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Constitutive Functions of Elastic Materials in Finite Growth and Deformation

YI-CHAO CHEN[1] and ANNE HOGER[2]
[1]*Department of Mechanical Engineering, University of Houston, Houston, TX 77204, U.S.A.*
[2]*Department of Applied Mechanics and Engineering Sciences, University of California, San Diego, LaJolla, CA 92093, U.S.A.*

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Dedicated to Roger Fosdick.

Abstract. The constitutive functions of soft biological tissues during growth are studied. A growth, treated as addition (often non-uniform) of material points, results in deformation, residual stresses, and evolution of the constitutive functions. A theory based on the concept of equivalent material points is developed with the current configuration taken as the reference. The residual stresses developed in a spherical shell undergoing spherical growths are studied.

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1. Introduction

The description of the biological process of growth and remodeling is of interest both because of its importance as a fundamental mechanical process that occurs in normal development and in a number of pathological conditions, and because it offers an interesting and unusual application of continuum mechanics. It is known that mechanical quantities such as stress and strain can modulate growth, and growth of a tissue can, in turn, induce the development of residual stress. These are among the reasons that the phenomena of growth and remodeling have attracted the attention of researchers in both the mechanics and the biology communities. A number of earlier authors constructed kinematical descriptions of growth using the methods of continuum mechanics, but the work of Hsu [1], Cowin and Hegedus [2], Skalak [3], and Skalak et al. [4] are the initial efforts made to formulate mechanical theories to describe the growth of biological tissues. An excellent article by Taber [5] reviews the progress in the field through 1995.

The most general 3-dimensional theory of stress-dependent volumetric growth for soft biological tissues was first presented by Rodriguez et al. [6]. That theory introduced two fundamental ideas. The shape change of the unloaded tissue during growth can be described by a mapping whose gradient is analogous to a deformation gradient tensor. The first fundamental idea in [6] was that this gradient could be decomposed into the product of a "growth stretch tensor", which describes the addition of material at a point and the orientation of its deposition, and a tensor that represents the gradient of the elastic accommodation of the body to the new material. This elastic portion of the gradient ensures compatibility of the total growth deformation. The second fundamental idea was that the stress due to growth of a tissue was a function of only the elastic portion of the gradient of the mapping.

Because at the time when [6] was written it was not yet known how to construct constitutive equations for residually stressed finite elastic materials, the theory presented in [6] was illustrated only by problems in which the initial configuration was stress-free. In addition, their interpretation of the theory required knowledge of a locally stress-free reference state for the grown material. That such a zero-stress state exists is suggested by the destructive experiments that are commonly used to determine the residual stress in biological tissues and other materials. When a piece of residually stressed material is cut into progressively smaller pieces, the residual stress is relieved as the cuts are made.

Johnson and Hoger [7] stated and proved a mathematical description of this destructive testing process, which they used to motivate the concept of a virtual configuration of a residually stressed material. They applied the concept of the virtual configuration to derive constitutive equations for residually stressed elastic materials [7, 8]. The derivation of these constitutive equations requires that the response function of the corresponding natural material be locally invertible and satisfy certain smoothness requirements.

Recently, Hoger [9] expanded the theory of growth originally presented in [6] to take advantage of the availability of constitutive equations for finite elastic residually stressed materials. That work included an example of growth where the original configuration of the body supports a residual stress. It also contains a discussion of the growth stretch tensor, and identifies broad classes of growth based on the form taken by the growth stretch tensor.

The theory of growth presented by Rodriguez et al. and Hoger incorporates the use of a locally stress-free configuration of the material either explicitly [6] or through the use of the virtual configuration in the construction of a constitutive equation [9]. In both cases, the locally stress-free state is used as a reference configuration in which the mechanical properties of the natural material are known. In this paper we will not rely on the existence of a locally stress-free configuration, but, instead, will present a more general approach to developing the appropriate constitutive equation for a material comprising a residually stressed growing body.

To accomplish this, we will first focus on the issue of how to determine whether two material points are equivalent in the sense that their intrinsic mechanical properties are the same. The two points may be in different elastic bodies, or they may be the same material point in two configurations of the body; they may be in different stress states and may have deformation histories that are unknown. We engage this question by defining the constitutive functions with the current configuration taken as reference. The material properties are completely characterized by a density function that gives the density in the current state, and a response function that gives the stress in any configuration obtained by an elastic deformation out of the current state. No information on the histories of the deformation and stress of the material is needed. In particular, it is not necessary to identify a zero-stress configuration.

We then will use this framework to construct a theory of growth in which the growth tensor and the total deformation are defined on the current configuration. The forms of the density function and the response function evolve as the material grows and deforms.

There are two advantages of the theory of growth that we present in this paper over the theory of Rodriguez et al. [6] and Hoger [9]. The first advantage is due to the use of the current configuration as reference. The growth theory of [6] and [9] is Lagrangian: every function is referred to the original configuration. This typically presents no difficulty in the mechanics of solid materials, but if a body is growing, it is gaining material points, so there is no easily identified fixed reference configuration. One consequence of this difficulty is that the theory can only approximate continuous growth by modeling the growth as taking place in discrete increments [9]. The theory of growth we present in this paper is constructed with the current configuration as reference, thereby making it ideal for the description of continuous growth processes.

Second, neither Rodriguez et al. [6], nor Hoger [9] provided a theoretically sound foundation for the two central ideas in their theory of growth: the decomposition of the gradient of the total deformation; and the related assertion that the response of the material should depend only on the elastic portion of the decomposed gradient. In this paper we provide that foundation. We present a derivation of the evolution of the response function during growth and deformation, and determine explicitly how the evolution depends on the growth and on the total deformation. In particular, we provide a mathematically rigorous justification for the statement made in [6]: “Intuitively, residual stress arises from the part of the total growth deformation that is responsible for accommodating the newly grown tissue to prevent discontinuities in the grown state of the body.”

