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Linearizability conditions for a cubic system

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Abstract

We obtain the necessary and sufficient conditions for linearizability of an eight-parameter family of two-dimensional system of differential equations in the form of linear canonical saddle perturbed by polynomials with four quadratic and four cubic terms.

Keywords: Isochronicity; Linearizability; Centers; Linear normal forms

1. Introduction

We consider a polynomial system of differential equations of the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q = P(x,y), \\ -\frac{\mathrm{d}y}{\mathrm{d}t} &= y - \sum_{(p,q)\in S} b_{qp} x^q y^{p+1} = -Q(x,y), \end{aligned} \tag{1}$$

where x, y, a_{pq}, b_{qp} are complex variables, $S = \{(p_m, q_m) | p_m + q_m \geqslant 1, m = 1, \ldots, l\}$ is a subset of $\{-1 \cup \mathbb{N}\} \times \mathbb{N}$, and $\mathbb{N}$ is the set of non-negative integers. The notation in (1) simply emphasizes that we take into account only nonzero coefficients of the polynomials. By (a, b) we will denote the ordered vector of the coefficients of system (1), $(a,b) = (a_{p_1q_1}, \ldots, a_{p_lq_l}, b_{q_lp_l}, \ldots, b_{q_1p_1})$. In the case when

$$x = \bar{y}, \quad a_{ij} = \bar{b}_{ji}, \quad \mathrm{i}\mathrm{d}t = \mathrm{d}\tau \tag{2}$$

(the bar stands for the complex conjugate numbers), the system (1) is equivalent to the equation

$$\mathrm{i}\frac{\mathrm{d}x}{\mathrm{d}\tau} = x - \sum_{(p,q)\in S} a_{pq} x^{p+1} \bar{x}^q, \tag{3}$$

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which has a center or a focus at the origin in the real plane $\{(u,v)|x = u + \mathrm{i}v\}$, where the system can be also written in the form

$$\dot{u} = -v + U(u,v), \quad \dot{v} = u + V(u,v). \tag{4}$$

For systems of the form (4) (where the power series expansions of U and V at the origin start with quadratic terms) the notions of center and isochronicity have a simple geometric meaning. Namely, the origin of system (4) is a *center* if all trajectories in its neighborhood are closed and it is an *isochronous* center if the period of oscillations is the same for all these trajectories. According to the Poincaré–Lyapunov Theorem system (4) has a center at the origin if and only if it admits a first integral of the form

$$\Psi(u,v) = u^2 + v^2 + \sum_{k+m=3}^{\infty} \theta_{km} u^k v^m.$$

Since after the complexification $x = u + \mathrm{i}v$, $y = \bar{x}$ the above integral has the form

$$\Psi(x,y) = xy + \sum_{k+m=3}^{\infty} \eta_{km} x^k y^m, \tag{5}$$

following to Dulac we say that system (1) has a center at the origin if it admits a first integral of the form (5).

We recall (see, for instance, [1,6] for the details) that the real analytical system (4) has an isochronous center if and only if it can be transformed to the linear system

$$\dot{u} = v, \quad \dot{v} = -u,$$

that is, if the normal form of (4) is linear. Thus for system (4) the notion of isochronicity is equivalent to the notion of linearizability. However the problem of linearizability arises also for more general system (1), which in the special case when the conditions (2) hold is equivalent to (3) and, therefore, to (4). In this paper we will study the problem of linearizability for system (1). Namely, we will consider the problem how to decide if a polynomial system (1) can be transformed to the linear system

$$\dot{z}_1 = z_1, \quad \dot{z}_2 = -z_2 \tag{6}$$

by means of a formal change of the phase variables

$$z_1 = x + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(a,b) x^m y^j, \quad z_2 = y + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(a,b) x^m y^j. \tag{7}$$

If for some values of the parameters a_{pq}, b_{qp} such transformation exists we say that the corresponding system (1) is *linearizable* (or has a *linearizable center* at the origin). It follows from a result of Poincaré and Lyapunov that if there is a formal transformation (7) linearizing (1) then it converges in a neighborhood of the origin.

