Exact Solutions of b-family Equation: Classical Lie Approach and Direct Method

K. Singh[1], R. K. Gupta[2*], Sachin Kumar[3]

[1] Department of Mathematics Jaypee University of Information Technology, Waknaghat, P.O. Dumehar Bani, Kandaghat, Distt. Solan, Pin-173215 (H.P.), INDIA.

[2,3] School of Mathematics and Computer Applications, Thapar University, Patiala-147004 (Punjab), INDIA.

Abstract: The b-family equation $u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}$ is introduced by D.D Holm and M.F Struley, which describes the balance between the convection and the stretching for small viscosity in the dynamics of $1D$ nonlinear waves in fluids. In this paper, we performed Lie classical method and Direct method for symmetries of the b-family equation. Using symmetries of the equation, similarity reductions and exact solutions are obtained.

Keywords: b-family equation; Lie classical method; Direct method; Symmetry reductions; Exact solutions

1 Introduction

Degasperis and Procesi [1] found, using the method of asymptotic integrability, that only three equations from the following six-parameter family

$$u_t + c_0u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \partial_x(c_1u^2 + c_2u_x^2 + c_3uu_{xx}), \tag{1}$$

where $c_0, c_1, ...\alpha, \gamma$ are real, were integrable up to third order: the KdV equation ($\alpha = c_2 = c_3 = 0$), the Camassa-Holm equation ($c_1 = \frac{-3c_3}{2\alpha^2}, c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = \frac{-2c_3}{\alpha^2}, c_2 = c_3$), which on proper scaling, shifting the dependent variable, and finally applying Galilean boost reads as [2, 3]

$$u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}. \tag{2}$$

KdV type of equations has been an important and well studied class of nonlinear evolution equations with numerous applications in physical sciences, engineering fields and arises in various physical contexts.

The Camassa-Holm equation was first introduced by Camassa and Holm as a shallow water equation [4]. Like KdV equation it admits solitary waves that are solitons. In addition to that, the Camassa -Holm equation models wave breaking which the KdV does not. It models the propagation of unidirectional shallow water waves on a flat bottom, and then represents the fluid velocity at time t in the horizontal direction x [4, 5]. The Camassa-Holm equation is a water wave equation at quadratic order in an asymptotic expansion for unidirectional shallow water waves described by the incompressible Euler equations, while the KdV equation appears at first order in this expansion [1, 5]. For further details on Camassa-Holm equation one may refer to a paper by Holm and Staley [6].

As mentioned above, the Degasperis-Procesi equation (2) was first introduced in [1] by an asymptotic integrability test within a family of third order dispersive equations. Then Degasperis et al. [3] proved the exact integrability of (2) by constructing a Lax pair. The n-peakon solutions of equation (2) are derived by Lundmark and Szmigielski [7] using inverse scattering approach. Mustafa [8] proved that smooth solutions to (2) have infinite speed of propagation, that is, they lose instantly the property of having compact support. Well-posedness (in terms of existence, uniqueness, and stability of solutions) of the Cauchy problem for the Degasperis-Procesi equation (2) was studied by Yin in a series of papers [9–12] and by Coclite and Karlsen [13].

Corresponding author. *E-mail address*: sachin1jan@yahoo.com, rajeshgupta@thapar.edu

In this paper, we investigate the symmetries of the following one parameter family of non-evolution equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{3}$$

where the parameter b is real. The equation is introduced by D. D. Holm and M. F. Struley [6], which describes the balance between the convection and the stretching for small viscosity in the dynamics of $1D$ nonlinear waves in fluids. In the particular case $b = 2$ equation (3) becomes the dispersionless version of the integrable Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \tag{4}$$

It may be noted that for $b = 3$, equation (3) takes the form of Degasperis-Procesi equation (2).

Exact solutions play a vital role in the study of nonlinear phenomena as these solutions provide much information on various aspects of the physical phenomena. Since equation (3) represents an important class of nonlinear partial differential equations including two physically relevant systems (Camassa-Holm and Degasperis-Procesi), its exact solutions are desirable. The present work, which is purely due to the intrinsic theoretical interest in the nonlinear system (3), is devoted to extract exact solutions of this system. We present some explicit exact solutions to equation (3) and from these solutions; one can easily derive the solutions of Camassa-Holm equation and Degasperis-Procesi equations as particular case.

