Misiurewicz parameters in the exponential family

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Abstract

A complex exponential map is said to be Misiurewicz if the forward trajectory of the asymptotic value 0 lies in the Julia set and is bounded. We prove that the set of Misiurewicz parameters in the exponential family $\lambda \exp(z)$, $\lambda \in \mathbb{C} \setminus \{0\}$, has Lebesgue measure zero.

1 Introduction

We consider the exponential family

$$f_\lambda(z) = \lambda \exp(z) \ , \ \lambda \in \mathbb{C} \setminus \{0\}, \tag{1.1}$$

of entire transcendental functions. It is well known class of functions, studied by many authors. It is impossible to list them all, but let us mention at least some: [2, 5, 6, 7, 12]. Note that the dependence on the parameter λ in the considered family is analytic. These maps have only one singular value 0, which is an asymptotic value, so the dynamics of f_λ is determined by the trajectory of 0. For a general description of the dynamics of transcendental functions and more references see e.g. [3].

The notion of Misiurewicz maps derives from the paper [13] by M. Misiurewicz, where the author studied e.g. the real quadratic family $g_a(x) = 1 - ax^2$ in the case when g_a is non-hyperbolic and the critical point 0 is non-recurrent. We refer to [1] for a nice discussion concerning various definitions of Misiurewicz condition in the complex case and more references. We are interested in defining Misiurewicz maps for the exponential family (1.1). The authors of [4] call a parameter λ_0 Misiurewicz if the only singular value 0 is eventually mapped by f_{λ_0} onto a periodic cycle in the Julia set $J(f_{\lambda_0})$. We want to generalize this notion and introduce the following definition which is an analogue of the Misiurewicz' original idea.

Definition 1.1 *Parameter λ_0 in the exponential family (1.1) is called Misiurewicz if the asymptotic value 0 belongs to the Julia set $J(f_{\lambda_0})$ and its f_{λ_0}-forward orbit is bounded.*

Note that, since f_{λ_0} is exponential, the boundedness of the orbit of 0 immediately implies that 0 is non-recurrent. Our definition of the Misiurewicz map obviously includes the case when 0 has finite trajectory (like in [4]).

Urbanski and Zdunik in [15] showed that every Misiurewicz parameter λ_0 in the exponential family is unstable, i.e. in any neighbourhood of λ_0 in the parameter space, there is some λ_1 such that f_{λ_0} and f_{λ_1} are not quasi-conformally (topologically) conjugate. One can ask about the Lebesgue measure of those parameters in the parameter space. Recently it was proved by M. Aspenberg in [1] that the set of Misiurewicz maps has Lebesgue measure zero in the space of rational functions of any fixed degree. In this paper we prove the following.

Theorem 1.2 *The set of Misiurewicz parameters in the exponential family (1.1) has the Lebesgue measure zero in $\mathbb{C}$.*

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Let us denote

$$\mathcal{M} = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda \text{ is a Misiurewicz parameter}\}.$$

For every $\lambda \in \mathcal{M}$, since 0 has bounded trajectory under f_λ, we can find some $\delta > 0$ such that

$$\overline{O_\lambda(0)} \subset B(0, 1/\delta) \setminus B(0, \delta), \tag{1.2}$$

where $O_\lambda(0) = \bigcup_{n \geq 1} f_\lambda^n(0)$ is the forward trajectory of the asymptotic value 0. Parameters for which (1.2) holds will be called δ-Misiurewicz. Denote also

$$\mathcal{M}_\delta = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda \text{ is } \delta\text{-Misiurewicz}\}.$$

Note that

$$\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_{1/n} \quad \text{and} \quad \delta_1 < \delta_2 \Rightarrow \mathcal{M}_{\delta_1} \supset \mathcal{M}_{\delta_2}.$$

Following Aspenberg's idea in [1] we will show that δ-Misiurewicz parameters are rare in any neighbourhood of λ_0.

Theorem 1.3 *If λ_0 is a Misiurewicz parameter, then for every $\delta > 0$, the set $\mathcal{M}_\delta$ has the Lebesgue density strictly smaller than one at λ_0.*

Obviously Theorem 1.3 implies that $\mu(\mathcal{M}_\delta) = 0$ for every $\delta > 0$, where μ is the 2-dimensional Lebesgue measure. Hence

$$\mu(\mathcal{M}) \leq \sum_{n \geq 1} \mu(\mathcal{M}_{1/n}) = 0,$$

which is exactly the statement of Theorem 1.2.

The proof of Theorem 1.3 in general follows the Aspenberg's approach from [1], however it differs in some crucial details. The main difficulty is the presence of essential singularity at ∞ and infinite degree of maps. Because of this our considerations have to be local in most cases. As we will see in Lemma 5.2 sometimes we need to be much more careful and our estimates need to be slightly more subtle to avoid obstacles which do not appear in the rational case.

We will focus on a Misiurewicz parameter λ_0 and its neighbourhood $B(\lambda_0, r)$ in the parameter plane. We will see how the Misiurewicz condition implies exponential expansion on Λ, the closure of the forward trajectory of 0 under f_{λ_0}, which leads to the existence of a holomorphic motion $h : \Lambda \times B(\lambda_0, r) \to \mathbb{C}$ conjugating the dynamics of f_{λ_0} and nearby maps f_λ, $\lambda \in B(\lambda_0, r)$, on a neighbourhood of Λ. Next, we will use the expansion property and instability of Misiurewicz exponential maps to derive nice distortion properties binding space and parameter derivatives in a small scale. This allows us to control the growth of a parameter ball $B(\lambda_0, r)$ to a big scale, where in turn we can estimate the measure of those parameters which are not δ-Misiurewicz.

2 Holomorphic motion and instability

In the following sections we will use the Euclidean metric and derivatives unless otherwise stated. Take a Misiurewicz parameter $\lambda_0 \in \mathcal{M}$, then, since the only singular value of f_{λ_0} is in the Julia set, we have that $J(f_{\lambda_0}) = \mathbb{C}$. Consider the set $\Lambda = \overline{O_{\lambda_0}(0)}$, closure of the forward trajectory of 0 under f_{λ_0}. It is compact, forward invariant, contains neither critical nor parabolic points. Hence, by Theorem 1 in [8] (compare also with [14]), Λ is a hyperbolic set, i.e. there are real constants $C > 0$ and $a > 1$ such that

$$|(f_{\lambda_0}^n)'(z)| \geq C a^n \quad \text{for all } z \in \Lambda \quad \text{and } n \geq 1.$$

Look now at the nearby exponential maps f_λ, $\lambda \in B(\lambda_0, r)$. If $r > 0$ is sufficiently small, since Λ is hyperbolic, there exists a holomorphic motion

$$h \colon \Lambda \times B(\lambda_0, r) \to \mathbb{C}$$

such that $h_{\lambda_0} = \mathrm{id}$, the map $h_\lambda := h(\cdot,\lambda)\colon \Lambda \to \Lambda_\lambda$ is quasiconformal for each parameter $\lambda \in B(\lambda_0, r)$, and $h(z,\cdot)\colon B(\lambda_0,r) \to \mathbb{C}$ is holomorphic at every $z \in \Lambda$. To see this, follow the proof of [11, Theorem III.1.6] locally in a neighbourhood of the hyperbolic set Λ. Moreover

$$h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda \ \text{ on } \ \Lambda.$$

Note that $\Lambda_\lambda := h_\lambda(\Lambda)$ is a hyperbolic set for f_λ.