Although the study of isochronicity goes back at least to Huygens who investigated the oscillations of cycloidal pendulum, at present the problem is of renewed interest. In particular, in recent years many studies has been devoted to the investigation of the linearizability (isochronicity) problem for different subfamilies of the cubic system (that is, the system (1) where the right hand sides are the polynomials of degree three, see, e.g. [2–4,6,8,7,10] and references therein). In this paper we study the problem of linearizability for the following eight-parameter cubic system:

$$\begin{aligned} \dot{x} &= x(1 - a_{10}x - a_{01}y - a_{20}x^2 - a_{02}y^2), \\ \dot{y} &= -y(1 - b_{10}x - b_{01}y - b_{02}y^2 - b_{20}x^2). \end{aligned} \tag{8}$$

2. Preliminaries

Taking derivatives with respect to t in both parts of each of the equalities in (7), we obtain

$$\dot{z}_1 = \dot{x} + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(mx^{m-1}y^j\dot{x} + jx^m y^{j-1}\dot{y}),$$

$$\dot{z}_2 = \dot{y} + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(mx^{m-1}y^j\dot{x} + jx^m y^{j-1}\dot{y}).$$

Equating coefficients of the terms $x^{q_1+1}y^{q_2}, x^{q_1}y^{q_2+1}$, correspondingly, yields the recurrence formulae

$$(q_1 - q_2)u^{(1)}_{q_1q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[(s_1+1)u^{(1)}_{s_1s_2}a_{q_1-s_1,q_2-s_2} - s_2u^{(1)}_{s_1s_2}b_{q_1-s_1,q_2-s_2}\right], \tag{9}$$

$$(q_1 - q_2)u^{(2)}_{q_1q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[s_1u^{(2)}_{s_1s_2}a_{q_1-s_1,q_2-s_2} - (s_2+1)u^{(2)}_{s_1s_2}b_{q_1-s_1,q_2-s_2}\right], \tag{10}$$

where $s_1, s_2 \geqslant -1, q_1, q_2 \geqslant -1, q_1 + q_2 \geqslant 0, u^{(1)}_{1,-1} = u^{(1)}_{-1,1} = 0, u^{(2)}_{1,-1} = u^{(2)}_{-1,1} = 0, u^{(1)}_{00} = u^{(2)}_{00} = 1$, and we set $a_{qm} = b_{mq} = 0$, if $(q,m) \notin S$.

Thus we see, that the coefficients $u^{(1)}_{q_1q_2}, u^{(2)}_{q_1q_2}$ of the transformation (7) can be computed step by step using the formulae (9) and (10). In the case $q_1 = q_2 = q$ the coefficients $u^{(1)}_{qq}, u^{(2)}_{qq}$ can be chosen arbitrary (we set $u^{(1)}_{qq} = u^{(2)}_{qq} = 0$). The system has a linearizable center only if the quantities in the right-hand side of (9) and (10) are equal to zero for all $q \in \mathbb{N}$. As a matter of definition, in the case $q_1 = q_2 = q$ we denote the polynomials in the right-hand side of (9) by i_{qq} and in the right-hand side of (10) by $-j_{qq}$ and call them qth linearizability quantities. We see that the system (1) with the given coefficients (a^*, b^*) is linearizable if and only if $i_{kk}(a^*, b^*) = j_{kk}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$.

By definition a Darboux linearization [4,6] of system (1) is a change of variables

$$z_1 = H_1(x,y), \quad z_2 = H_2(x,y) \tag{11}$$

which transforms the system to the linear system (6), and such that at least one of the functions H_1, H_2 is of the form $H = f_1^{\alpha_1} \ldots f_k^{\alpha_k}$, with α_js being complex numbers, where $f_i(x,y)$s are either invariant algebraic curves of system (1) defined by $f_i(x,y) = 0$, that is, polynomials satisfying the equation

$$\frac{\partial f_i}{\partial x}P + \frac{\partial f_i}{\partial y}Q = K_i f_i, \tag{12}$$