The outline of this paper is as follows: In section 2, the classical Lie method is utilized to obtain optimal system of the subalgebras for the equation (3) and In section 3, we used Direct method to obtain similarity reductions of the equation (3). In section 4, we have given some explicit exact solutions of the equation. In the last section, we have drawn some conclusions.

2   Classical Lie Symmetries Analysis

Lie method [14, 15] of infinitesimal transformation groups which essentially reduces the number if independent variables in partial differential equation (PDE) and reduces the order of ordinary differential equation (ODE) has been widely used in equations of mathematical physics, some recent and important contributions are in [15–20]. The classical method of for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations and the associated determining equations are an over determined linear system. As mentioned in [14], We let the group of infinitesimal transformations be is defined as

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2) \end{aligned} \tag{5}$$

and impose the condition of invariance on (3). The invariance under (5) means that if u is solution of equation (3), then u^* is also a solution of it.

Herein, too, on invoking the invariance criterion as mentioned in [14], the following relation from the coefficients of the first order of ϵ is deduced:

$$-\eta^t + \eta^{txx} - (b+1)[\eta u_x + u\eta^x] + b[\eta^x u_{xx} + \eta^{xx}u_x] + u\eta^{xxx} + \eta u_{xxx} = 0 \tag{6}$$

where $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{txx} are extended (prolonged) infinitesimals acting on an enlarged space corresponding to $u_t, u_x, u_{xx}, u_{xxx}$ and u_{txx} respectively. The method for determining the symmetry group of (3) mainly consists of finding the infinitesimals ξ, τ and η, which are functions of x, t and u. The general solution of equation (6) provides the infinitesimal elements ξ, τ and η, for which the equation (3) possesses Lie symmetry. Using the expressions for $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{txx} in equation (6) and u_t must be replaced by equation (3). On substituting the coefficients of different differentials equal to zero lead to number of PDEs in ξ, τ and η, that need to be satisfied. The set of determining equations for the group infinitesimals ξ, τ and η, which is obtained from (6), after equating the coefficients of various derivative terms to zero, is as follows:

$$\xi_u = 0$$
$$\tau_x = \tau_u = 0$$
$$\eta_{uu} = 0$$
$$2\eta_{xu} - \xi_{xx} = 0$$
$$\eta_{xxu} - 2\xi_x = 0$$
$$u\tau_t - \xi_t + \eta - u\xi_x = 0$$
$$-\tau_t + \xi_x - \eta_u = 0$$
$$\eta_{tu} - 2\xi_{tx} + b\eta_x - 3u\eta_{xu} = 0$$
$$\eta_t - \eta_{txx} + (b+1)u\eta_x - u\eta_{xxx} = 0 u\eta_u - bu\eta_{xxu} + \xi_t - (b+1)\eta + b\eta_{xx} = 0. \tag{7}$$

The set of equations (7) helps us to obtain the infinitesimals ξ, τ and η, as follows:

$$\xi = a_1$$
$$\tau = -a_3 t + a_2 \tag{8}$$
$$\eta = a_3 u,$$

where a_1, a_2 and a_3 are arbitrary constants. The Lie algebra associated with equation (3) consists of the following three vector fields

$$V_1 = \frac{\partial}{\partial x}, \ V_2 = \frac{\partial}{\partial t} \text{ and } V_3 = u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t}.$$

The similarity variable and form can be obtained by solving the characteristic equations

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta}. \tag{9}$$

The general solution of these equations involves two constants; one becomes the new independent variable ξ and the other, say F, plays the role of new dependent variable. On substituting these solutions of (9) in equation (3), one gets the reduced ordinary differential equation (ODE).

As mentioned in Olver [14], the commutator Table-1 and the adjoint Table-2 for above Lie algebra can be easily constructed as follows:

Table 1: Commutator Table

	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	$-V_2$
V_3	0	V_2	0

Table 2: Adjoint Table

	V_1	V_2	V_3
V_1	V_1	V_2	V_3
V_2	V_1	V_2	$V_3 + \epsilon V_3$
V_3	V_1	$V_2 e^{-\epsilon}$	V_3

The optimal sub algebra comprises of two vector fields viz. (i)$V_1 + \mu V_3$ and (ii)V_3. Now we primary focus on the reductions associated with these vector fields and attempt to find some exact solutions.