Now, we want to obtain so-called transversality condition (cf. [1]), which says that the asymptotic value 0 cannot follow its holomorphic motion $h_\lambda(0)$ in the whole parameter ball $B(\lambda_0, r)$. As we will see, it is an immediate consequence of the instability of Misiurewicz parameters (more general Collet-Eckmann maps) in the exponential family proved by Urbański and Zdunik in [15]. It follows also from the non-existence of invariant line-fields for Misiurewicz maps proved by Graczyk, Kotus and Świątek in [8, Theorem 2]. For the convenience of the reader, we will use notation analogous to [1].

Recall that 0 is the only singular value of each f_λ, thus its trajectory determines the dynamics of the map. Consider a holomorphic function $x : B(\lambda_0, r) \to \mathbb{C}$ given by

$$x(\lambda) = 0 - h_\lambda(0)$$

which is exactly the difference between the asymptotic value and its holomorphic motion. Note that $h_\lambda(0)$ always belongs to the hyperbolic set Λ_λ. We obviously have that $x(\lambda_0) = 0$. Our aim is to show that λ_0 is an isolated zero of x.

Lemma 2.1 *The function x is not identically zero in any ball $B(\lambda_0, r)$ in the parameter plane.*

Proof. Suppose that $x(\lambda) \equiv 0$ on some ball $B(\lambda_0, r)$. Then $\xi_n(\lambda) := f_\lambda^n(0)$ would form a normal family on $B(\lambda_0, r)$ and we could extend h_λ to a quasiconformal conjugacy on the whole Julia set $J(f_{\lambda_0}) = \mathbb{C}$ between two Misiurewicz maps f_{λ_0} and f_λ for any $\lambda \in B(\lambda_0, r)$ (cf. [10, Theorem 4.2] and [9, Theorem 3.1]). But this means that λ_0 would be a stable parameter. This is however a contradiction since every Misiurewicz parameter in the exponential family is unstable (see [15]). □

Therefore we have that

$$x(\lambda) = \alpha_K(\lambda - \lambda_0)^K + \ldots \tag{2.1}$$

for some $K \geq 1$ and $\alpha_K \neq 0$. This property will be crucial to obtain distortion estimates in the next section.

3 Expansion and distortion estimates

In this section we will derive distortion estimates based on the expansion property near the hyperbolic set Λ. It is rather technical and mainly follows analogous proofs in [1]. We decided however to keep it in a very detailed form for the convenience of the reader and also because of changes which are minor but crucial.

Since Λ is a hyperbolic set, we can take a small neighbourhood $\mathcal{N}$ of Λ such that

$$|(f_{\lambda_0}^m)'(z)| > \tilde{a} > 1 \ \text{ for some } m \geq 1 \ \text{ and all } z \in \mathcal{N}.$$

Moreover, for some small radius $r > 0$, decreasing slightly $\tilde{a}$ if necessarily, we also have

$$|(f_\lambda^m)'(z)| > \tilde{a} > 1 \ \text{ for all } z \in \mathcal{N} \ \text{ and } \lambda \in B(\lambda_0, r).$$

We may also assume that the set $\mathcal{N}$ is closed, bounded (hence compact) and for some $\delta > 0$, $\mathcal{N} \subset B(0, 1/\delta) \setminus B(0, \delta)$. We get therefore the following lemma.

Lemma 3.1 *There are constants $C > 0$, $a > 1$ and radius $r > 0$ such that whenever $f_\lambda^j(z) \in \mathcal{N}$ for $j = 0, \ldots, k$ and $\lambda \in B(\lambda_0, r)$, then*

$$|(f_\lambda^k)'(z)| \geq C a^k.$$

If we now take some $\delta' > 0$ for which $\{z : \mathrm{dist}(z, \Lambda) \le 11\delta'\} \subset \mathcal{N}$, then we will always assume $r > 0$ to be so small that $\{z : \mathrm{dist}(z, \Lambda_\lambda) \le 10\delta'\} \subset \mathcal{N}$ for each $\lambda \in B(\lambda_0, r)$. This means that Λ_λ, the hyperbolic set for f_λ, is well inside $\mathcal{N}$.

Recall that we have chosen $\mathcal{N}$ so that for some $m \ge 1$, $\tilde{a} > 1$ and for all $z \in \mathcal{N}$, $\lambda \in B(\lambda_0, r)$, we have $|(f_\lambda^m)'(z)| \ge \tilde{a}$. Thus for every $z \in \mathcal{N}$ we can find some radius $r(z) > 0$ such that

$$|f_\lambda^m(z) - f_\lambda^m(w)| \ge \tilde{a}|z - w| \tag{3.1}$$

for all $w \in \mathcal{N}$ with $|z - w| \le r(z)$ (decreasing slightly $\tilde{a} > 1$ if necessarily). Since $\mathcal{N}$ is compact and $r(z)$ changes continuously, we can find a universal $\tilde{r} > 1$ such that (3.1) holds for every $z, w \in \mathcal{N}$ with $|z - w| \le \tilde{r}$. This implies exponential expansion in a small scale.

Lemma 3.2 *There are constants $\tilde{\delta}, C > 0$ and $a > 1$ such that for every $\lambda \in B(\lambda_0, r)$ and every $z, w \in \mathcal{N}$, if $f_\lambda^j(z), f_\lambda^j(w) \in \mathcal{N}$ and $|f_\lambda^j(z) - f_\lambda^j(w)| \le \tilde{\delta}$ for $j = 0, \ldots, k$, then*

$$|f_\lambda^k(z) - f_\lambda^k(w)| \ge Ca^k|z - w|.$$

Proof. Every integer k can be written in the form $k = pm + q$, where $q \le m - 1$. For some $\tilde{C}, \tilde{\delta} > 0$ we can estimate for all $\lambda \in B(\lambda_0, r)$

$$|f_\lambda(z) - f_\lambda(w)| \ge \tilde{C}|z - w| \quad \text{for all } z, w \in \mathcal{N} \quad \text{with } |z - w| \le \tilde{\delta}.$$

If we now take $z, w \in \mathcal{N}$ for which assumptions of the lemma are satisfied, then

$$|f_\lambda^k(z) - f_\lambda^k(w)| \ge \tilde{a}^p|f_\lambda^q(z) - f_\lambda^q(w)| \ge \tilde{a}^p\tilde{C}^q|z - w| \ge a^kC|z - w|$$

for $a = \tilde{a}^{\frac{1}{m}}$ and some $C > 0$. □

We will use the expansion property in the following distortion estimates to show that in a small scale parameter and space derivatives are comparable. For $\lambda \in B(\lambda_0, r)$ and $n \ge 0$ put

$$\xi_n(\lambda) = f_\lambda^n(0) \quad \text{and} \quad \mu_n(\lambda) = f_\lambda^n(h_\lambda(0)) = h_\lambda(f_{\lambda_0}^n(0)).$$

Then $\xi_n(\lambda)$ is the forward orbit of the asymptotic value for f_λ while $\mu_n(\lambda)$ is the holomorphic motion of the asymptotic orbit for f_{λ_0}, hence $\mu_n(\lambda) \in \Lambda_\lambda$. In particular $x(\lambda) = \xi_0(\lambda) - \mu_0(\lambda)$.