or Darboux functions, that is, some functions satisfying (12) with K_is being some polynomials (thus, an algebraic invariant curve is a particular case of Darboux function). The polynomial $K_i(x,y)$ is called the cofactor of the invariant curve $f_i(x,y)$ or the Darboux function $f_i(x,y)$. A simple computation shows that if there are Darboux functions $f_1, f_2, \ldots, f_k$ with the cofactors $K_1, K_2, \ldots, K_k$ satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0, \tag{13}$$

then $H = f_1^{\alpha_1} \ldots f_k^{\alpha_k}$, is a first integral of (1), and if

$$\sum_{i=1}^{k} \alpha_i K_i + P'_x + Q'_y = 0 \tag{14}$$

then the system admits the integrating factor $\mu = f_1^{\alpha_1} \ldots f_k^{\alpha_k}$.

Similar idea leads to the so-called Darboux linearization. Namely, if there are $\alpha_1, \ldots, \alpha_k$ such that

$$P(x,y)/x + \sum_{i=1}^{k} \alpha_i K_i = 1, \tag{15}$$

then by means of the substitution $z_1 = xf_1^{\alpha_1} \ldots f_k^{\alpha_k}$ the first equation of (1) is transformed to the linear equation $\dot{z}_1 = z_1$, and if

$$Q(x,y)/y + \sum_{i=1}^{k} \alpha_i K_i = -1, \tag{16}$$

then the second equation of the system is linearized by the change $z_2 = yf_1^{\alpha_1} \ldots f_k^{\alpha_k}$.

If system (1) is such that only one of the conditions (15), (16) is satisfied, let say (16), but it has a Lyapunov first integral $\Psi(x,y)$ of the form (5) then (1) is linearizable by the change

$$z_1 = \Psi(x,y)/H_2(x,y), \quad z_2 = H_2(x,y), \tag{17}$$

and, correspondingly, if (15) holds, then the linearizing transformation is given by

$$z_1 = H_1(x,y), \quad z_2 = \Psi(x,y)/H_1(x,y). \tag{18}$$

It is shown in [9,8] that the Lyapunov first integral (5) can play the role of a Darboux function, that is, can be used in order to construct a linearization of the form

$$z = x^a y^b \Psi^c f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_k^{\alpha_k}.$$

This observation leads to the following theorem.

Theorem 1. *Assume that system (1) has a first integral of the form*

$$\Psi(x,y) = xy\left(1 + \sum_{k+j=1}^{\infty} v_{k,j} x^k y^j\right) \tag{19}$$

and the Darboux functions $f_1, f_2, \ldots, f_s$ of the form $1 + \text{h.o.t.}$ *with the cofactors $K_1, \ldots, K_s$. In such case, if*

$$(1-c)\frac{P}{x} - c\frac{Q}{y} + \sum_{j=1}^{s} \alpha_j K_j = 1 \tag{20}$$

then the first equation of (1) is linearized by the substitution

$$z_1 = x^{1-c} y^{-c} \Psi^c f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_s^{\alpha_s}, \tag{21}$$

and if

$$-c\frac{P}{x} + (1-c)\frac{Q}{y} + \sum_{j=1}^{s} \alpha_j K_j = -1 \tag{22}$$

then the second equation of the system is linearizable by the substitution

$$z_2 = x^{-c} y^{1-c} \Psi^c f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_s^{\alpha_s}. \tag{23}$$

Proof. Since the first integral has the form (19) the substitutions (21) and (23) are of the form $z_1 = x + \text{h.o.t.}$ and $z_2 = y + \text{h.o.t.}$, respectively. Direct calculations show that (21) and (23) are indeed the linearizations. □

3. The linearizability conditions

In this section we will obtain the necessary and sufficient conditions for linearizability of system (8).