Vector field $V_1 + \mu V_3$
For this vector field, on using the characteristic equations (9), the similarity variable and the form of the similarity solution are as follows:

$$\xi(t, x) = te^{\mu x}, \quad u(t, x) = \frac{1}{t}F(\xi).$$

On using these in equation (3), the reduced ODE is given by

$$-\xi F'(\xi) + F + \mu^2\xi^3 F'''(\xi) + 2\mu^2\xi^2 F'' - \mu(b+1)\xi FF' + b\mu^3\xi^3 F'F'' + b\mu^3\xi^2(F')^2$$

$$+\mu^3\xi^3 FF''' + 3\mu^3\xi^2 FF'' + \mu^3\xi FF' = 0, \tag{10}$$

where $'$ prime denotes the differentiation with respect to the variable ξ. On transforming the independent variable by the relation , $\xi = exp(\zeta)$, the ODE (10) becomes

$$-\dot{F} + F + \mu^2\ddot{F} - \mu^2\dddot{F} - \mu(b+1)F\dot{F} + b\mu^3\dot{F}\ddot{F} + \mu^3 F\dddot{F} = 0. \tag{11}$$

Vector field V_3

In this case, the form of the similarity variable and similarity solution is as follows:

$$\xi = x, \qquad u(t,x) = \frac{1}{t}F(\xi).$$

The reduced ODE in this case is as follows:

$$FF''' + bF''F'' - (b+1)FF' - F'' + F = 0. \tag{12}$$

3 Similarity Reductions by Direct Method

In this section, we use direct method introduced by Clarkson and Kruksal [21] to obtain similarity reductions of the equation (3). The novel features of it are entirely straightforward without group analysis.The direct method to find similarity reductions is a very simple method that does not use group theory. The main idea is to seek a reduction of a given PDE in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)), \tag{13}$$

where $\alpha(x,t), \beta(x,t)$ and $z(x,t)$ are to be determined.

Substituting (13) into (3) and collecting monomials of w and its derivatives yields

$$
\begin{aligned}
&-\beta^2 z_x^3 ww''' - (\beta z_t z_x^2 + \alpha\beta z_x^3)w''' - (3\beta^2 z_x z_{xx} + 3\beta_x\beta z_x^2 + b\beta\beta_x z_x^2)ww'' \\
&-b\beta^2 z_x^3 w'w'' + (-2\beta_x z_x z_t - 3\alpha\beta_x z_x^2 - \beta_t z_x^2 - 3\alpha\beta z_x z_{xx} - 2\beta z_x z_{xt} \\
&-\beta z_t z_{xx} - b\alpha_x\beta z_x^2)w'' + (-2b\beta_x^2 z_x - b\beta\beta_x z_{xx} - \beta^2 z_{xxx} - 3\beta\beta_x z_{xx} - 3\beta\beta_{xx}z_x \\
&+(b+1)\beta^2 z_x - b\beta\beta_{xx}z_x)ww' + (-2\beta_x z_{xt} - b\alpha_x\beta z_{xx} + b\alpha\beta z_x - 2\beta_{xt}z_x \\
&+\alpha\beta z_x - 3\alpha\beta_{xx}z_x - \beta_{xx}z_t + \beta z_t - \alpha\beta z_{xxx} - b\alpha_{xx}\beta z_x - \beta z_{xxt} - 3\alpha\beta_x z_{xx} \\
&-2b\alpha_x\beta_x z_x)w' + ((b+1)\beta\beta_x - \beta\beta_{xxx} - b\beta_x\beta_{xx})w^2 + (-b\beta^2 z_x z_{xx} - 2b\beta\beta_x z_x^2)w'^2 \\
&+(-b\alpha_x\beta_{xx} - b\alpha_{xx}\beta_x + \alpha\beta_x + \alpha_x\beta - \alpha\beta_{xxx} - \alpha_{xxx}\beta + b\alpha\beta_x + b\alpha_x\beta + \beta_t)w \\
&+(b+1)\alpha\alpha_x - \alpha\alpha_{xxx} + \alpha_t - \alpha_{xxt} - b\alpha_x\alpha_{xx} = 0.
\end{aligned}
\tag{14}
$$

Demand that result be a ODE, impose conditions upon w and z and their derivatives that enable one to solve for α, β and z. However, before doing this we make some remarks about this direct mehtod of seeking similarity reductions (using the simplified ansatz (13)).