In the next section we focus on the general issue of the characterization of the mechanical properties of a compressible material in the context of continuum mechanics. The density function and the response function are introduced with the current configuration taken as reference. Our goal in the section is to provide a rigorous definition for the statement that two material points have the same intrinsic material properties. We do this through the definitions of identical material points and equivalent material points, both of which are constructed using the density and response functions. The definition of equivalent material points requires the introduction of the equivalence transformation tensor, whose properties are examined. The relationship of this transformation tensor to the locally-stress-free configuration of [6] and the virtual configuration of [7–9] is also discussed.

In Section 3, we construct a theory of growth for a compressible elastic body. Only growth processes in which the intrinsic mechanical properties of the material do not change are considered. This allows the equivalence transformation tensor associated with the growth at a point to be defined through the requirement of material equivalence. The growth tensor, which describes the amount and the orientation of the material added at a point during a growth process, is defined. The dependence of the equivalence transformation tensor on the growth tensor and the total deformation gradient is then derived, and used to obtain the equations that explicitly describe the evolution of the density function and the response function during a growth process.

In Section 4, we outline the analogous theory of growth for incompressible elastic materials. The concepts of identical material points and equivalent material points are defined for incompressible materials; the appropriate equivalence transformation tensor is introduced; and the dependence of the Cauchy stress on the growth tensor and on the total deformation gradient is given.

We conclude the paper in Section 5 with an example in which we examine the spherically symmetric continuous growth of a spherical shell that is initially stress-free. The shell remains unloaded during the growth process. This growth boundary value problem is solved for the general class of homogeneous, isotropic, incompressible, elastic materials. We obtain expressions for the components of the residual stress generated by the growth process in terms of the growth tensor. In addition, explicit expressions for the time rates of change of the residual stress components are derived in terms of the growth rate. The section concludes with a discussion of the dependence of the growth-induced residual stress on the form of the growth tensor.

2. Characterization of Material. Equivalence of Material Points

Consider a body composed of a compressible elastic material that occupies the domain $\Omega \subset \mathfrak{R}^3$ at the current time. The body may undergo deformations due purely to external loads or deformations that include growth. It is usual in continuum mechanics to characterize the material of the body, especially the relation between the deformation and the stress, in terms of its mechanical properties in a fixed reference configuration. This approach is not convenient, however, if a fixed reference configuration cannot be easily identified, or if there is no fixed reference configuration in which the properties of the material are known.

An alternate approach is to use the current configuration Ω as the reference configuration, and to define the response function of the material in terms of the relation between deformations out of Ω and the stresses in the deformed configurations. If the body is subject to a time dependent deformation or growth process, then Ω will vary in time continuously, as will the characterization of the material.

For the present study, we will characterize the material by a density function* ρ: $\Omega \rightarrow \Re_+$ and a response function $\widehat{\mathbf{T}}$: $\mathrm{Lin}^+ \times \Omega \rightarrow \mathrm{Sym}$, where Lin^+ is the set of all linear transformations on $\Re^3$ that have positive determinant, and Sym the set of all symmetric linear transformations on $\Re^3$. These functions are assumed to possess the smoothness needed in the analysis. The value of the density function $\rho(\mathbf{x})$ gives the current density of the material at $\mathbf{x} \in \Omega$. We will term a deformation that is produced purely by external loads a *pure deformation* to distinguish it from a deformation in which growth occurs. Let the mapping $\mathbf{w}$: $\Omega \rightarrow \Re^3$ represent a pure deformation of the body out of the current configuration. Then the value of the response function $\widehat{\mathbf{T}}(\nabla\mathbf{w}(\mathbf{x}), \mathbf{x})$ gives the Cauchy stress tensor $\mathbf{T}(\mathbf{w})$ at $\mathbf{w} = \mathbf{w}(\mathbf{x})$. Note that the Cauchy stress $\mathbf{T}(\mathbf{x})$ at $\mathbf{x}$ in the current configuration Ω is given by

$$\mathbf{T}(\mathbf{x}) = \widehat{\mathbf{T}}(\mathbf{I}, \mathbf{x}), \tag{1}$$

where $\mathbf{I}$ is the identity tensor. If the body is not loaded in the current configuration, this stress is typically called the residual stress.

As a body deforms, the expressions for the density function and the response function at a point will evolve to reflect the changing reference configuration. We now examine how these functions evolve over a pure deformation. To be definite, let the configurations of the body before and after a pure deformation $\mathbf{w}$ be denoted by Ω and Ω_w, respectively, where $\Omega_w \equiv \mathbf{w}(\Omega)$. In addition, let $(\rho, \widehat{\mathbf{T}})$ and $(\rho_w, \widehat{\mathbf{T}}_w)$ be the density and response function pairs in Ω and Ω_w, respectively. By the conservation of mass, the density of the deformed body at $\mathbf{w} = \mathbf{w}(\mathbf{x})$ is given by $[\det\nabla\mathbf{w}(\mathbf{x})]^{-1}\rho(\mathbf{x})$.

Now consider a further pure deformation $\mathbf{z}$: $\Omega_w \rightarrow \Re^3$ of the body out of configuration Ω_w. The Cauchy stress at $\mathbf{z} = \mathbf{z}(\mathbf{w}(\mathbf{x}))$ is given by $\widehat{\mathbf{T}}(\nabla\mathbf{z}\nabla\mathbf{w}, \mathbf{x})$. Hence, the density function ρ_w and the response function $\widehat{\mathbf{T}}_w$ are given by

$$\rho_w(\mathbf{w}) = \det\big[\nabla\mathbf{w}(\mathbf{x})\big]^{-1}\rho(\mathbf{x}), \tag{2}$$

and

$$\widehat{\mathbf{T}}_w(\mathbf{F}, \mathbf{w}) = \widehat{\mathbf{T}}\big(\mathbf{F}\nabla\mathbf{w}(\mathbf{x}), \mathbf{x}\big), \tag{3}$$

where $\mathbf{w} = \mathbf{w}(\mathbf{x})$.