The following lemma will be used several times in our distortion estimates. See [1] for references.

Lemma 3.3 *Let $u_n \in \mathbb{C}$ for $n = 1, \ldots, N$. Then*

$$\left|\prod_{n=1}^{N}(1 + u_n) - 1\right| \le \exp\left(\sum_{n=1}^{N}|u_n|\right) - 1.$$

Let us begin with the Main Distortion Lemma concerning control of the space derivative in a neighbourhood of the hyperbolic set.

Lemma 3.4 *For every $\varepsilon > 0$ we can find $\delta' > 0$ and $r > 0$ arbitrarily small with the following property. For any $a, b \in B(\lambda_0, r)$ if $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for all $k \le n$ and $\lambda = a, b$, then*

$$\left|\frac{(f_a^n)'(0)}{(f_b^n)'(0)} - 1\right| < \varepsilon.$$

Proof. First we will show that for an arbitrarily small $\varepsilon_1 = \varepsilon_1(\delta')$, it is possible to choose $\delta' > 0$ so that

$$\left|\frac{(f_\lambda^n)'(h_\lambda(0))}{(f_\lambda^n)'(0)} - 1\right| \le \varepsilon_1 \tag{3.2}$$

provided $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for all $k \le n$.

By the expansion property and since $|f'_\lambda| > C_\delta^{-1}$ on $\mathcal{N}$ for some $C_\delta > 0$, we can estimate for any $\lambda \in B(\lambda_0, r)$:

$$\sum_{j=0}^{n-1} \left| \frac{f'_\lambda(\mu_j(\lambda)) - f'_\lambda(\xi_j(\lambda))}{f'_\lambda(\xi_j(\lambda))} \right| \le C_\delta \sum_{j=0}^{n-1} |f'_\lambda(\mu_j(\lambda)) - f'_\lambda(\xi_j(\lambda))| \le$$
$$\le C_\delta \max_{z\in\mathcal{N}} |f''_\lambda(z)| \sum_{j=0}^{n-1} |\mu_j(\lambda) - \xi_j(\lambda)| \le \tilde{C} \sum_{j=0}^{n-1} Ca^{j-n} |\mu_n(\lambda) - \xi_n(\lambda)| \le C'\delta'$$

Using Lemma 3.3 we obtain the inequality (3.2) if δ' is small enough.

Secondly, for any $\varepsilon_2 > 0$, if $\delta' > 0$ and $r > 0$ are chosen sufficiently small, then for every $t, s \in B(\lambda_0, r)$

$$\left| \frac{(f_t^n)'(h_t(0))}{(f_s^n)'(h_s(0))} - 1 \right| \le \varepsilon_2. \tag{3.3}$$

Put $a_{\lambda,j} = f'_\lambda(\mu_j(\lambda))$. Since each $a_{\lambda,j}$ is analytic with respect to λ, it can be expressed as follows: $a_{\lambda,j} = a_{\lambda_0,j}(1 + c_j(\lambda - \lambda_0)^j + \ldots)$. Moreover, by Lemma 3.2 and (2.1), we have that

$$n \le -C \log |x(\lambda)| \le -\tilde{C} \log |\lambda - \lambda_0|, \tag{3.4}$$

where constants depend only on δ' and not on n. Thus, if $c = \sum_{j=0}^{n-1} c_j$, then

$$\frac{(f_t^n)'(h_t(0))}{(f_s^n)'(h_s(0))} = \prod_{j=0}^{n-1} \frac{a_{t,j}}{a_{s,j}} = \prod_{j=0}^{n-1} \frac{a_{\lambda_0,j}(1 + c_j(t - \lambda_0)^l + \ldots)}{a_{\lambda_0,j}(1 + c_j(s - \lambda_0)^l + \ldots)} = \frac{1 + cn(t - \lambda_0)^l + \ldots}{1 + cn(s - \lambda_0)^l + \ldots}.$$

Now, both the numerator and the denominator can be made arbitrarily close to one if only $r > 0$ is small enough, since they are of order $1 + \mathcal{O}(|t - \lambda_0|^l \log |t - \lambda_0|)$ and $1 + \mathcal{O}(|s - \lambda_0|^l \log |s - \lambda_0|)$.

Putting together (3.2) and (3.3) we obtain the statement of the lemma. □

Next we want to compare space and parameter derivatives.

Lemma 3.5 *Let $\varepsilon > 0$. If $\delta' > 0$ is sufficiently small, then for every $0 < \delta'' < \delta'$, there exists an $r > 0$ such that the following holds. For any $\lambda \in B(\lambda_0, r)$, if $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for $k \le n$ and $|\xi_n(\lambda) - \mu_n(\lambda)| \ge \delta''$, then*

$$\left| \frac{\xi'_n(\lambda)}{(f_\lambda^n)'(h_\lambda(0))x'(\lambda)} - 1 \right| \le \varepsilon.$$

Proof. Note that we have

$$\xi_n(\lambda) = \mu_n(\lambda) + (f_\lambda^n)'(h_\lambda(0))x(\lambda) + E_n(\lambda),$$

where $|E_n(\lambda)| \le \varepsilon' |\xi_n(\lambda) - \mu_n(\lambda)|$ independently of n, for any small $\varepsilon' > 0$, if only $\delta' > 0$ was chosen small enough. To see this we will proceed similarly as in the first part of the proof of Lemma 3.4. First we can write

$$\frac{(f_\lambda^n)'(h_\lambda(0))x(\lambda)}{\xi_n(\lambda) - \mu_n(\lambda)} = \prod_{j=0}^{n-1} \frac{f'_\lambda(\mu_j(\lambda))(\xi_j(\lambda) - \mu_j(\lambda))}{\xi_{j+1}(\lambda) - \mu_{j+1}(\lambda)}.$$

By the expansion property - Lemma 3.2 - we can estimate as follows

$$\left| \frac{f'_\lambda(\mu_j(\lambda))(\xi_j(\lambda) - \mu_j(\lambda))}{\xi_{j+1}(\lambda) - \mu_{j+1}(\lambda)} - 1 \right| \le \frac{1}{Ca} \left| f'_\lambda(\mu_j(\lambda)) - \frac{\xi_{j+1}(\lambda) - \mu_{j+1}(\lambda)}{\xi_j(\lambda) - \mu_j(\lambda)} \right| \le$$
$$\le \frac{1}{Ca} \max_{z\in\mathcal{N}} |f''_\lambda(z)|\, |\xi_j(\lambda) - \mu_j(\lambda)| \le \frac{M''}{Ca} C^{-1} a^{j-n} |\xi_n(\lambda) - \mu_n(\lambda)|,$$

for $M'' = \max\{|f''_\lambda(z)| : z \in \mathcal{N}, \lambda \in B(\lambda_0, r)\}$. Applying Lemma 3.3 we obtain the estimate we were looking for.