*Remark*1. We substitute (13) into the partial differential equation and then require that the resulting equation is an ordinary differential equation for $w(z)$, so it is necessary that the ratios of different derivatives and powers of $w(z)$ be function of z only. This gives a set of conditions for $\alpha(x,t), \beta(x,t), z(x,t)$ in the form of an overdetermined system of equations, any solution of which yield a similarity reduction. (These conditions are both necessary and sufficient for (13) to reduce the partial differential equation for $u(x,t)$ to an ordinary differential equation for $w(z)$).

*Remark*2. We use the coefficient of $w'''w$ as normalizing coefficient and therefore require that the other coefficients be of the form $\beta^2 z_x^3\Gamma(z)$, where Γ is function to be determined.

*Remark*3. We reserve uppercase greek letters for undetermined functions of z so that after performing operations the result can be denoted by the same letter [e.g, the derivative of $\Gamma(z)$ will be called $\Gamma(z)$].

*Remark*4. There are three freedoms in the determination of α, β, z and w we can exploit, without loss of generality, that are valuable in keeping the method manageable: (i) if $\alpha(x,t)$ has the form $\alpha = \alpha_0(x,t) + \beta(x,t)\Omega(z)$, then we can choose $\Omega \equiv 0$ [by substituting $w(z) \rightarrow w(z) - \Omega(z)$] (ii) if $\beta(x,t)$ has the form $\beta = \beta_0(x,t)\Omega(z)$, then we can take

$\Omega \equiv 1$ [by substituting $w(z) \to w(z)/\Omega(z)$]; and (iii) if $z(x, t)$ is determined by the equation of the form $\Omega(z) = z_0(x, t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ [by substituting $z \to \Omega^{-1}(z)$].

We shall now proceed to find general symmetry reduction of b-family equation using this method. Use the coefficient of $w'''w$ as normalizing coefficient and using the freedoms in aforementioned remarks (1-4) as explained in [21–23], we find that

$$\alpha = 0, \ \beta = \frac{\sigma'(t)}{\theta'(x)} \text{ and } z = \theta(x) + \sigma(t).$$

Put these values in (13), on simplification we get

$$
\begin{aligned}
&-\sigma'^2\theta'(ww''' + w''' + bw'w'') - (b+6)\sigma'^2\frac{\theta''}{\theta'}ww'' + (\sigma'^2\frac{\theta''}{\theta'} - \frac{\sigma''}{\theta'})w'' \\
&+(-3(b+1)\sigma'^2\frac{\theta''^2}{\theta'^3} + (b+1)\frac{\sigma'^2}{\theta'} - (b+4)\sigma'^2\frac{\theta'''}{\theta'^2})ww' + (\sigma''\frac{\theta''}{\theta'} + \sigma'^2\frac{\theta'''}{\theta'^2} + \frac{\sigma'^2}{\theta'})w' + \\
&(-(b+1)\sigma'^2\frac{\theta''}{\theta'^3} + (2b+6)\sigma'^2\frac{\theta''^3}{\theta'^5} - (b+6)\sigma'^2\frac{\theta''\theta'''}{\theta'^4} + \sigma'^2\frac{\theta''''}{\theta'^3})w^2 + b\sigma'^2\frac{\theta''}{\theta'}w'^2 \\
&+\frac{\sigma''}{\theta'}w = 0.
\end{aligned}
\tag{15}
$$