Up to this point we have obtained the relations between the density functions and the response functions for the points $\mathbf{x}$ and $\mathbf{w}$ that are related by a pure deformation. We now turn to the more general problem of describing the relation between the constitutive functions of any two points that have the same intrinsic material properties. By intrinsic properties of a material point, we mean the density and response functions at this point in a configuration with a prescribed stress state.

* Here we have chosen to include the density function in the characterization of a material, because mass change is an important aspect of growth, and because the evolution of density must be taken into account when studying a full dynamical problem of growth. The density function is, however, not needed in a static or quasistatic problem, such as the one presented in Section 5.

If a material has a stress-free natural state, it is convenient to use the properties of this underlying natural material as the intrinsic material properties. In any case, the intrinsic properties of a material do not change when the material undergoes pure deformations.

Two material points will be said to be *identical* if their density functions and their response functions are the same. Two such identical material points may belong to the same body or to different bodies, or may be the same material point at two different deformation or growth configurations.⋆ In any case, it is not possible to distinguish between two identical material points with mechanical experiments.

During all pure deformations and for certain types of growth, the intrinsic properties of a material point remain unchanged, although the forms of the density function and the response function at the material point will typically change. In order to identify materials which share intrinsic material properties, we now introduce the concept of *equivalent material points*: two material points $\mathbf{x}_1$ and $\mathbf{x}_2$ will be said to be *equivalent* if there exists a pure deformation at $\mathbf{x}_1$ such that after this deformation, $\mathbf{x}_1$ is identical to $\mathbf{x}_2$. More precisely, let $(\rho_1, \widehat{\mathbf{T}}_1)$ and $(\rho_2, \widehat{\mathbf{T}}_2)$ be the density and response function pairs associated with the material points $\mathbf{x}_1$ and $\mathbf{x}_2$. These two material points are equivalent if there exists $\overline{\mathbf{F}} \in \mathrm{Lin}^+$ such that

$$\rho_2(\mathbf{x}_2) = \left(\det\overline{\mathbf{F}}\right)^{-1} \rho_1(\mathbf{x}_1) \tag{4}$$

and

$$\widehat{\mathbf{T}}_2(\mathbf{F}, \mathbf{x}_2) = \widehat{\mathbf{T}}_1\left(\mathbf{F}\overline{\mathbf{F}}, \mathbf{x}_1\right) \quad \forall \mathbf{F} \in \mathrm{Lin}^+. \tag{5}$$

The tensor $\overline{\mathbf{F}}$ will be termed the *equivalence transformation*.

By comparing equations (4) and (5) with equations (2) and (3), we observe that the equivalence transformation tensor can be thought of as the gradient of the homogeneous deformation under which the material point $\mathbf{x}_1$ becomes identical to the material point $\mathbf{x}_2$.

It is obvious that if two material points are identical, then they are equivalent with $\overline{\mathbf{F}} = \mathbf{I}$. It is also clear that a material point is equivalent to itself after any pure deformation. For example, let $\mathbf{w}\colon \Omega \to \mathfrak{R}^3$ be a pure deformation of Ω. The material point $\mathbf{x} \in \Omega$ is equivalent to $\mathbf{w} = \mathbf{w}(\mathbf{x})$ with

$$\overline{\mathbf{F}} = \nabla \mathbf{w}(\mathbf{x}). \tag{6}$$

It follows from equation (6) that if the material undergoes a further pure deformation $\mathbf{z}\colon \mathbf{w}(\Omega) \to \mathfrak{R}^3$, then the new equivalence transformation $\overline{\mathbf{F}}^*$ for the point $\mathbf{z} = \mathbf{z}(\mathbf{w}(\mathbf{x}))$ is given by the product of $\nabla\mathbf{z}$ and the equivalence transformation associated with $\mathbf{w}$:

$$\overline{\mathbf{F}}^* = \nabla(\mathbf{z} \circ \mathbf{w})(\mathbf{x}) = \nabla\mathbf{z}\nabla\mathbf{w} = \nabla\mathbf{z}\overline{\mathbf{F}}. \tag{7}$$

⋆ In this latter case, one must redefine the “current times” of the two configurations accordingly.

If the stresses at two equivalent material points $\mathbf{x}_1$ and $\mathbf{x}_2$ are the same, then it follows from (1) and (5) that

$$\widehat{\mathbf{T}}_1(\mathbf{I}, \mathbf{x}_1) = \widehat{\mathbf{T}}_2\big(\mathbf{I}, \mathbf{x}_2\big) = \widehat{\mathbf{T}}_1(\overline{\mathbf{F}}, \mathbf{x}_1).$$

A solution of the above equation is $\overline{\mathbf{F}} = \mathbf{I}$. We shall assume that this solution is unique. Hence, if the stresses and densities at two equivalent material points are the same, these two material points are identical.

In the next section we shall use the concept of material equivalence developed above to study growth of soft biological tissues. Of course, growth does not happen in discrete increments, but is a continuous process. Using the current configuration as reference leads to a natural formulation for a continuous growth process, as displayed at the end of Section 3.