Theorem 2. *System (8) has an isochronous center at the origin if and only if one of the following conditions holds*:

(1) $a_{02} + b_{02} = a_{20} + b_{20} = a_{01} + b_{01} = a_{10} + b_{10} = b_{02}b_{10}^2 + b_{20}b_{01}^2 = 0$,
(2) $b_{10} = b_{20} = a_{02} = a_{01} = 0$,
(3) $b_{10} = b_{20} = a_{20} = a_{01} = 0$,
(4) $b_{10} = b_{20} = a_{20} = a_{10} = 0$,
(5) $b_{10} = b_{20} = a_{10} = 2a_{01}^2 - a_{01}b_{01} + a_{02} = 0$,
(6) $b_{01} = b_{20} = a_{02} + 2b_{02} = a_{20} = a_{01} = a_{10} + 2b_{10} = 0$,
(7) $b_{01} = b_{20} = a_{02} - 2b_{02} = a_{01} = a_{10} - 6b_{10} = 8b_{10}^2 + a_{20} = 0$,
(8) $b_{10} = a_{02} = 2a_{20} - b_{20} = 6a_{01} - b_{01} = a_{10} = 2b_{01}^2 + 9b_{02} = 0$,
(9) $b_{10} = b_{02} = a_{02} = 2a_{20} + b_{20} = 2a_{01} + b_{01} = a_{10} = 0$,
(10) $b_{10} = a_{02} + b_{02} = a_{20} + b_{20} = a_{01} = 0$,
(11) $b_{10} = b_{02} = a_{02} = a_{01} = 0$,
(12) $b_{01} = a_{02} = a_{01} = a_{10}b_{10} - 2b_{10}^2 - b_{20} = 0$,
(13) $b_{01} = b_{02} = a_{02} = a_{01} = 0$.

Proof. To solve the linearizability problem for (8) we have computed eight first pairs of the linearizability quantities i_{kk}, j_{kk}. The polynomials are too long, so we do not present them here, however one can easily compute them using Mathematica (for instance, with a modification of the code from Appendix of [8]) or any other computer algebra system and formulae (9) and (10). Then we find the primary decomposition of the ideal $\langle i_{11}, j_{11}, \ldots, i_{88}, j_{88}\rangle$ using the routine *minAssChar* of the computer algebra program Singular [5] and obtain the necessary conditions of linearizability (the 13 conditions presented in the statement of the theorem). To prove that each of the obtained conditions is also the sufficient condition for the linearizability we look for Darboux linearizations.

Case 1. In this case system (8) has the form

$$\begin{aligned}\dot{x} &= \frac{1}{b_{10}^2}x\left(b_{10}^2 + b_{10}^3x + b_{10}^2b_{20}x^2 + b_{01}b_{10}^2y - b_{01}^2b_{20}y^2\right) = P(x,y),\\ \dot{y} &= -\frac{1}{b_{10}^2}y\left(b_{10}^2 - b_{10}^3x - b_{10}^2b_{20}x^2 - b_{01}b_{10}^2y + b_{01}^2b_{20}y^2\right) = Q(x,y).\end{aligned}\tag{24}$$

If $b_{10} \neq 0$ then among the trajectories of the system (24) there two invariant lines:

$$\begin{aligned}f_1 &= 1 + \frac{1}{2}\left(b_{10} - \sqrt{b_{10}^2 - 4b_{20}}\right)x + \frac{1}{2}\left(\frac{b_{01}}{b_{10}}\sqrt{b_{10}^2 - 4b_{20}} - b_{01}\right)y,\\ f_2 &= 1 + \frac{1}{2}\left(b_{10} + \sqrt{b_{10}^2 - 4b_{20}}\right)x - \frac{1}{2}\left(\frac{b_{01}}{b_{10}}\sqrt{b_{10}^2 - 4b_{20}} + b_{01}\right)y,\end{aligned}$$

with the corresponding cofactors

$$\begin{aligned}K_1 &= \frac{1}{2}\left(b_{10} - \sqrt{b_{10}^2 - 4b_{20}}\right)x + b_{20}x^2 + \frac{1}{2}\left(b_{01} - \frac{b_{01}}{b_{10}}\sqrt{b_{10}^2 - 4b_{20}}\right)y - \frac{b_{01}^2b_{20}y^2}{b_{10}^2},\\ K_2 &= \frac{1}{2}\left(b_{10} + \sqrt{b_{10}^2 - 4b_{20}}\right)x + b_{20}x^2 + \frac{1}{2}\left(\frac{b_{01}}{b_{10}}\sqrt{b_{10}^2 - 4b_{20}} + b_{01}\right)y - \frac{b_{01}^2b_{20}y^2}{b_{10}^2}.\end{aligned}$$