We continue to make this an ordinary differential equation for $w(z)$. Then the remaining coefficients yield

$$-\sigma'^2\theta'\Gamma_1(z) = -\sigma'^2\frac{\theta''}{\theta'} \tag{16}$$

$$-\sigma'^2\theta'\Gamma_2(z) = -\sigma'^2\frac{\theta''}{\theta'} - \frac{\sigma''}{\theta'} \tag{17}$$

$$-\sigma'^2\theta'\Gamma_3(z) = -3(b+1)\sigma'^2\frac{\theta''^2}{\theta'^3} + (b+1)\frac{\sigma'^2}{\theta'} - (b+4)\sigma'^2\frac{\theta'''}{\theta'^2} \tag{18}$$

$$-\sigma'^2\theta'\Gamma_4(z) = \sigma''\frac{\theta''}{\theta'} - 2\sigma'^2\frac{\theta''^2}{\theta'^3} + \sigma'^2\frac{\theta'''}{\theta'^2} + \frac{\sigma'^2}{\theta'} \tag{19}$$

$$-\sigma'^2\theta'\Gamma_5(z) = -(b+1)\sigma'^2\frac{\theta''}{\theta'^3} + (6+2b)\sigma'^2\frac{\theta''^3}{\theta'^5} - (b+6)\sigma'^2\frac{\theta'\theta'''}{\theta'^4} + \sigma'^2\frac{\theta'\theta''''}{\theta'^3} \tag{20}$$

$$-\sigma'^2\theta'\Gamma_6(z) = \sigma'^2\frac{\theta''}{\theta'} \tag{21}$$

$$-\sigma'^2\theta'\Gamma_7(z) = \frac{\sigma''}{\theta'}, \tag{22}$$

where $\Gamma_1(z), \Gamma_2(z), \Gamma_3(z), \Gamma_4(z), \Gamma_5(z), \Gamma_6(z)$ and $\Gamma_7(z)$ has to be determined.

First consider (16), equation (15) will be ordinary differential equation if
$\theta'' = A\theta'$, where A is a arbitrary constant.

Now from equation (22), equation (15) will reduce to ODE only if

$$\theta' = 1. \tag{23}$$

So from equation (17),
$$\sigma'' = B\sigma' \tag{24}$$

where B is a arbitrary constant. Similarly other equations will be satisfied by taking
$\Gamma_2 = B, \Gamma_7 = -B$ and $\Gamma_3(z) = \Gamma_4(z) = C,$
where C is a arbitrary constant. From equation (24), we get

$$\sigma = -\frac{\ln(Bt + D)}{B} + E, \tag{25}$$

where D and E are arbitrary constants.

Using equation (23) and (25), we get

$$z = x - \frac{\ln(Bt+D)}{B} + C_1 \text{ and } \beta = -\frac{1}{Bt+D},$$

where C_1 is constant.

So b-family equation possesses similarity reduction of the form

$$u(x,t) = -\frac{1}{Bt+D}w(z) \text{ and } z = x - \frac{\ln(Bt+D)}{B} + C_1,$$

where $w(z)$ satisfies

$$ww''' + w''' + bw'w'' + Bw'' - (b+1)ww' - w' - Bw = 0. \tag{26}$$

Now we seek solutions of the b-family equation (3) in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)(y(x)). \tag{27}$$

Substituting this into (3) yields

$$
\begin{aligned}
&-\beta^2 yy''' - \alpha\beta y''' - b\beta^2 y'y'' - (b+3)\beta\beta_x yy'' + (-\beta_t - 3\alpha\beta_x - b\alpha_x\beta)y'' \\
&-2b\beta\beta_x y'^2 + (-(b+3)\beta\beta_{xx} + (b+1)\beta - 2b\beta_x{}^2)yy' + ((b+1)\alpha\beta - b\alpha_{xx}\beta \\
&-3\alpha\beta_{xx} - 2\beta_{xt} - 2b\alpha_x\beta_x)y' + ((b+1)\beta\beta_x - b\beta_x\beta_{xx} - \beta\beta_{xxx})y^2 + ((b+1)\alpha_x\beta \\
&+(b+1)\alpha\beta_x - \alpha\beta_{xxx} - \beta_{xxt} - b\beta_x\alpha_{xx} - \alpha_{xxx}\beta - b\alpha_x\beta_{xx})y + (b+1)\alpha\alpha_x + \alpha_t \\
&-\alpha\alpha_{xxx} - b\alpha_x\alpha_{xx} - \alpha_{xxt} = 0.
\end{aligned}
\tag{28}
$$

This is an ordinary differential equation for $y(x)$ if the ratios of coefficients of different powers and derivatives of y are functions of x only. There are three cases to consider (since the calculations are similar to those done in the more general case above, details are omitted).