3. Growth

Changes in size and shape of a biological tissue can involve both growth, a change in mass, and remodeling, which includes changes in internal structure, density and material properties. Often growth and remodeling are linked, since the process of mass alteration can change the mechanical properties of the tissue. For example, the newly deposited tissue may have different material properties than the original tissue; or only one component of the tissue may grow, thus changing the mechanical properties of the tissue as a whole. Here we will focus exclusively on growth, which can involve both the addition and removal of material, sometimes occurring simultaneously. See Rodriguez et al. [6] and Hoger [9] for a full discussion of the kinematical description of growth.

In this work, we will restrict our attention to growth processes that meet the following two requirements. First, the material points must be dense during growth. This implies that in any arbitrary neighborhood in the grown body, there will always be material points that existed before the growth took place. The second requirement is that the intrinsic mechanical properties of the material should not change during growth. In other words, the new material has the same properties as does the original material. This implies that not only is a material point equivalent (in the sense of Section 2) to itself after growth, but also that a new material point added during growth is equivalent to an original material point in an arbitrarily small neighborhood of the new material point. Growth processes that meet these restrictions are special forms of volumetric growth, which takes place in the volume of a tissue rather than on the surface.

Kinematically, the addition of material alone can be described by a tensor valued function $\mathbf{G}\colon \Omega \to \mathrm{Lin}^+$, termed the growth tensor, that describes the amount and the orientation of material deposition [6, 9].

To set ideas, we first consider the simple case of a homogeneous elastic body with constant density and zero stress in the current configuration. Let the body undergo a homogeneous growth, with no applied external loads. For this case, the shape change of the grown body can be described by an affine mapping similar to a homogeneous deformation in continuum mechanics. The growth tensor in this simple case would correspond to the gradient of this mapping, which is uniform. The Cauchy stress in the grown body is maintained at zero, and the density of the body remains constant.

The situation just described is atypical. The growth tensor is not homogeneous in general, and the Cauchy stress in the body is usually inhomogeneous. To generalize the idea introduced in the above simple example, consider an inhomogeneous elastic body supporting an inhomogeneous stress in the current configuration Ω, and consider a small spherical neighborhood centered at point $\mathbf{x} \in \Omega$. Let the body undergo an inhomogeneous growth, and suppose that loads are applied on the grown body so that the stress at point $\mathbf{x}$ remains the same in the grown state as it was in Ω. Due to the growth, the sphere grows into a shape which can be approximated by an ellipsoid. The growth tensor $\mathbf{G}(\mathbf{x})$ describes the geometric change of the sphere to the ellipsoid.

The above argument is local. Globally, it is generally impossible to maintain, by applying the surface loads alone, the same inhomogeneous stress in the entire body during the growth. The shape change of a body subjected to an arbitrary growth process can be described by a mapping $\mathbf{y}\colon \Omega \to \Re^3$, which will be referred to as the *total deformation*. In general, the gradient of this total deformation will be different from the growth tensor. Hence, the total deformation cannot be obtained from the growth tensor alone. In fact, the growth tensor will in general not be the gradient of a vector field. However, at any material point where the gradient of the total deformation happens to be equal to the value of the growth tensor, the stress at that point will remain unchanged after the growth process.

A growth process can also be accompanied by an additional pure deformation due to a change of the applied loads on the body. Of course, in this case the additional pure deformation will contribute to the total deformation.

Let $\rho(\mathbf{x})$ and $\widehat{\mathbf{T}}(\mathbf{F}, \mathbf{x})$ denote the density and response functions for the material in the current configuration. And let $\rho_y(\mathbf{y})$ and $\widehat{\mathbf{T}}_y(\mathbf{F}, \mathbf{y})$ denote the density and response functions of the material at some later time after growth and deformation has occurred, as described by the total deformation $\mathbf{y}(x)$. We now examine how $\rho_y(\mathbf{y})$ and $\widehat{\mathbf{T}}_y(\mathbf{F}, \mathbf{y})$ are related to $\rho(\mathbf{x})$ and $\widehat{\mathbf{T}}(\mathbf{F}, \mathbf{x})$.

By the definition of material equivalence and by the requirement that a material point be equivalent to itself at any time during a growth or deformation, there exists an equivalence transformation $\overline{\mathbf{F}}\colon \Omega \times \mathrm{Lin}^+ \times \mathrm{Lin}^+ \to \mathrm{Lin}^+$, such that

$$\rho_y\big(\mathbf{y}(\mathbf{x})\big) = \big[\det\overline{\mathbf{F}}\big(\mathbf{x}, \mathbf{G}(\mathbf{x}), \nabla\mathbf{y}(\mathbf{x})\big)\big]^{-1}\rho(\mathbf{x}), \tag{8}$$

and

$$\widehat{\mathbf{T}}_y(\mathbf{F}, \mathbf{y}(\mathbf{x})) = \widehat{\mathbf{T}}\big(\mathbf{F}\overline{\mathbf{F}}\big(\mathbf{x}, \mathbf{G}(\mathbf{x}), \nabla\mathbf{y}(\mathbf{x})\big), \mathbf{x}\big) \quad \forall\mathbf{F} \in \mathrm{Lin}^+. \tag{9}$$

Here we have temporarily assumed that the equivalence transformation depends on the material point, the growth tensor, and the total deformation gradient.