Put again $f'_\lambda(\mu_j(\lambda)) = a_{\lambda,j}$, then $(f^n_\lambda)'(h_\lambda(0)) = \prod_{j=0}^{n-1} a_{\lambda,j}$. Now, differentiate ξ_n with respect to λ. By the Chain Rule we get

$$\begin{aligned}\xi'_n(\lambda) =&\mu'_n(\lambda) + x'(\lambda)\prod_{j=0}^{n-1} a_{\lambda,j} + x(\lambda)\sum_{j=0}^{n-1} a'_{\lambda,j}\frac{\prod_{k=0}^{n-1} a_{\lambda,k}}{a_{\lambda,j}} + E'_n(\lambda) = \\ =&\prod_{j=0}^{n-1} a_{\lambda,j}\left(x'(\lambda) + x(\lambda)\sum_{j=0}^{n-1}\frac{a'_{\lambda,j}}{a_{\lambda,j}} + \frac{\mu'_n(\lambda)+E'_n(\lambda)}{\prod_{j=0}^{n-1} a_{\lambda,j}}\right).\end{aligned}$$

In the following we want to show that $x'(\lambda)$ is the leading term in the above expression.

Recall that $\delta'' \le |\xi_n(\lambda) - \mu_n(\lambda)| \le \delta'$, thus if $r > 0$ is small enough, then by Lemma 3.4, for an arbitrarily small $\varepsilon_1 > 0$

$$\delta' \ge |\xi_n(\lambda) - \mu_n(\lambda)| = |f^n_\lambda(0) - f^n_\lambda(h_\lambda(0))| \ge |(f^n_\lambda)'(h_\lambda(0))||x(\lambda)|\frac{1}{1+\varepsilon_1}$$

and

$$\delta'' \le |\xi_n(\lambda) - \mu_n(\lambda)| = |f^n_\lambda(0) - f^n_\lambda(h_\lambda(0))| \le |(f^n_\lambda)'(h_\lambda(0))||x(\lambda)|\frac{1}{1-\varepsilon_1}.$$

So we have

$$(1-\varepsilon_1)\delta'' \le |x(\lambda)|\prod_{j=0}^{n-1}|a_{\lambda,j}| \le (1+\varepsilon_1)\delta' \tag{3.5}$$

Now we need to estimate $|\sum\frac{a'_{\lambda,j}}{a_{\lambda,j}}|$. Note that, since $\mu_j(\lambda) = f^j_\lambda(h_\lambda(0)) \in \Lambda_\lambda$, we get that

$$|a_{\lambda,j}| = |f'_\lambda(\mu_j(\lambda))| \le \max_{z\in\Lambda_\lambda, \lambda\in B(\lambda_0,r)} |f'_\lambda(z)| \quad \text{and} \quad |a_{\lambda,j}| \ge Ca\,,\ C, a > 0.$$

Since $a_{\lambda,j}$ are uniformly bounded for every j and $\lambda \in B(\lambda_0, r)$, therefore, by Cauchy's formula, also $a'_{\lambda,j}$ are uniformly bounded by some $M' > 0$ on a slightly smaller ball $B(\lambda_0, r')$. We get the following

$$\left|\sum_{j=0}^{n-1}\frac{a'_{\lambda,j}}{a_{\lambda,j}}\right| \le \sum_{j=0}^{n-1}\left|\frac{a'_{\lambda,j}}{a_{\lambda,j}}\right| \le n\frac{M'}{Ca} =: n\tilde{C}.$$

Thus, using (3.4),

$$|x(\lambda)|\left|\sum\frac{a'_{\lambda,j}}{a_{\lambda,j}}\right| \le |x(\lambda)|n\tilde{C} \le |x(\lambda)|C'(-\log|x(\lambda)|)\tilde{C},$$

where $C' > 0$ depends only on δ', and, up to a multiplicative constant,

$$\frac{-|x(\lambda)|\log|x(\lambda)|}{|x'(\lambda)|} \asymp \frac{-|(\lambda-\lambda_0)^K|\log|\lambda-\lambda_0|}{|(\lambda-\lambda_0)^{K-1}|} \asymp -|\lambda-\lambda_0|\log|\lambda-\lambda_0|. \tag{3.6}$$

Let us estimate

$$\begin{aligned}\frac{\xi'_n(\lambda)}{(f^n_\lambda)'(h_\lambda(0))x'(\lambda)} - 1 =&\frac{\prod a_{\lambda,j}\left(x'(\lambda) + x(\lambda)\sum\frac{a'_{\lambda,j}}{a_{\lambda,j}} + \frac{\mu'_n(\lambda)+E'_n(\lambda)}{\prod a_{\lambda,j}}\right)}{\prod a_{\lambda,j}\,x'(\lambda)} - 1 = \\ =&\frac{x(\lambda)\sum\frac{a'_{\lambda,j}}{a_{\lambda,j}}}{x'(\lambda)} + \frac{\mu'_n(\lambda)+E'_n(\lambda)}{\prod a_{\lambda,j}\,x'(\lambda)}.\end{aligned}$$

By (3.6) the first summand tends uniformly to zero as $\lambda \to \lambda_0$. To see what happens with the second summand note that $|\mu'_n(\lambda) + E'_n(\lambda)|$ is uniformly bounded by Cauchy's formula, since

$\mu_n(\lambda)$ and $E_n(\lambda)$ are bounded. We have also seen that $|\prod a_{\lambda,j}\, x(\lambda)|$ is bounded (from both sides) independently of n. Therefore, by (3.5), we get

$$\left|\frac{1}{\prod a_{\lambda,j}\, x'(\lambda)}\right| = \left|\frac{1}{\prod a_{\lambda,j}\, x(\lambda)}\right| \left|\frac{x(\lambda)}{x'(\lambda)}\right| \le \frac{1}{\delta''(1-\varepsilon_1)} \left|\frac{x(\lambda)}{x'(\lambda)}\right| \asymp |\lambda - \lambda_0|,$$

thus also the second summand tends uniformly to zero as $\lambda \to \lambda_0$. This finishes the proof. □

Binding together Lemma 3.4 and Lemma 3.5 we obtain the following result.

Corollary 3.6 *Let $\varepsilon > 0$. If $\delta' > 0$ is small enough and $0 < \delta'' < \delta'$, we can find a radius $r > 0$ such that for every $\lambda \in B(\lambda_0, r)$ if $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for $k \le n$ and $|\xi_n(\lambda) - \mu_n(\lambda)| \ge \delta''$, then*

$$\left|\frac{\xi_n'(\lambda)}{(f_\lambda^n)'(0)\, x'(\lambda)} - 1\right| \le \varepsilon.$$

4 Distortion in an annulus

As we have seen in the previous section, we need to move away from λ_0 in the parameter ball $B(\lambda_0, r)$ in order to have nice distortion estimates. That is why we will restrict our considerations to an annular domain. This approach will give us a powerful tool which is bounded distortion of ξ_n and will lead to the control of the growth of $B(\lambda_0, r)$ under ξ_n.

Consider an annulus in the parameter space:

$$A = A(\lambda_0; r_1, r_2) = \{\lambda : r_1 < |\lambda - \lambda_0| < r_2\}.$$

Note that, by (2.1), for some constant $C \ge 1$ and any $\lambda_1, \lambda_2 \in A$

$$C^{-1}\left(\frac{r_1}{r_2}\right)^{K-1} \le \left|\frac{x'(\lambda_1)}{x'(\lambda_2)}\right| \le C\left(\frac{r_2}{r_1}\right)^{K-1},$$

where K is the degree of $x(.)$ at λ_0. Therefore from Corollary 3.6 and Lemma 3.4 we conclude that if $r_2 > 0$ is small enough, then

$$\tilde{C}^{-1}\left(\frac{r_1}{r_2}\right)^{K-1} \le \left|\frac{\xi_n'(\lambda_1)}{\xi_n'(\lambda_2)}\right| \le \tilde{C}\left(\frac{r_2}{r_1}\right)^{K-1},$$

for some $\tilde{C} \ge 1$ and all $\lambda_1, \lambda_2 \in A$, as long as $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for $k \le n$ and $|\xi_n(\lambda) - \mu_n(\lambda)| \ge \delta''$ for all $\lambda \in A$.