case (i) $\beta_t = 0$

In this case, we get solution in the form of $u = e^x y(x)$, where $y(x)$ is given be

$$yy''' + (b+3)yy'' + 2(b+1)yy' + byy'' + 2by'^2 = 0. \tag{29}$$

case (ii) $\beta_x = 0, \ \beta_t \neq 0$

In this case, we get solution in the form of $u = -\frac{1}{t+C_1}y(x)$, where C_1 is constant and $y(x)$ is given by

$$yy''' + byy'' + y'' - (b+1)yy' - y = 0. \tag{30}$$

case (iii) $\beta_x \neq 0, \ \beta_t \neq 0$

In this case, we get $\beta = \frac{1}{t+C_2}$, where C_2 is constant, which contradicts the initial assumption that $\beta_x \neq 0$. Therefore there are no special solutions of the b-family equation in this case.

Now we seek solutions of the b-family equation (3) in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)(y(t)).$$

In this case, we will get solution of equation (3) as

$$u(x,t) = (\theta_1(t)e^x + \theta_2(t)e^{-x})y(t)$$

or

$$u(x,t) = (\theta_1(t)\sinh(x) + \theta_2(t)\cosh(x))y(t), \tag{31}$$

where $\theta_1(t)$ and $\theta_2(t)$ are arbitrary functions of t.

4   Some Exact Solutions of b-family Equation

In above section, we have given various reductions of b-family equation using Lie classical method and Direct method. In this section we have given some exact solution corresponding to the ODEs that are obtained by the reduction of b-family equation.

For equation (11), let us assume a special solution of the form

$$F(\zeta) = \frac{k \tanh(\zeta) + l + m \sec h(\zeta)}{tanh(\zeta) + p + q \sec h(\zeta)},$$

where k, l, m, p and q are constants to be found out.

The substitution of the form of $F(\zeta)$ in equation (11) brings forth the following four possibilities:

(i) $p = -1, k = l = -qm, \mu = 1$
(ii) $p = -1, k = l = -qm, \mu = -1$
(iii) $p = -1, k = -qm, l = mq, \mu = 1$
(iv) $p = -1, k = -qm, l = mq, \mu = -1$

The solution to equation (3) for the above cases can be obtained, respectively, in the following forms

$$
\begin{aligned}
(i)\ \ & u(x,t) = \frac{m(-q\tanh(x+\ln(t))-q+\sec h(x+\ln(t)))}{t(\tanh(x+\ln(t))-1+\sec h(x+\ln(t)))} \\
(ii)\ \ & u(x,t) = \frac{m(-q\tanh(-x+\ln(t))-q+\sec h(-x+\ln(t)))}{t(\tanh(-x+\ln(t))-1+\sec h(-x+\ln(t)))} \\
(iii)\ \ & u(x,t) = \frac{m(-q\tanh(x+\ln(t))+q+\sec h(x+\ln(t)))}{t(\tanh(x+\ln(t))+1+\sec h(x+\ln(t)))} \\
(iv)\ \ & u(x,t) = \frac{m(-q\tanh(-x+\ln(t))+q+\sec h(-x+\ln(t)))}{t(\tanh(-x+\ln(t))+1+\sec h(-x+\ln(t)))} .
\end{aligned}
\tag{32}
$$

Proceeding in a similar manner as in the previous case, and assuming a solution of the reduced ODE (12) in the form

$$F(\xi) = \frac{k \tanh(\xi) + l + m \sec h(\xi)}{tanh(\xi) + p + q \sec h(\xi)},$$

where k, l, m, p and q are constants to be found.

In this case, following two possibilities arise:

(i) $p = 1, k = -qm, l = mq$
(ii) $p = -1, k = -qm, l = -mq$

and the final solution to equation (3) can be expressed, in respective order of cases, as follows:

$$
\begin{aligned}
(i)\ \ & u(x,t) = \frac{m(-q\tanh(x)+q+\sec h(x))}{t(\tanh(x)+1+q\sec h(x))} \\
(ii)\ \ & u(x,t) = \frac{m(-q\tanh(x)-q+\sec h(x))}{t(\tanh(x)-1+q\sec h(x))} .
\end{aligned}
\tag{33}
$$

For equation (26), let us assume a special solution of the form

$$w(z) = \frac{k \tanh(z) + l + m \sec h(z)}{tanh(z) + p + q \sec h(z)},$$

where k, l, m, p and q are constants to be found out.