Using these invariant lines we can construct a Darboux linearization of system (24). Indeed, in this case the Eq. (15) is $P(x,y)/x + \alpha_1K_1 + \alpha_2K_2 = 1$ and the Eq. (16) is $Q(x,y)/y + \alpha_1K_1 + \alpha_2K_2 = -1$. Both equations are satisfied by

$$\alpha_1 = -\frac{1}{2} + \frac{b_{10}}{2\sqrt{b_{10}^2 - 4b_{20}}}, \quad \alpha_2 = -\frac{1}{2} - \frac{b_{10}}{2\sqrt{b_{10}^2 - 4b_{20}}}.$$

Therefore system (24) is linearized by the substitutions:

$$z_1 = xf_1^{\alpha_1}f_2^{\alpha_2}, \quad z_2 = yf_1^{\alpha_1}f_2^{\alpha_2}.\tag{25}$$

If $b_{10} = 0$ then either $b_{01} = 0$ or $b_{20} = 0$. The first case is considered in [4] and the second one is a particular case of system (26).

Case 3. The conditions of this case yield the system

$$\dot{x} = x(1 - a_{10}x - a_{02}y^2) = P(x,y), \quad \dot{y} = -y(1 - b_{01}y - b_{02}y^2) = Q(x,y)\tag{26}$$

with the invariant lines

$$f_1 = x, \quad f_2 = y, \quad f_3 = 1 - \frac{1}{2}\left(b_{01} - \sqrt{b_{01}^2 + 4b_{02}}\right)y, \quad f_4 = 1 - \frac{1}{2}\left(b_{01} + \sqrt{b_{01}^2 + 4b_{02}}\right)y,$$

and the corresponding cofactors

$$\begin{aligned}K_1 &= \frac{P(x,y)}{x}, \quad K_2 = \frac{Q(x,y)}{y}, \quad K_3 = \frac{1}{2}\left(b_{01} - \sqrt{b_{01}^2 + 4b_{02}}\right)y + b_{02}y^2,\\ K_4 &= \frac{1}{2}\left(b_{01} + \sqrt{b_{01}^2 + 4b_{02}}\right)y + b_{02}y^2.\end{aligned}$$

Using (16) we see that the second equation is linearizable by the substitution

$$z_2 = 2y\left(2 - b_{01}y - \sqrt{b_{01}^2 + 4b_{02}}\right)^{-\frac{1}{2}-\frac{b_{01}}{2\sqrt{b_{01}^2+4b_{02}}}}\left(2 - b_{01}y + \sqrt{b_{01}^2 + 4b_{02}y}\right)^{-\frac{1}{2}+\frac{b_{01}}{2\sqrt{b_{01}^2+4b_{02}}}}. \tag{27}$$

In order to find a linearizing transformation for the first equation we solve the Eq. (14) and find the solution

$$\alpha_1 = -2, \quad \alpha_2 = -2.$$

$$\alpha_3 = -\frac{(a_{02}+b_{02})\left(b_{01} + \sqrt{b_{01}^2+4b_{02}}\right)}{2b_{02}\sqrt{b_{01}^2+4b_{02}}}, \quad \alpha_4 = \frac{(a_{02}+b_{02})\left(b_{01} - \sqrt{b_{01}^2+4b_{02}}\right)}{2b_{02}\sqrt{b_{01}^2+4b_{02}}}.$$