Lemma 4.1 *Let $\varepsilon > 0$. If $\delta' > 0$ and $\frac{\delta''}{\delta'}$ are sufficiently small, $0 < \delta'' < \delta'$, there exists an $r > 0$ such that for any ball $B = B(0, r_2) \subset B(0, r)$ we have the following. Let n be maximal for which $|\xi_n(\lambda) - \mu_n(\lambda)| \le \delta'$ for all $\lambda \in B$. Let $r_1 < r_2$ be minimal such that $|\xi_n(\lambda) - \mu_n(\lambda)| \ge \delta''$ for all $\lambda \in A = A(\lambda_0; r_1, r_2)$. Then $\frac{r_1}{r_2} \le \frac{1}{10}$ and there is some $\delta'' < \delta_1' < \delta'$ such that*

$$A(\mu_n(\lambda_0); \delta'' + \varepsilon, \delta_1' - \varepsilon) \subset \xi_n(A) \subset A(\mu_n(\lambda_0); \delta'' - \varepsilon, \delta_1' + \varepsilon).$$

Moreover, ξ_n is at most K-to-1 on B.

Proof. Note that a parameter circle $\gamma_r = \{\lambda : |\lambda - \lambda_0| = r\}$, for small $r > 0$, is mapped under $x(.)$ onto a curve that encircles λ_0 K-times so that $x(\gamma_r)$ is close to a circle of radius $\alpha_K r^K$. Moreover, $|\mu_n(\lambda) - \mu_n(\lambda_0)| = |h_\lambda(f_{\lambda_0}^n(0)) - f_{\lambda_0}^n(0)|$ is arbitrarily small for small radius in the parameter space, since Λ and Λ_λ can be very close to each other for $\lambda \in B(\lambda_0, r)$. Thus if r is small and $|\xi_n(\lambda) - \mu_n(\lambda)| \ge \delta''$, then

$$|\xi_n(\lambda) - \mu_n(\lambda)| > P|\mu_n(\lambda) - \mu_n(\lambda_0)| \tag{4.1}$$

for some big $P \gg 1$ depending only on δ'' and r. Arguing again like in the proof of Lemma 3.5, we get that for every $\varepsilon_1 > 0$ we can choose $\delta' > 0$ and $r > 0$ so that

$$|\xi_n(\lambda) - \mu_n(\lambda) - (f_\lambda^n)'(0)x(\lambda)| < \varepsilon_1 |\xi_n(\lambda) - \mu_n(\lambda)| \tag{4.2}$$

for all $\lambda \in B(0, r)$.

If r_1 is minimal so that $|\xi_n(\lambda) - \mu_n(\lambda)| \geq \delta''$ for all $\lambda \in A(\lambda_0; r_1, r_2)$, then for some λ_1 with $|\lambda_1 - \lambda_0| = r_1$ we have

$$|\xi_n(\lambda_1) - \mu_n(\lambda_1)| = \delta''. \tag{4.3}$$

On the other hand, from the definition of n, we have for some λ_2 with $|\lambda_2 - \lambda_0| = r_2$ that $|\xi_{n+1}(\lambda_2) - \mu_{n+1}(\lambda_2)| \geq \delta'$. But

$$|\xi_{n+1}(\lambda_2) - \mu_{n+1}(\lambda_2)| = |f_{\lambda_2}(\xi_n(\lambda_2)) - f_{\lambda_2}(\mu_n(\lambda_2))| \leq M' |\xi_n(\lambda_2) - \mu_n(\lambda_2)|,$$

where $M' = \max\{|f_\lambda'(z)| : z \in \mathcal{N}, \lambda \in B(\lambda_0, r)\}$. Therefore we get that

$$|\xi_n(\lambda_2) - \mu_n(\lambda_2)| \geq \frac{\delta'}{M'}. \tag{4.4}$$

Moreover, by (4.2), for every $\lambda \in B(\lambda_0, r)$, if $r > 0$ and $\delta' > 0$ were small enough, then

$$\frac{1}{1+\varepsilon_1} |(f_\lambda^n)'(0)x(\lambda)| \leq |\xi_n(\lambda) - \mu_n(\lambda)| \leq \frac{1}{1-\varepsilon_1} |(f_\lambda^n)'(0)x(\lambda)|. \tag{4.5}$$

Using (4.3), (4.4), (4.5) and Lemma 3.4 we can estimate as follows

$$\frac{\delta'}{\delta''} \leq \frac{M'|\xi_n(\lambda_2) - \mu_n(\lambda_2)|}{|\xi_n(\lambda_1) - \mu_n(\lambda_1)|} \leq M' \frac{1+\varepsilon_1}{1-\varepsilon_1} \left| \frac{(f_{\lambda_2}^n)'(0)x(\lambda_2)}{(f_{\lambda_1}^n)'(0)x(\lambda_1)} \right| \leq M' \frac{(1+\varepsilon_1)^2}{1-\varepsilon_1} \left| \frac{x(\lambda_2)}{x(\lambda_1)} \right|.$$

Thus we can choose $\delta'' > 0$ so small that $\frac{r_1}{r_2} \leq \frac{1}{10}$ independently of n.

Now we want to see how many times $\xi_n(\lambda) - \mu_n(\lambda)$ orbits around 0, as the parameter λ moves along the circle γ_r, $r > r_1$. To see this let us look at the expression $\frac{\xi_n(\lambda) - \mu_n(\lambda)}{|\xi_n(\lambda) - \mu_n(\lambda)|}$. But by (4.2) we have that

$$\left| \frac{\xi_n(\lambda) - \mu_n(\lambda)}{|\xi_n(\lambda) - \mu_n(\lambda)|} - \frac{(f_\lambda^n)'(0)x(\lambda)}{|\xi_n(\lambda) - \mu_n(\lambda)|} \right| \leq \varepsilon_1,$$

so it is the same to ask how many times $(f_\lambda^n)'(0)x(\lambda)$ encircles 0. By Lemma 3.4, $(f_\lambda^n)'(0)$ is essentially constant on $B(\lambda_0, r_2)$, so the number we are looking for is K, the same as for $x(\lambda)$ only. Further, recall after (4.1) that $|\mu_n(\lambda) - \mu_n(\lambda_0)|$ is much smaller than $|\xi_n(\lambda) - \mu_n(\lambda)|$. This means that $\xi_n(\lambda)$ orbits around $\mu_n(\lambda_0) = \xi_n(\lambda_0)$ also K times close to some circle centered at $\mu_n(\lambda_0)$. By the Argument Principle, the degree of ξ_n is at most K.