The substitution of the form of $w(z)$ in equation (26) brings forth the following four possibilities:

(i) $p = -1, k = l = -qm,$
(ii) $p = -1, k = -qm, l = mq.$

The solution to equation (3) for the above cases can be obtained, respectively, in the following forms

$$
\begin{aligned}
(i)\ \ & u(x,t) = -\frac{1}{Bt+D} \frac{m(-q\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)-q+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1))}{(\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)-1+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1))} \\
(ii)\ \ & u(x,t) = -\frac{1}{Bt+D} \frac{m(-q\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)+q+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1))}{t(\tanh(x-\frac{\ln(Bt+D)}{B}+C_1)+1+\sec h(x-\frac{\ln(Bt+D)}{B}+C_1))} ,
\end{aligned}
\tag{34}
$$

where B, D and C_1 are arbitrary constants.

Solution of ODE (29) are

$$
\begin{aligned}
&(i)\ \ y\,(x) = C_1\,\mathrm{e}^{\left(-2\,b-3/2-1/2\,\sqrt{16\,b^2+16\,b+1}\right)x} \\
&(ii)\ \ y\,(x) = C_1\,\mathrm{e}^{\left(-2\,b-3/2+1/2\,\sqrt{16\,b^2+16\,b+1}\right)x},
\end{aligned}
\tag{35}
$$

where C_1 is a arbitrary constants.

In this case, we get stationary solution of equation (3) as

$$
\begin{aligned}
&(i)\ \ u(x) = e^x\,C_1\,\mathrm{e}^{\left(-2\,b-3/2-1/2\,\sqrt{16\,b^2+16\,b+1}\right)x} \\
&(i)\ \ u(x) = e^x\,C_1\,\mathrm{e}^{\left(-2\,b-3/2+1/2\,\sqrt{16\,b^2+16\,b+1}\right)x}.
\end{aligned}
\tag{36}
$$

Now consider the ODE (30). Solutions of ODE (30) are given as

$$
\begin{aligned}
&(i)\ \ y\,(x) = C_2\,\mathrm{e}^x \\
&(ii)\ \ y\,(x) = C_3 - \tfrac{C_1+C_2\,x}{(b+1)\,C_2},
\end{aligned}
\tag{37}
$$

where C_2 and C_3 are arbitrary constants and final solution of equation (3) can be expressed as

$$
\begin{aligned}
&(i)\ \ u(x,t) = -\tfrac{1}{t+C_1}\,C_2\,\mathrm{e}^x \\
&(ii)\ \ u(x,t) = -\tfrac{1}{t+C_1}\left(C_3 - \tfrac{C_4+C_2\,x}{(b+1)\,C_2}\right).
\end{aligned}
\tag{38}
$$

Let us assuming a solution of the reduced ODE (30) in the form

$$
y(x) = \frac{k\tanh(x) + l + m\sec h(x)}{tanh(x) + p + q\sec h(x)},
$$

where k, l, m, p and q are constants to be found.

In this case, following two possibilities arise:
$(i)\ \ p = 1, k = -qm, l = mq$
$(ii)\ \ p = -1, k = -qm, l = -mq$
and the final solution to equation (3) can be expressed, in respective order of cases, as follows:

$$
\begin{aligned}
&(i)\ \ u(x,t) = -\tfrac{1}{t+C_1}\,\tfrac{m(-q\tanh(x)+q+\sec h(x))}{(\tanh(x)+1+q\sec h(x))} \\
&(ii)\ \ u(x,t) = -\tfrac{1}{t+C_1}\,\tfrac{m(-q\tanh(x)-q+\sec h(x))}{t(\tanh(x)-1+q\sec h(x))},
\end{aligned}
\tag{39}
$$

where C_1 is arbitrary constant.
From (31), We get the solution of (3) as follows

$$
u(x,t) = (\theta_1(t)e^x + \theta_2(t)e^{-x})y(t)
$$

or

$$
u(x,t) = (\theta_1(t)\sinh(x) + \theta_2(t)\cosh(x))y(t),
$$

where $\theta_1(t)$ and $\theta_2(t)$ are arbitrary functions of t.