Therefore $\mu = (xy)^{-2} f_3^{\alpha_3} f_4^{\alpha_4}$ is the Darboux integrating factor of (26). Then a first integral $\Psi(x,y)$ of (26) should satisfy the equations

$$\Psi'_x = \mu Q, \quad \Psi'_y = -\mu P. \tag{28}$$

Integration of the first equation of (28) gives $\Psi(x,y) = -xy\mu K_2 + \psi(y)$. Using the second equation we obtain $\psi'_y = a_{10}\mu x^2$. The integration yields the first integral

$$\Psi(x,y) = -xy\mu K_2 + \int a_{10}\mu x^2\,\mathrm{d}y,$$

which can be expressed in terms of Appell's functions. Since

$$-xy\mu K_2 = \frac{1}{xy}(1 + \text{h.o.t}) \quad \text{and} \quad \int \mu x^2\,\mathrm{d}y = -\frac{1}{y} + \frac{a_{02}+b_{02}}{2}y + \text{h.o.t.},$$

the integral $\Psi_1 = 1/\Psi$ is a first integral of the form (19). Thus, the first equation is linearizable by the substitution

$$z_1 = \frac{\Psi_1}{z_2}.$$

The above expressions are defined if $b_{01}^2 + 4b_{02} \neq 0$ and $b_{02} \neq 0$. Consider now the case when $b_{02} = -\frac{1}{4}b_{01}^2$. We find immediately the algebraic invariant line $f_1 = 1 - \frac{b_{01}}{2}y$. It turns out in this case there is also a Darboux function which is not algebraic function. Namely, we look for a Darboux function in the form $f_2 = \mathrm{e}^{(h_2x+h_3y)/f_1}$. Then K_2 should satisfy the equation $D(f_2)/f_2 - K_2 = 0$. This yields

$$f_2 = \mathrm{e}^{\frac{2b01y}{2-b_{01}y}}, \quad K_2 = -b_{01}y.$$

Using (16) we see that the second equation of the system is linearizable by the substitution

$$z_2 = \frac{2y\mathrm{e}^{\frac{b_{01}y}{2-b_{01}y}}}{(2-b_{01}y)}.$$

Again, from (14) we obtain that $\mu = \frac{f_1^{\alpha_1}f_2^{\alpha_2}}{x^2y^2}$ (where $\alpha_1 = \frac{4a_{02}}{b_{01}^2} - 1$, $\alpha_2 = \frac{2a_{02}}{b_{01}^2} - \frac{1}{2}$) is an integrating factor and

$$\Psi(x,y) = \frac{f_1^{\alpha_1+2}f_2^{\alpha_2}}{xy} + a_{10}\int \frac{f_1^{\alpha_1}f_2^{\alpha_2}}{y^2}\,\mathrm{d}y = \frac{f_1^{\alpha_1+2}f_2^{\alpha_2}}{xy} - a_{10}\left(\frac{1}{y} + \text{h.o.t.}\right)$$

is the first integral of the form (5) yielding the linearization of the first equation

$$z_1 = \frac{1}{z_2\Psi}.$$

The integral is defined when $b_{01} \neq 0$, however if $b_{01} = 0$ then we can construct a first integral using the Darboux function $\mathrm{e}^{a_{02}y^2/2}$.

Consider now the case when $b_{02} = 0$. Obviously, the second equation is linearizable by the substitution

$$z_2 = \frac{y}{1 - b_{01}y}.$$

To find a linearizing transformation for the first equation of (26) we divide the first equation of the system by the second one and obtain the Bernoulli equation,

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{x(1-a_{02}y^2)}{y(1-b_{01}y)} + \frac{a_{10}x^2}{y(1-b_{01}y)}.$$

Solving this equation we have a first integral in the form

$$\Psi = (-1)^{\frac{a_{02}}{b_{01}^2}}\left(-\frac{e^{-\frac{a_{02}y}{b_{01}}}(-1+b_{01}y)^{1-\frac{a_{02}}{b_{01}^2}}}{xy} + a_{10}\int \frac{e^{-\frac{a_{02}y}{b_{01}}}(-1+b_{01}y)^{1-\frac{a_{02}}{b_{01}^2}}}{y^2}\mathrm{d}y\right).$$

Therefore the linearization of the first equation of the system is given by $z_1 = 1/(z_2\Psi)$.