We now determine the form of the equivalence transformation $\overline{\mathbf{F}}(\mathbf{x}, \mathbf{G}, \nabla\mathbf{y})$, and show that the form of $\overline{\mathbf{F}}$ is actually independent of the material. By the properties of the growth tensor, we observe that if the values of the total deformation gradient and the growth tensor are the same at a point, then the stress and density at this point do not change. Consequently, such a material point is identical to itself after the growth and deformation. That is,

$$\overline{\mathbf{F}}\big(\mathbf{x}, \mathbf{G}(\mathbf{x}), \mathbf{G}(\mathbf{x})\big) = \mathbf{I}. \tag{10}$$

If the material, after the total deformation $\mathbf{y}$, undergoes a further pure deformation $\mathbf{z}$: $\mathbf{y}(\Omega) \to \mathfrak{R}^3$, then the transformation tensor for the combined deformations can be obtained by considering $\mathbf{z}$ as a pure deformation imposed after the total deformation $\mathbf{y}$. The new equivalence transformation can be obtained by using equation (7):

$$\overline{\mathbf{F}}(\mathbf{x}, \mathbf{G}, \nabla\mathbf{z}\nabla\mathbf{y}) = \nabla\mathbf{z}\overline{\mathbf{F}}(\mathbf{x}, \mathbf{G}, \nabla\mathbf{y}) \quad \forall\mathbf{z}. \tag{11}$$

For a given growth tensor $\mathbf{G}$, the value of the total deformation gradient at a particular point $\mathbf{x} \in \Omega$ can be made equal to any given tensor in Lin^+ by applying appropriate loads to the body. Suppose that loads are chosen so that, at the point $\mathbf{x}$, $\nabla\mathbf{y}(\mathbf{x}) = \mathbf{G}(\mathbf{x})$. Then by (11),

$$\overline{\mathbf{F}}(\mathbf{x}, \mathbf{G}, \nabla\mathbf{z}\mathbf{G}) = \nabla\mathbf{z}\overline{\mathbf{F}}(\mathbf{x}, \mathbf{G}, \mathbf{G}) \quad \forall\mathbf{z},$$

which, with (10) and $\mathbf{F} = \nabla\mathbf{z}\mathbf{G}$, can be written as

$$\overline{\mathbf{F}}(\mathbf{x}, \mathbf{G}, \mathbf{F}) = \mathbf{F}\mathbf{G}^{-1} \quad \forall\mathbf{F} \in \mathrm{Lin}^+. \tag{12}$$

Equation (12) gives the explicit form of the equivalence transformation. In particular, it indicates that $\overline{\mathbf{F}}$ does not depend explicitly on either position or on the material. Rather, for a given growth tensor $\mathbf{G}$ and the total deformation $\mathbf{y}$ on Ω, the value of $\overline{\mathbf{F}}$ depends on $\mathbf{x}$ implicitly through $\mathbf{G}(\mathbf{x})$ and $\nabla\mathbf{y}(\mathbf{x})$.

Substitution of (12) into (8) and (9) yields

$$\rho_y\big(\mathbf{y}(\mathbf{x})\big) = \det\mathbf{G}(\mathbf{x})\big[\det\nabla\mathbf{y}(\mathbf{x})\big]^{-1}\rho(\mathbf{x}), \tag{13}$$

and

$$\widehat{\mathbf{T}}_y\big(\mathbf{F}, \mathbf{y}(\mathbf{x})\big) = \widehat{\mathbf{T}}\big(\mathbf{F}\nabla\mathbf{y}(\mathbf{x})\mathbf{G}^{-1}(\mathbf{x}), \mathbf{x}\big) \quad \forall\mathbf{F} \in \mathrm{Lin}^+.$$

These equations describe the changes in the density function and the response function after growth. In particular, the Cauchy stress in the grown state is given by

$$\mathbf{T}(\mathbf{y}) = \widehat{\mathbf{T}}_y(\mathbf{I}, \mathbf{y}) = \widehat{\mathbf{T}}\big(\nabla\mathbf{y}(\mathbf{x})\mathbf{G}^{-1}(\mathbf{x}), \mathbf{x}\big), \tag{14}$$

where $\mathbf{y} = \mathbf{y}(\mathbf{x})$.

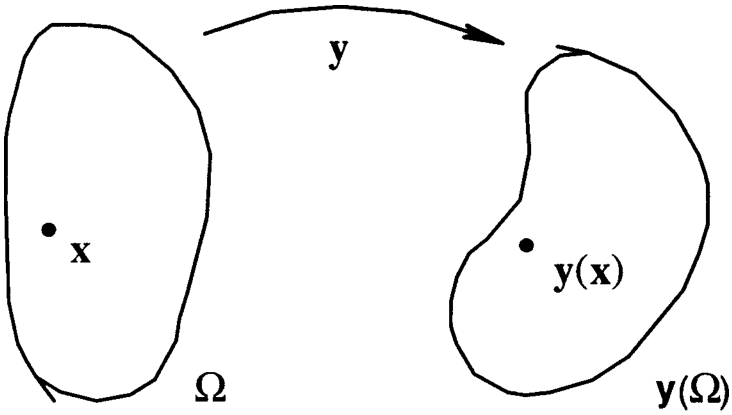

Figure 1. The body originally occupies domain Ω. In this configuration the material is characterized by the density function ρ: $\Omega \to \Re_+$ and the response function $\widehat{\mathbf{T}}$: $\mathrm{Lin}^+ \times \Omega \to \mathrm{Sym}$. The density of the material at $\mathbf{x} \in \Omega$ is given by $\rho(\mathbf{x})$, and the Cauchy stress $\mathbf{T}(\mathbf{x})$ in the configuration Ω is given by $\mathbf{T}(\mathbf{x}) = \widehat{\mathbf{T}}(\mathbf{I}, \mathbf{x})$. After a growth process, represented here by the total deformation $\mathbf{y}(\mathbf{x})$, the body occupies the region $\mathbf{y}(\Omega)$. The density $\rho_y(\mathbf{y})$ and the stress $\mathbf{T}(\mathbf{y})$ at point $\mathbf{y} \in \mathbf{y}(\Omega)$ are given by $\rho_y(\mathbf{y}) = \det\mathbf{G}(\mathbf{x})[\det\nabla\mathbf{y}(\mathbf{x})]^{-1}\rho(\mathbf{x})$ and $\mathbf{T}(\mathbf{y}) = \widehat{\mathbf{T}}(\nabla\mathbf{y}(\mathbf{x})\mathbf{G}^{-1}(\mathbf{x}), \mathbf{x})$.

