + \beta_i$, $x \in]c_{i-1}, c_i[$ where $\alpha_i \in \mathbb{R}^\star_+, \beta_i \in \mathbb{R}$ are constants. Then for every $\delta \in S^1$, $x \in]c_{i-1}, c_i[$, we have

$$h \circ R_\delta \circ h^{-1}(x) = h \circ R_\delta(\frac{1}{\lambda} \log(\frac{\lambda}{\alpha_i}(x - \beta_i)))$$
$$= h(\frac{1}{\lambda} \log(\frac{\lambda}{\alpha_i}(x - \beta_i)) + \delta) = \frac{\alpha_j}{\alpha_i} e^{\lambda\delta}(x - \beta_i) + \beta_j.$$

Therefore, $S \subset PL_+(S^1)$ and since $h \notin PL_+(S^1)$, S is an exotic circle of $PL_+(S^1)$. This completes the proof. □

Remark 2. Let $h_\sigma \in Homeo_+(S^1)$ as in Lemma 5.1 with one break point 0 such that $h_\sigma(0) = 0$ and $\sigma_{h_\sigma}(0) = \sigma$. Then

$$h_\sigma(x) = \frac{\sigma^x - 1}{\sigma - 1}, \ x \in [0, 1[.$$

Indeed, by Lemma 5.1,

$$h_\sigma(x) = \frac{\alpha}{\lambda} e^{\lambda x} + \beta, \ x \in [0, 1].$$

Since $h_\sigma(0) = 0$ and $h_\sigma(1) = 1$, we have $\beta = \frac{1}{1-e^\lambda}$ and $\alpha = -\frac{\lambda}{1-e^\lambda}$. Hence,

$$h_\sigma(x) = \frac{-1}{1 - e^\lambda} e^{\lambda x} + \frac{1}{1 - e^\lambda} = \frac{e^{\lambda x} - 1}{e^\lambda - 1}.$$

Or

$$\sigma_{h_\sigma}(0) = \frac{D(h_\sigma)_-(0)}{D(h_\sigma)_+(0)}$$
$$= \frac{D(h_\sigma)_-(1)}{D(h_\sigma)_+(0)} = e^\lambda,$$

hence $e^\lambda = \sigma$ and $h_\sigma(x) = \frac{\sigma^x - 1}{\sigma - 1}$.

Proof of Minakawa's Theorem. Under the hypothesis of Minakawa's Theorem, $S_\sigma = h_\sigma \circ SO(2) \circ h_\sigma^{-1}$ is an exotic circle of $PL_+(S^1)$ by Remark 2.

Now let $S = h \circ SO(2) \circ h^{-1}$ be an exotic circle of $PL_+(S^1)$. By Theorem 2.10, $h \in \mathcal{P}^\infty(S^1)$. Letting $\pi_s(h) = \sigma$, we have $S = L \circ S_\sigma \circ L^{-1}$ where $L = h \circ h_\sigma^{-1}$ and $\pi_s(L) = 1$. Let's show that $L \in PL_+(S^1)$:

By Lemma 5.1, there exists $\lambda \in \mathbb{R}^\star$ and a subdivision $c_0, c_1, ..., c_{p-1}$ of S^1 such that $h(x) = \frac{\alpha_i}{\lambda} e^{\lambda x} + \beta_i,\ x \in]c_{i-1}, c_i[$ where $\alpha_i \in \mathbb{R}^\star_+,\ \beta_i \in \mathbb{R}$ are constants. One can suppose that $c_0 = 0$ by replacing h with $h \circ R_{c_0}$ since

$$S = h \circ SO(2) \circ h^{-1} = h \circ R_{c_0} \circ SO(2) \circ R_{c_0}^{-1} \circ h^{-1}.$$

For $i = 1, ..., p-1$, we have

$$\sigma_h(c_i) = \frac{Dh_-(c_i)}{Dh_+(c_i)} = \frac{\alpha_i e^{\lambda c_i}}{\alpha_{i+1} e^{\lambda c_i}} = \frac{\alpha_i}{\alpha_{i+1}},$$

and

$$\sigma_h(0) = \frac{D_-h(0)}{D_+h(0)} = \frac{D_-h(1)}{D_+h(0)} = \frac{\alpha_p e^\lambda}{\alpha_1}.$$

Hence,

$$\begin{aligned} \pi_s(h) &= \sigma_h(0) \prod_{1 \leq i \leq p-1} \sigma_h(c_i) \\ &= \frac{\alpha_p e^\lambda}{\alpha_1} \prod_{1 \leq i \leq p-1} \frac{\alpha_i}{\alpha_{i+1}} = \frac{\alpha_p e^\lambda}{\alpha_1} \frac{\alpha_1}{\alpha_p} = e^\lambda \end{aligned}$$

So, $\pi_s(h) = e^\lambda = \sigma$. Since $\lambda \neq 0,\ \sigma \neq 1$.

It follows that $h(x) = \frac{\alpha_i}{\log \sigma} \sigma^x + \beta_i,\ x \in]c_{i-1}, c_i[$. On the other hand, we have $h_\sigma^{-1}(x) = \frac{1}{\log \sigma} \log((\sigma - 1)x + 1)$. We compute

$$h \circ h_\sigma^{-1}(x) = \frac{\alpha_i}{\log \sigma}((\sigma - 1)x + 1) + \beta_i.$$

Moreover, $\frac{\alpha_i}{\log \sigma}(\sigma - 1) > 0$, hence $L \in PL_+(S^1)$. This completes the proof. □

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