Case 4. In this case the system has the form

$$\dot{x} = x(1-a_{01}y-a_{02}y^2), \quad \dot{y} = -y(1-b_{01}y-b_{02}y^2). \tag{29}$$

There are two invariant lines

$$f_1 = 1-\frac{1}{2}\left(b_{01}-\sqrt{b_{01}^2+4b_{02}}\right)y, \quad f_2 = 1-\frac{1}{2}\left(b_{01}+\sqrt{b_{01}^2+4b_{02}}\right)y,$$

with the corresponding cofactors

$$K_1 = \frac{1}{2}\left(b_{01}-\sqrt{b_{01}^2+4b_{02}}\right)y + b_{02}y^2, \quad K_2 = \frac{1}{2}\left(b_{01}+\sqrt{b_{01}^2+4b_{02}}\right)y + b_{02}y^2.$$

The first equation of (26) is linearizable by the substitution

$$z_1 = xf_1^{\alpha_1}f_2^{\alpha_2}$$

where

$$\alpha_1 = \frac{a_{02}b_{01}-2a_{01}b_{02}+a_{02}\sqrt{b_{01}^2+4b_{02}}}{2b_{02}\sqrt{b_{01}^2+4b_{02}}}, \quad \alpha_2 = \frac{-a_{02}b_{01}+2a_{01}b_{02}+a_{02}\sqrt{b_{01}^2+4b_{02}}}{2b_{02}\sqrt{b_{01}^2+4b_{02}}},$$

and the linearizing transformation for the second equation is

$$z_2 = yf_1^{\beta_1}f_2^{\beta_2},$$

where

$$\beta_1 = -\frac{1}{2}+\frac{b_{01}}{2\sqrt{b_{01}^2+4b_{02}}}, \quad \beta_2 = -\frac{1}{2}-\frac{b_{01}}{2\sqrt{b_{01}^2+4b_{02}}}.$$

Case 5. When the conditions of this case are fulfilled the system takes the form

$$\begin{aligned}\dot{x} &= x(1-a_{01}y-a_{20}x^2+2a_{01}^2y^2-a_{01}b_{01}y^2) = P(x,y),\\ \dot{y} &= -y(1-b_{01}y-b_{02}y^2) = Q(x,y).\end{aligned} \tag{30}$$

The invariant lines of (30) are

$$\begin{aligned}&f_1 = x, \quad f_2 = y,\\ &f_3 = 1-\frac{1}{2}\left(b_{01}-\sqrt{b_{01}^2+4b_{02}}\right)y, \quad f_4 = 1-\frac{1}{2}\left(b_{01}+\sqrt{b_{01}^2+4b_{02}}\right)y,\end{aligned}$$

with the corresponding cofactors

$$\begin{aligned}&K_1 = \frac{P(x,y)}{x}, \quad K_2 = \frac{Q(x,y)}{y},\\ &K_3 = \frac{1}{2}\left(b_{01}-\sqrt{b_{01}^2+4b_{02}}\right)y + b_{02}y^2, \quad K_4 = \frac{1}{2}\left(b_{01}+\sqrt{b_{01}^2+4b_{02}}\right)y + b_{02}y^2.\end{aligned}$$

Using (16) we find that the second equation of (30) is linearizable by the substitution (27). However, the Eq. (15) has no solution, therefore there is no Darboux linearization of the first equation involving the lines f_1, f_2, f_3, f_4. However the system has the Darboux integrating factor

$$\mu = x^{\alpha_1} y^{\alpha_2} f_3^{\alpha_3} f_4^{\alpha_3},$$

where

$$\alpha_1 = -3, \quad \alpha_2 = -3,$$

$$\alpha_3 = \frac{2a_{01}^2 b_{01} - a_{01} b_{01}^2 + 2a_{01} b_{02} - b_{01} b_{02} + 2a_{01}^2 \sqrt{b_{01}^2 + 4b_{02}} - a_{01} b_{01} \sqrt{b_{01}^2 + 4b_{02}}}{b_{02} \sqrt{b_{01}^2 + 4b_{02}}},$$

$$\alpha_4 = -\frac{2a_{01}^2 b_{01} - a_{01} b_{01}^2 + 2a_{01} b_{02} - b_{01} b_{02} - 2a_{01}^2 \sqrt{b_{01}^2 + 4b_{02}} + a_{01} b_{01} \sqrt{b_{01}^2 + 4b_{02}}}{b_{02} \sqrt{b_{01}^2 + 4b_{02}}}.$$