In order to prove that the shape of the considered set is really close to round let us take λ_1, λ_2 with $|\lambda_1 - \lambda_0| = |\lambda_2 - \lambda_0| = r$. Then again by (4.5) and Lemma 3.4 we obtain the following estimates

$$\left| \frac{\xi_n(\lambda_1) - \mu_n(\lambda_0)}{\xi_n(\lambda_2) - \mu_n(\lambda_0)} \right| \leq \frac{1+\varepsilon}{1-\varepsilon} \left| \frac{\xi_n(\lambda_1) - \mu_n(\lambda_1)}{\xi_n(\lambda_2) - \mu_n(\lambda_2)} \right| \leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} \left| \frac{(f_{\lambda_1}^n)'(0)x(\lambda_1)}{(f_{\lambda_2}^n)'(0)x(\lambda_2)} \right| \leq$$
$$\leq \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2} \left| \frac{(f_{\lambda_2}^n)'(0)x(\lambda_1)}{(f_{\lambda_2}^n)'(0)x(\lambda_2)} \right| = \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2} \left| \frac{x(\lambda_1)}{x(\lambda_2)} \right|.$$

The last expression can be arbitrarily close to 1 independently of n for small r. This means that the set $\xi_n(\gamma_r)$ is close to a circle centered at $\xi_n(\lambda_0) = \mu_n(\lambda_0)$ and of radius $|\xi_n(\lambda) - \mu_n(\lambda_0)|$ for any $|\lambda - \lambda_0| = r$, so the annulus A is mapped onto a slightly distorted annulus whose shape can be controlled independently of n. This finishes the proof of the lemma. □

With the notation of the previous lemma, we obtain from its proof and Lemma 3.4 the following important corollary.

Corollary 4.2 *If n is maximal for which $|\xi_n(\lambda) - \mu_n(\lambda)| \leq \delta'$, $\lambda \in B(0, r_2)$, then for all λ with $|\lambda - \lambda_0| = r_2$ we have $|\xi_n(\lambda) - \mu_n(\lambda)| \geq \frac{\delta'}{2M'}$, if $\delta' > 0$ and $r > 0$ were chosen small enough.*

5 Measure estimates

By now we know how to control the behaviour of ξ_n in a small scale. In this section we will see how does it act in a large scale. This will help us to estimate the Lebesgue measure of those parameters λ for which $f_\lambda^n(0)$ either turns back to a neighbourhood of zero or escapes close to infinity.

Let $U_\delta = \overline{\mathbb{C}} \setminus B(0, \frac{1}{\delta}) \cup B(0,\delta)$, for some small $\delta > 0$, be a neighbourhood of infinity and the asymptotic value 0. We want to estimate the number of iterates of f_λ, $\lambda \in B(\lambda_0, r)$ for some $r > 0$, after which the image of a small disk intersecting the Julia set covers a big part of U_δ. Note however that, since f_λ is entire and U_δ is unbounded, it is impossible to cover whole U_δ in the finite number of steps. Consider then a bounded set instead:

$$\tilde{U}_\delta = U_\delta \cap B(0, 2/\delta) = B(0,\delta) \cup A(0; 1/\delta, 2/\delta). \tag{5.1}$$

Lemma 5.1 *Let D be a bounded set disjoint from U_δ containing a disk of radius $d > 0$ centered at the Julia set of some $f = f_\lambda$. Then we can choose an $\tilde{N}$, depending only on d and f, such that*

$$\inf\left\{m \in \mathbb{N} : f^m(D) \supset \overline{\tilde{U}_\delta}\right\} \leq \tilde{N}.$$

Proof. Cover $\overline{J(f) \setminus U_\delta}$ with a collection of open disks D_z of diameter d centered at $z \in \overline{J(f) \setminus U_\delta}$. Since the family $\{f^n\}$ is not normal on $J(f)$, for every D_z there is a minimal $n = n(z)$ such that

$$f^n(D_z) \supset \overline{\tilde{U}_\delta}.$$

But $n(z)$ is constant in some neighbourhood of z since f^n is continuous, moreover $\overline{J(f) \setminus U_\delta}$ is compact, therefore we can find an integer $\tilde{N}$ such that $n(z) \leq \tilde{N}$ for every z. □

Note that we can choose a radius $r > 0$ so that the statement holds for every f_λ, $\lambda \in B(\lambda_0, r)$ and say $2\tilde{N}$, which depends only on $d > 0$ for r small enough. It is possible since the dependence on λ is analytic hence continuous.

We know now that $f^m(D) \Supset \tilde{U}_\delta$ for some $m \leq \tilde{N}$. The next step is to estimate the measure of those points from D that get mapped into U_δ under f^j for some $j \leq m$. Recall that $f = f_\lambda$ is an exponential map and D is an open set disjoint from U_δ. In particular $D \subset B(0, \frac{1}{\delta})$. The following lemma is similar to an analogous one in the rational case (cf. [1, Lemma 4.2]), however to prove it we have to be much more careful. Recall that μ denotes the Lebesgue measure, U_δ is the δ-neighbourhood of 0 and ∞, while $\tilde{U}_\delta$ is given by (5.1).

Lemma 5.2 *Assume that D is an open set disjoint from U_δ and the integer m is such that $f^m(D) \Supset \tilde{U} := \tilde{U}_\delta$. Then there exists a constant $N > 0$, only depending on f, m and U_δ, such that*

$$\mu\left(\left\{z \in D\colon f^j(z) \in U_\delta \text{ for some } 1 \leq j \leq m\right\}\right) \geq N\mu(D).$$

Proof. Let us define

$$F = \{z \in D : f^j(z) \in U_\delta \text{ for some } 1 \leq j \leq m\}$$

and

$$\tilde{F} = \{z \in D : f^j(z) \in \tilde{U} \text{ for some } 1 \leq j \leq m\}.$$

Divide $\tilde{F}$ into m pairwise disjoint subsets, i.e. domains of the first entry map to $\tilde{U}$:

$$\begin{aligned} F_1 &= \{z \in D : f(z) \in \tilde{U}\} = f^{-1}(\tilde{U}) \cap D, \\ F_2 &= \{z \in D : f^2(z) \in \tilde{U} \text{ but } f(z) \notin \tilde{U}\} = f^{-2}(\tilde{U}) \cap f^{-1}(\overline{\mathbb{C}} \setminus \tilde{U}) \cap D, \\ F_3 &= \{z \in D : f^3(z) \in \tilde{U} \text{ but } f(z) \notin \tilde{U}, f^2(z) \notin \tilde{U}\}, \\ &\vdots \\ F_m &= \{z \in D : f^m(z) \in \tilde{U} \text{ but } f^j(z) \notin \tilde{U} \text{ for } j \leq m-1\} = \\ &= f^{-m}(\tilde{U}) \cap \bigcap_{j=1}^{m-1} f^{-j}(\overline{\mathbb{C}} \setminus \tilde{U}) \cap D. \end{aligned}$$

Then $\tilde{F} = F_1 \cup F_2 \cup \ldots \cup F_m$ and the sum is disjoint. But, since $D \subset B(0, \frac{1}{\delta})$ and the set $\bigcup_{j=0}^{m} f^j(B(0, \frac{1}{\delta}))$ is bounded, we can take some bounded set A disjoint from $\tilde{U}$ instead of $\overline{\mathbb{C}} \setminus \tilde{U}$ in the definition of sets $F_2, \ldots, F_m$, i.e.

$$\begin{aligned} F_2 &= f^{-2}(\tilde{U}) \cap f^{-1}(A) \cap D \\ &\vdots \\ F_m &= f^{-m}(\tilde{U}) \cap \bigcap_{j=1}^{m-1} f^{-j}(A) \cap D. \end{aligned}$$

Note also that

$$D \setminus \tilde{F} = \{z \in D : f(z) \notin \tilde{U}, \ldots, f^m(z) \notin \tilde{U}\} = \bigcap_{j=1}^{m} f^{-j}(\overline{\mathbb{C}} \setminus \tilde{U}) \cap D = \bigcap_{j=1}^{m} f^{-j}(A) \cap D.$$

To estimate the degree of f^m on $D \setminus \tilde{F}$ recall that f is $2\pi i$-periodic, A is a bounded set in $\mathbb{C} \setminus \tilde{U}$, hence $A \cap B(0, \delta) = \emptyset$. Note that the set $f^{-1}(A) \cap A$ intersects finitely many, say n_A, fundamental strips for f and is bounded. Next, $f^{-1}(f^{-1}(A) \cap A) \cap A$ intersects at most n_A fundamental strips and is bounded as well, and so on. Since on every strip f is injective, we conclude that the degree of f^m on $D \setminus \tilde{F}$ is at most $(n_A)^m$ and this number depends only on f, m and A (which in turn depends on f, m and δ).