The derivations leading to equations (12) and (14) provide a rigorous foundation for two suggestions originally proposed by Rodrigue et al. [6] and elaborated by Hoger [9]. Equation (12) can be rewritten as

$$\overline{\mathbf{F}}(\mathbf{x}, \mathbf{G}, \nabla\mathbf{y}) = \nabla\mathbf{y}\mathbf{G}^{-1},$$

or

$$\nabla\mathbf{y} = \overline{\mathbf{F}}\mathbf{G}.$$

This is the decomposition of the total deformation originally presented in [6, equation 9]. The decomposition was introduced in the context where $\mathbf{G}$ is defined as the growth from the original stress-free reference state to a new locally stress-free state, and $\overline{\mathbf{F}}$ is viewed as an elastic deformation that ensures the continuity of the body. In the present work, $\mathbf{G}$ corresponds to the growth of the body from the configuration Ω, which is not necessarily stress free. The introduction of the equivalence transformation $\overline{\mathbf{F}}$ is based on the requirement of material equivalence for growth rather than being assumed, as it was in [6, 7, 8, 9]. Of course, $\overline{\mathbf{F}}$ is generally not the gradient of a deformation of the body, as the notion of equivalence of material points is local. In [6, 7, 8, 9], it is conjectured that the stress which arises from the growth depends only on the elastic deformation that ensures the continuity of the body. The derivation of (14) provided here gives a rigorous proof of this conjecture.

A continuous growth process can be described by a time-dependent growth tensor $\mathbf{G}(\mathbf{x}, t)$ and a time-dependent total deformation $\mathbf{y}(\mathbf{x}, t)$. Equations (13) and (14) can be applied to this growth process, yielding

$$\rho_y(\mathbf{y}, t) = \det\mathbf{G}(\mathbf{x}, t)\left[\det\nabla\mathbf{y}(\mathbf{x}, t)\right]^{-1}\rho(\mathbf{x}),$$

and

$$\mathbf{T}(\mathbf{y}, t) = \widehat{\mathbf{T}}\big(\nabla\mathbf{y}(\mathbf{x}, t)\mathbf{G}^{-1}(\mathbf{x}, t), \mathbf{x}\big),$$

where $\mathbf{y} = \mathbf{y}(\mathbf{x}, t)$, and $\rho_y(\mathbf{y}, t)$ and $\mathbf{T}(\mathbf{y}, t)$ are the density and the stress at $\mathbf{y}$ in the grown configuration at time t.

4. Incompressible Materials

The concepts developed in Sections 2 and 3 can also be used for incompressible materials. In this section we present the parallel theory for the growth of an incompressible elastic body. For brevity, we shall present the basic equations with little or no derivation.

Let the body composed of an incompressible elastic material occupy the domain $\Omega \in \mathfrak{R}^3$ at the present time. As in the compressible case, the material is characterized by a density ρ function and a response function $\widehat{\mathbf{T}}$, but for incompressible materials $\widehat{\mathbf{T}}$: Unim $\times\, \Omega \to$ Sym, where Unim is the set of all linear transformations on $\mathfrak{R}^3$ whose determinant is unity. The value of the response function $\widehat{\mathbf{T}}$ gives the Cauchy stress only up to an arbitrary hydrostatic pressure. Let $\mathbf{w}$: $\Omega \to \mathfrak{R}^3$ with $\nabla\mathbf{w} \in$ Unim be a pure deformation of the body. The Cauchy stress in the deformed configuration is

$$\mathbf{T}(\mathbf{w}) = -p_w(\mathbf{w})\mathbf{I} + \widehat{\mathbf{T}}(\mathbf{F}, \mathbf{x}), \tag{15}$$

where $\mathbf{w} = \mathbf{w}(\mathbf{x})$, $\mathbf{F} = \nabla\mathbf{w}(\mathbf{x})$, and p_w: $\mathbf{w}(\Omega) \to \mathfrak{R}$ is the hydrostatic pressure required by the incompressibility constraint. The Cauchy stress $\mathbf{T}(\mathbf{x})$ in the current configuration Ω is given by

$$\mathbf{T}(\mathbf{x}) = -p(\mathbf{x})\mathbf{I} + \widehat{\mathbf{T}}(\mathbf{I}, \mathbf{x}),$$

where p: $\Omega \to \mathfrak{R}$ is in general different from p_w appearing in (15).

The notions of identical material points and of equivalent material points can be defined for incompressible bodies in a way similar to that used for compressible bodies. However, since for incompressible bodies the response function determines only those components of the Cauchy stress that are orthogonal to the identity tensor $\mathbf{I}$, only the orthogonal components of the response function need be considered when making the definitions. In the following, $\overline{\mathbf{T}}$ will denote the part of $\widehat{\mathbf{T}}$ that is orthogonal to $\mathbf{I}$:

$$\overline{\mathbf{T}} \equiv \widehat{\mathbf{T}} - \frac{1}{3}\big(\mathrm{tr}\widehat{\mathbf{T}}\big)\mathbf{I}.$$

Consider two bodies Ω_1 and Ω_2 (which may represent two configurations of the same body), with associated density and response function pairs $(\rho_1, \widehat{\mathbf{T}}_1)$ and $(\rho_2, \widehat{\mathbf{T}}_2)$. Two material points $\mathbf{x}_1 \in \Omega_1$ and $\mathbf{x}_2 \in \Omega_2$ are said to be identical if

$$\rho_1(\mathbf{x}_1) = \rho_2(\mathbf{x}_2), \tag{16}$$

and

$$\overline{\mathbf{T}}_1(\mathbf{F}, \mathbf{x}_1) = \overline{\mathbf{T}}_2(\mathbf{F}, \mathbf{x}_2) \quad \forall \mathbf{F} \in \text{Unim}.$$

If instead, the two material points satisfy (16) and

$$\overline{\mathbf{T}}_1(\mathbf{F}\overline{\mathbf{F}}, \mathbf{x}_1) = \overline{\mathbf{T}}_2(\mathbf{F}, \mathbf{x}_2) \quad \forall \mathbf{F} \in \text{Unim} \tag{17}$$

for some equivalence transformation tensor $\overline{\mathbf{F}} \in \text{Unim}$, then the points are said to be equivalent.