Using it we find the first integral

$$\Psi = -xy\mu K_2 + 2a_{20} \int \mu x^3 \, \mathrm{d}y.$$

This integral is not of the form (5), however $\Psi_1 = 1/\sqrt{\Psi}$ is also the first integral of (30) and has the form (5). Therefore the first equation is linearizable by the substitution $z_1 = \Psi_1/z_2$, where z_2 is defined by (27).

Case 6. In this case the system has the form

$$\dot{x} = x(1 + 2b_{10}x + 2b_{02}y^2), \quad \dot{y} = -y(1 - b_{10}x - b_{02}y^2).$$

It is easy to find that a Darboux linearization of the system is given by

$$z_1 = \frac{x}{1 + 2b_{10}x - b_{02}y^2}, \quad z_2 = \frac{y}{\sqrt{1 + 2b_{10}x - b_{02}y^2}}.$$

Case 7. The corresponding system

$$\dot{x} = x(1 - 6b_{10}x + 8b_{10}^2x^2 - 2b_{02}y^2), \quad \dot{y} = -y(1 - b_{10}x - b_{02}y^2) \tag{31}$$

has invariant lines

$$f_1 = 1 - 4b_{10}x + 8b_{02}b_{10}xy^2, \quad f_2 = 1 - 12b_{10}x + 48b_{10}^2x^2 - 64b_{10}^3x^3 + 24b_{02}b_{10}xy^2$$

with the cofactors $K_1 = -4b_{10}x + 8b_{10}^2x^2$, $K_2 = -12b_{10}x + 24b_{10}^2x^2$, respectively. In order to obtain a linearizing transformation using the Eq. (13) we find the first integral $\Phi(x,y) = f_1^{-3}f_2$. Then

$$\Psi(x,y) = \frac{1}{8\sqrt{3b_{02}b_{10}^2}} \sqrt{\frac{1 - 12b_{10}x + 48b_{10}^2x^2 - 64b_{10}^3x^3 + 24b_{02}b_{10}xy^2}{(1 + 4b_{10}x + 8b_{02}b_{10}xy^2)^3} - 1}$$

is a first integral of the form (19). Eqs. (20) and (21) have the solutions $c = 2, \alpha_1 = 1, \alpha_2 = 0$ and $c = -1$, $\alpha_1 = -1, \alpha_2 = 0$, respectively. Therefore, according to Theorem 1 system (31) a is linearizable by the substitutions

$$z_1 = \frac{f_1 \Psi^2}{xy^2}, \quad z_2 = \frac{xy^2}{f_1 \Psi}.$$

Case 8. In this case the system has the form

$$\dot{x} = x(1 - a_{01}y - a_{20}x^2), \quad \dot{y} = -y(1 - 6a_{01}y - 2a_{20}x^2 + 8a_{01}^2y^2). \tag{32}$$

Thus after the involution

$$x \leftrightarrow y, \quad a_{ij} \leftrightarrow b_{ji} \tag{33}$$

we obtain from (32) the system (31). It means that this case is dual to case 7 under the involution (33) in the sense that all invariant curves, integrals and linearizations of (31) are mapped to invariant curves, integrals and linearizations, respectively, of (32) by the involution (33).

Cases (2), (10) and (11) are considered in [8], where the corresponding linearized substitutions are obtained. Cases (9), (12), (13) are dual (with respect to the involution (33)) to Cases (6), (5) and (4), respectively.

Note that some of obtained linearization are not defined for certain values of parameters. For example, the linearization (25) is defined if $4b_{02} \neq b_{10}^2$. It is possible to find linearizations also for these ''degenerated'' cases as we did in case 3. However we can also conclude that linearizations exist without computing them explicitly observing that each system (1)–(13) in the statement of the theorem defines an irreducible variety. □

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