Moreover, on every F_j the modulus of the derivative $|(f^j)'|$ is bounded from above by some constant $c_j = c_j(f, m, \delta)$. On the other hand on $D \setminus \tilde{F}$, $|(f^m)'|$ is bounded from below by a constant $a = a(f, m, \delta) > 0$. Since μ is the 2-dimensional Lebesgue measure, we get the following estimates.

$$\mu(\tilde{U}) \leq \sum_{j=1}^{m} \int_{F_j} |(f^j)'(z)|^2 d\mu(z) \leq \sum_{j=1}^{m} c_j^2 \mu(F_j) \leq \max_{j=1,\ldots,m} c_j^2 \sum_{j=1}^{m} \mu(F_j) =: C\mu(\tilde{F}), \tag{5.2}$$

Denote $g(w) = \{z \in D \setminus \tilde{F} : f^m(z) = w\}$ for $w \in \mathbb{C} \setminus \tilde{U}$. Actually it is enough to consider $w \in A$, since $f^m : D \setminus \tilde{F} \to A$. Then:

$$\begin{aligned} \mu(D \setminus \tilde{F}) = \int_{\mathbb{C} \setminus \tilde{U}} \sum_{z \in g(w)} |(f^m(z))|^{-2} d\mu(w) = \int_{A} \sum_{z \in g(w)} |(f^m(z))|^{-2} d\mu(w) \leq \\ \leq (n_A)^m a^{-2} \mu(A) =: \kappa\mu(A). \end{aligned} \tag{5.3}$$

Finally, for some constant $M_{\tilde{U},m} = M_{\tilde{U},m}(f, m, \tilde{U}) > 0$ we have that

$$\mu(\tilde{U}) \geq M_{\tilde{U},m} \mu(A). \tag{5.4}$$

Putting together (5.2), (5.3) and (5.4) we obtain the following

$$\mu(\tilde{F}) \geq \frac{1}{C} \mu(\tilde{U}) \geq \frac{M_{\tilde{U},m}}{C} \mu(A) \geq \frac{M_{\tilde{U},m}}{C\kappa} \mu(D \setminus \tilde{F}),$$

which implies that

$$\mu(F) \geq \mu(\tilde{F}) \geq N\mu(D)$$

for some constant $N = N(f, m, \delta)$. □

6 Conclusion

Consider f_λ, $\lambda \in B(\lambda_0, \varepsilon)$ for some small $\varepsilon > 0$. For $r \leq \varepsilon$ denote

$$A(z) = \{\lambda \in B(\lambda_0, r) : \xi_n(\lambda) = z\}, \text{ for } z \in D = \xi_n(B(\lambda_0, r)).$$

Proposition 6.1 *There exist $\delta' > 0$ and $0 < \tilde{r} < \varepsilon$, only depending on f_{λ_0}, such that for any $0 < r < \tilde{r}$, if we take the maximal integer n for which $\mathrm{diam}(\xi_n(B(\lambda_0, r))) \le \delta'$, then $\xi_n(B(\lambda_0, r))$ contains a ball centered at $\mu_n(\lambda_0) \in J(f_{\lambda_0})$ of diameter $\frac{\delta'}{2M'}$, where $M' = \max\{|f'_\lambda| : z \in \mathcal{N}, \lambda \in B(\lambda_0, \tilde{r})\}$. The degree of ξ_n on $B(\lambda_0, r)$ is bounded by K only depending on the family f_λ, $\lambda \in B(\lambda_0, \tilde{r})$.*

Moreover, if $\delta > 0$, $U_\delta = B(0, \delta) \cup (\overline{\mathbb{C}} \setminus B(0, \frac{1}{\delta}))$ and $D = \xi_n(B(\lambda_0, r))$, there are constants $C > 0$ and $\tilde{N}$ such that

$$\mu\left(\left\{z \in D : \xi_{n+j}(a(z)) \in U_\delta \text{ for all } a(z) \in A(z) \text{ and some } 1 \le j \le \tilde{N}\right\}\right) \ge C\mu(D),$$

where C depends only on f_{λ_0} and U_δ.

Proof. First part follows from Lemma 4.1 and Corollary 4.2, if only $\tilde{r}$ is chosen small enough so that z and its holomorphic motion $h_\lambda(z)$ are sufficiently close for all $z \in \Lambda$ and $\lambda \in B(\lambda_0, \tilde{r})$.

To prove the second part we apply Lemma 5.1 for f_{λ_0} and $U_{\delta/2}$, and next Lemma 5.2. It follows that there exists an integer $\tilde{N}$ and a constant $C > 0$, depending only on f_{λ_0}, $\tilde{N}$ and $\delta > 0$, such that

$$\mu\left(\left\{z \in D : f^j_{\lambda_0}(z) \in \tilde{U}_{\delta/2} \text{ for some } 1 \le j \le \tilde{N}\right\}\right) \ge C\mu(D),$$

where, recalling our notation, $\tilde{U}_{\delta/2} = B(0, \delta/2) \cup A(0; 2/\delta, 4/\delta)$. Now, since we have only finitely many steps to consider, we can decrease $\tilde{r} > 0$ if necessarily to have that for any $\lambda \in B(\lambda_0, \tilde{r})$

$$f^j_{\lambda_0}(\xi_n(\lambda)) \in \tilde{U}_{\delta/2} \quad \Longrightarrow \quad \xi_{n+j}(\lambda) = f^j_\lambda(\xi_n(\lambda)) \in U_\delta$$

for any $j < \tilde{N}$. □

To conclude with the proof of Theorem 1.3, recall that f_{λ_0} was a Misiurewicz exponential map and consider $f_\lambda = \lambda e^z$, $\lambda \in B(\lambda_0, r)$ for some small $r > 0$. Take an arbitrarily small $\delta > 0$ (such that e.g. f_{λ_0} is 2δ-Misiurewicz). We want to show that the set of δ-Misiurewicz maps in $B(\lambda_0, r)$ has the Lebesgue density less than one at λ_0.

Let $\delta' > 0$ and $\tilde{r} > 0$ be chosen so that the statement of the Proposition 6.1 is satisfied and all our expansion and distortion properties hold. Consider a parameter ball $B = B(\lambda_0, r_2)$ for any $r_2 \le \tilde{r}$. Let N be the largest possible integer for which $\xi_N(B)$ has the diameter at most δ' and let $\tilde{N}$ be as in the Proposition 6.1.