We consider the growth of an incompressible body that meets the requirements that the material points are dense during the growth, and that the intrinsic mechanical properties of the material do not change. Such a growth is described, in analogy to compressible materials, by a growth tensor $\mathbf{G}: \Omega \to \text{Lin}^+$, along with a total deformation $\mathbf{y}: \Omega \to \mathfrak{R}^3$ which provides the total shape change of the body.

Let $\rho_y(\mathbf{y})$ and $\widehat{\mathbf{T}}_y(\mathbf{F}, \mathbf{y})$ be the density and response functions of the material in the grown body. Since a material point is equivalent to itself after a total deformation, it follows from (16) and (17) that there exits an equivalence transformation tensor $\overline{\mathbf{F}}: \Omega \times \text{Lin}^+ \times \text{Lin}^+ \to \text{Unim}$ such that

$$\rho_y\big(\mathbf{y}(\mathbf{x})\big) = \rho(\mathbf{x}),$$

and

$$\overline{\mathbf{T}}_y\big(\mathbf{F}, \mathbf{y}(\mathbf{x})\big) = \overline{\mathbf{T}}\Big(\mathbf{F}\overline{\mathbf{F}}\big(\mathbf{x}, \mathbf{G}(\mathbf{x}), \nabla\mathbf{y}(\mathbf{x})\big), \mathbf{x}\Big) \quad \forall \mathbf{F} \in \text{Unim}. \tag{18}$$

The arguments that lead to (12) for compressible materials are equally valid for incompressible materials; so the representation (12) for the equivalence transformation holds for incompressible materials as well. Since $\overline{\mathbf{F}} \in \text{Unim}$, equation (12) implies that

$$\det\mathbf{G}(\mathbf{x}) = \det\nabla\mathbf{y}(\mathbf{x}). \tag{19}$$

Substitution of (12) into (18) gives

$$\overline{\mathbf{T}}_y\big(\mathbf{F}, \mathbf{y}(\mathbf{x})\big) = \overline{\mathbf{T}}\big(\mathbf{F}\nabla\mathbf{y}(\mathbf{x})\mathbf{G}^{-1}(\mathbf{x}), \mathbf{x}\big) \quad \forall \mathbf{F} \in \text{Unim}.$$

The Cauchy stress in the grown state is then given by

$$\mathbf{T}(\mathbf{y}) = -p_y(\mathbf{y})\mathbf{I} + \widehat{\mathbf{T}}_y(\mathbf{I}, \mathbf{y}) = -p(\mathbf{x})\mathbf{I} + \widehat{\mathbf{T}}\big(\nabla\mathbf{y}(\mathbf{x})\mathbf{G}^{-1}(\mathbf{x}), \mathbf{x}\big),$$

where $\mathbf{y} = \mathbf{y}(\mathbf{x})$.

Continuous growth of an incompressible body can be described by a time-dependent growth tensor $\mathbf{G}(\mathbf{x}, t)$ and a time-dependent total deformation $\mathbf{y}(\mathbf{x}, t)$. For such a growth process the Cauchy stress becomes

$$\mathbf{T}(\mathbf{y}, t) = -p(\mathbf{x}, t)\mathbf{I} + \widehat{\mathbf{T}}\big(\nabla\mathbf{y}(\mathbf{x}, t)\mathbf{G}^{-1}(\mathbf{x}, t), \mathbf{x}\big), \tag{20}$$

where $\mathbf{y} = y(\mathbf{x}, t)$.

5. Growth of a spherical shell

In this section we examine the spherically symmetric growth of a spherical shell which, in the initial configuration, is composed of a stress-free, homogeneous, isotropic, incompressible, elastic material. The shell remains unloaded during the growth process. Here we investigate the generation of residual stresses by growth.

The initial spherical configuration Ω is defined, in spherical coordinates $(R$, Φ, $\Theta)$, by

$$A \geqslant R \geqslant B, \qquad 0 \geqslant \Phi \geqslant \pi, \qquad 0 \geqslant \Theta \geqslant 2\pi,$$

where A and B are the inner and outer radii of the shell, respectively. We consider the growth to be spherically symmetric and continuous, so in spherical coordinates the growth tensor $\mathbf{G}(R, t)$ has the component form

$$\mathbf{G}(R, t) = \begin{pmatrix} G_1(R, t) & 0 & 0 \\ 0 & G_2(R, t) & 0 \\ 0 & 0 & G_2(R, t) \end{pmatrix}. \tag{21}$$

The smooth growth functions G_1 and G_2, which represent the radial and circumferential growth of the shell, respectively, must satisfy the initial conditions

$$G_1(R, 0) = G_2(R, 0) = 1. \tag{22}$$

When the value of a growth function is greater than 1, it represents growth; a value less than 1 represents resorption [9], which is the removal of material. Following the nomenclature introduced by Hoger [9], we will refer to the case where $G_1 > 1$ and $G_2 = 1$ as fiber growth in the radial direction; the case where $G_1 = 1$ and $G_2 > 1$ will be termed area growth perpendicular to the radial direction; and the case where $G_1 = G_2 > 1$ is called isotropic growth.