5   Conclusion

We have investigated the symmetries and invariant solutions of b-family equation. Firstly, the Lie group method is utilized for the purpose of obtaining the group infinitesimals. The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations.Secondly, we used direct method introduced by Clarkson and Krusksal to find symmetries of b-family equation. We obtain the exact solutions of b-family equation corresponding to reduced ODEs, which have been verified by putting them back into the original equation using Maple. One can easily derive the exact solutions of Camassa-Holm equation and Degasperis-Procesi equations as particular case.

References

[1] A. Degasperis and M. Procesi, Asymptotic Integrability, in: Symmetry and Perturbation Theory Ed.A. Degasperis and G. Gaeta, World Scientific (1999):23-37.

[2] A. Degasperis, D.D. Holm and A. N. W. Hone, Nonlinear Physics: Theory and Experiment, II (Gallipoli 2002), World Sci. Publ., River Edge, NJ (2003):37-43.

[3] A. Degasperis, D. D. Holm and A. N. I. Khon, A new integrable equation with peakon solutions, *Teoret. Mat. Fiz.* 133(2)(2002):170-183.

[4] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71(1993):1661-1664.

[5] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves , *J. Fluid Mech.* 455(2002):63-82.

[6] D. D. Holm and M. F. Staley, Wave Structure and Nonlinear Balances in a Family of Evolutionary PDEs, *SIAM J. Appl. Dyn. Systems* 2(2003):323-380.

[7] H. Lundmark and J. Szmigielski, G. Multi-peakon solutions of the Degasperis-Procesi Equation, *Inverse Problems* 19(2003):1241-1245.

[8] O. G. Mustafa, A note on the Degasperis-Procesi equation, *J. Nonlinear Math. Phys.* 12(2005):10-14.

[9] Z. Yin, Global existence for a new periodic integrable equation, *J. Math. Anal. Appl.* 283(2003):129-139.

[10] Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, *Illinois J. Math.* 47(2003):649-666.

[11] Z. Yin, Global solutions to a new integrable equation with peakons, *Indiana Univ. Math. J.* 53(2004):1189-1209.

[12] Z. Yin, Global weak solutions for a new periodic integrable equation with peakon solutions, *J. Funct. Anal.* 212(2004):182-194.

[13] G. M. Coclite and K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, *J. Funct. Anal.* 233(2006):60-91.

[14] P. J. Olver, Applications of Lie Groups to Differential Equations, Graduate Texts Math. Vol. 107, Springer Verlag, New York, 1993.

[15] K. Singh and R. K. Gupta, Lie symmetries and exact solutions of a new generalized Hirota-Satsuma coupled KdV system with variable coefficients, *Int. J. Engng. Sci.* 44(2006):241-255.

[16] R. K. Gupta and Anupma, The Dullin-Gottwald-Holm Equation: Classical Lie Approach and Exact Solutions, *Int. J. Nonlinear Sci.* 9(2010)(accepted).

[17] R. O. Popovych, Sophocleous and O. O. Vaneeva, Exact Solutions of a Remarkable Fin Equation, *Appl. math. lett.* 21(2008):209-214.

[18] M. Oberlack and A. F. Cheviakov, Higher Order Symmetries and Conservation laws of the G-Equation for Premixed Combustion and Resulting Numerical Schemes, *J. Engg. Math.* 66:(2010):121-140.

[19] G. Bluman, P. Broadbridge, J. R. King and M. J. Ward, Similarity:Generalizations, Applications and Open Problems, *J. Engng. Math.* 66(2010):1-9.

[20] O. Bogoyavlenskij, Restricted Lie Point Symmetries and Reductions for Ideal Magnetohydynamics Equilibria, *J. Engng. Math* 66(2010):141-152.

[21] P. A. Clarkson and M. D. Kruskal, New Similarity Reduction of the Boussinesq Equation, *J. Math. Phys.* 30(1989):2201-2213.

[22] P. A. Clarkson, New Similarity Reductions and Painlevé Analysis for the Symmetric Regularised Long Wave and Modified Benjamin-Bona-Mahoney equations, *J. Phys. A: Math. Gen* 22(1939):3821-3848.

[23] M. A. Hong-Cai, Generating Lie Point Symmetry Groups of (2+1)-Dimensional Broer Kaup Equation, *Commun. Theor. Phys* 43(2005):1047-1052.