Lemma 6.2 *It is possible to choose $\delta'' \in (0, \delta')$ so that for every $\lambda \in B(\lambda_0, r)$*

$$\xi_{N+j}(\lambda) \in U_{3\delta/4} \text{ for some } j \le \tilde{N} \quad \Longrightarrow \quad \lambda \in A(\lambda_0; r_1, r_2),$$

where $r_1 > 0$ is minimal for which $|\xi_n(\lambda) - \mu_n(\lambda)| \ge \delta''$ for all $\lambda \in A(\lambda_0; r_1, r_2)$.

Proof. Note that in the Proposition 6.1 we could choose $\delta'' > 0$ as small as desired, provided $\tilde{r} > 0$ was small enough. Thus, to have that for any $\lambda \in B(\lambda_0, \tilde{r})$ with $|\xi_N(\lambda) - \mu_N(\lambda)| \le \delta''$ and for $j \le \tilde{N}$

$$|\xi_{N+j}(\lambda) - \mu_{N+j}(\lambda)| \le b^j |\xi_N(\lambda) - \mu_N(\lambda)| \le \delta'$$

it is sufficient to choose δ'' so small that $b^{\tilde{N}} \le \frac{\delta'}{\delta''}$, where

$$b = \max\{|f'_\lambda(z)| : z \in \mathcal{N}, \lambda \in B(\lambda_0, \tilde{r})\}, \ b > 1.$$

Next, we know that $\mu_{N+j}(\lambda) \in \mathcal{N}$ and $\mathcal{N} \cap U_\delta = \emptyset$ (if $\tilde{r}$ small). Thus, if $\delta' < \delta/4$, then $\xi_{N+j}(\lambda) \notin U_{3\delta/4}$ for all λ with $|\xi_N(\lambda) - \mu_N(\lambda)| \le \delta''$. □

Recall that inside the annulus $A = A(\lambda_0; r_1, r_2)$ we have bounded distortion:

$$\frac{1}{C'}\left(\frac{r_1}{r_2}\right)^{K-1} \le \left|\frac{\xi'_N(\lambda_1)}{\xi'_N(\lambda_2)}\right| \le C'\left(\frac{r_2}{r_1}\right)^{K-1}.$$

Moreover, if $\tilde{r}$ was chosen small enough and we take λ_i with $|\lambda_i - \lambda_0| = r_i$, $i = 1, 2$, since $\mathrm{diam}(\xi_N(B)) \leq \delta'$,

$$|\xi_N(\lambda_1) - \mu_N(\lambda_1)| \geq \delta'' \quad \text{and} \quad |\xi_N(\lambda_2) - \mu_N(\lambda_2)| \leq \frac{1}{1-\varepsilon}\delta',$$

consequently, applying Lemma 3.4 and (4.2), we get similarly like in the proof of Lemma 4.1,

$$\begin{aligned}\frac{\delta''}{\delta'} \leq & \frac{1}{1-\varepsilon}\left|\frac{\xi_N(\lambda_1) - \mu_N(\lambda_1)}{\xi_N(\lambda_2) - \mu_N(\lambda_2)}\right| \leq \frac{1+\varepsilon}{(1-\varepsilon)^2}\left|\frac{(f_{\lambda_1})'(0)x(\lambda_1)}{(f_{\lambda_2})'(0)x(\lambda_2)}\right| \leq \\ \leq & \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\left|\frac{(f_{\lambda_2})'(0)x(\lambda_1)}{(f_{\lambda_2})'(0)x(\lambda_2)}\right| = \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\left|\frac{x(\lambda_1)}{x(\lambda_2)}\right| \leq \frac{(1+\varepsilon)^3}{(1-\varepsilon)^3}\left(\frac{r_1}{r_2}\right)^K,\end{aligned}$$

and therefore

$$\left(\frac{r_1}{r_2}\right)^K \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^3 \frac{\delta''}{\delta'}.$$

As a consequence we obtain uniform bounds on the distortion of ξ_N on the annulus A:

$$\tilde{C}^{-1} \leq \left|\frac{\xi_N'(\lambda_1)}{\xi_N'(\lambda_2)}\right| \leq \tilde{C} \tag{6.1}$$

for all $\lambda_1, \lambda_2 \in A$, where $\tilde{C}$ depends only on δ'' and δ', if $\tilde{r}$ was chosen small enough. On the other hand, we have also had $\frac{r_1}{r_2} \leq \frac{1}{10}$, thus $\mu(A) \geq \frac{99}{100}\mu(B)$.

Recall our notation: $A(z) = \{\lambda \in B : \xi_N(\lambda) = z\}$ for $z \in \xi_N(B) =: D$. Let

$$E = \left\{z \in D : \xi_{N+j}(a(z)) \in U_{3\delta/4} \text{ for all } a(z) \in A(z) \text{ and some } j \leq \tilde{N}\right\}.$$

By Proposition 6.1 there is some $C > 0$, independent of D, such that $\mu(E) \geq C\mu(D)$. We want to estimate the measure of the following set $G = \{\lambda \in B : \xi_N(\lambda) \in E\}$. But, by Lemma 6.2, we have that

$$G = H := \{\lambda \in A : \xi_N(\lambda) \in E\}.$$

Take any point $z_0 \in A$. By (6.1) we get

$$\mu(E) \leq \int_H |\xi_N'(z)|^2 d\mu(z) \leq \tilde{C}^2 |\xi_N'(z_0)|^2 \mu(H).$$

On the other hand, since the degree of ξ_N is bounded by K on A,

$$\mu(A) = \int_D \sum_{z \in \xi_N^{-1}(w) \cap A} |\xi_N'(z)|^{-2} d\mu(w) \leq \tilde{C}^2 K |\xi_N'(z_0)|^{-2} \mu(D).$$

Therefore

$$\begin{aligned}\mu(H) \geq & \tilde{C}^{-2}|\xi_N'(z_0)|^{-2}\mu(E) \geq \tilde{C}^{-2}|\xi_N'(z_0)|^{-2} C\mu(D) \geq \\ \geq & \frac{C\tilde{C}^{-4}}{K}\mu(A) \geq \frac{C\tilde{C}^{-4}}{K}\frac{99}{100}\mu(B).\end{aligned}$$

Thus for some $q \in (0, 1)$, $q = q(\delta', \delta'', U_\delta)$, we have that

$$\mu\left(\{\lambda \in B : \xi_N(\lambda) \in E\}\right) \geq q\mu(B).$$

But, by the definition of E, this implies that

$$\mu\left(\{\lambda \in B : \xi_n(\lambda) \in U_{3\delta/4} \text{ for some } n \geq N\}\right) \geq q\mu(B).$$

If the asymptotic value 0 falls under f_λ to a slightly smaller set $U_{3\delta/4} \subset U_\delta$, then f_λ cannot be δ-Misiurewicz, so

$$\mu\left(\{\lambda \in B(\lambda_0, r_2) : f_\lambda \text{ is not } \delta\text{-Misiurewicz}\}\right) \geq q\mu(B(\lambda_0, r_2)).$$

Since it holds for every $r_2 \leq \tilde{r}$, the Lebesgue density of δ-Misiurewicz maps at λ_0 is at most $1 - q < 1$. This finishes the proof of Theorem 1.3.

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