Because we have restricted our attention to the situation where the spherical shell grows into a spherical shell, the total deformation $\mathbf{y}(\mathbf{x}, t)$ associated with the growth function is spherically symmetric, and its components in spherical coordinates are given by

$$r = r(R, t), \qquad \phi = \Phi, \qquad \theta = \Theta,$$

where the radial deformation $r(R, t)$ is a smooth function satisfying

$$r(R, 0) = R. \tag{23}$$

With a prime denoting the derivative with respect to R, the corresponding deformation gradient $\nabla\mathbf{y}(\mathbf{x}, t)$ has the component form

$$\nabla\mathbf{y}(\mathbf{x}, t) = \begin{pmatrix} r'(R,t) & 0 & 0 \\ 0 & \dfrac{r(R,t)}{R} & 0 \\ 0 & 0 & \dfrac{r(R,t)}{R} \end{pmatrix}. \tag{24}$$

Substitution of (21) and (24) into the incompressibility condition (19) yields

$$G_1 G_2^2 = \frac{r^2 r'}{R^2}.$$

This equation can be integrated to give the deformation $r(R, t)$ in terms of the growth functions $G_1(R, t)$ and $G_2(R, t)$:

$$r(R,t) = \left[a^3(t) + \int_A^R 3\xi^2 G_1(\xi, t) G_2^2(\xi, t)\,\mathrm{d}\xi\right]^{1/3}, \tag{25}$$

where

$$a(t) \equiv r(A, t)$$

is the radial position of the inner surface during the growth process.

Recall that we assumed the material in the initial configuration Ω to be homogeneous and isotropic. Thus, the response function $\widehat{\mathbf{T}}(F, x)$ has the following representation (see [10, Section 49]):

$$\widehat{\mathbf{T}}(\mathbf{F}, \mathbf{x}) = f_1(I_1, I_2)\mathbf{V} + f_2(I_1, I_2)\mathbf{V}^2, \tag{26}$$

where $\mathbf{V} \equiv (\mathbf{F}\mathbf{F}^{\mathrm{T}})^{1/2}$, I_1 and I_2 are the first and second principal invariants of $\mathbf{V}$, and f_1 and f_2 are two scalar functions of the principal invariants. By substituting (26) into (20), we obtain the Cauchy stress

$$\mathbf{T} = -p\mathbf{I} + f_1(I_1, I_2)\mathbf{V} + f_2(I_1, I_2)\mathbf{V}^2.$$

For the total deformation gradient (24) and the growth tensor (21), we have

$$\mathbf{V} = \begin{pmatrix} \dfrac{r'}{G_1} & 0 & 0 \\ 0 & \dfrac{r}{RG_2} & 0 \\ 0 & 0 & \dfrac{r}{RG_2} \end{pmatrix},$$

$$I_1 = \frac{r'}{G_1} + \frac{2r}{RG_2}, \qquad I_2 = \frac{r^2}{R^2 G_2^2} + \frac{2rr'}{RG_1G_2}. \tag{27}$$

The material is assumed to satisfy the Baker–Ericksen inequality, which asserts that the greater principal stress occurs in the direction of the greater principal stretch. For the present problem, the Baker–Ericksen inequality states

$$f_1 + (v_1 + v_2) f_2 > 0 \quad \text{if } v_1 \neq v_2, \tag{28}$$

where the principal stretches v_1 and v_2 are given by

$$v_1 = \frac{r'}{G_1}, \qquad v_2 = \frac{r}{RG_2}. \tag{29}$$

In the absence of body forces, the only nontrivial component of the equation of equilibrium is

$$\frac{\partial T_1}{\partial r} + \frac{2(T_1 - T_2)}{r} = 0, \tag{30}$$

where T_1 and T_2 are the radial and circumferential stresses, respectively, given by

$$T_1 = -p + v_1 f_1 + v_1^2 f_2, \qquad T_2 = -p + v_2 f_1 + v_2^2 f_2. \tag{31}$$

We shall examine the case where the spherical shell is unloaded during the growth, so the boundary conditions are

$$T_1\big(r(A,t),t\big) = 0, \qquad T_1\big(r(B,t),t\big) = 0. \tag{32}$$

Equation (30) can be written, with the help of (31), as

$$T_1' = \frac{2(v_2 - v_1)r'}{r}\big[f_1 + (v_1 + v_2) f_2\big],$$

which can be integrated to yield

$$\begin{aligned} T_1\big(r(R,t),t\big) \; = \; & \int_A^R \frac{2[v_2(\xi,t) - v_1(\xi,t)]r'(\xi,t)}{r(\xi,t)} \Big\{ f_1\big(I_1(\xi,t), I_2(\xi,t)\big) \\ & + \big[v_1(\xi,t) + v_2(\xi,t)\big] f_2\big(I_1(\xi,t), I_2(\xi,t)\big)\Big\} \mathrm{d}\xi. \end{aligned} \tag{33}$$

This solution satisfies the boundary condition $(32)_1$ on the inner surface. It also follows from (31) that

$$\begin{aligned} T_2\big(r(R,t),t\big) \; = \; & T_1\big(r(R,t),t\big) + \big[v_2(R,t) - v_1(R,t)\big] \\ & \times\Big\{ f_1\big(I_1(R,t), I_2(R,t)\big) \\ & + \big[v_1(R,t) + v_2(R,t)\big] f_2\big(I_1(R,t), I_2(R,t)\big)\Big\}. \end{aligned} \tag{34}$$

Equations (33) and (34) give the stresses in the growing shell as a function of time. Since the shell is unloaded, these equations provide the residual stress due solely to the growth that evolves during the growth process.

A special case of isotropic growth is when the growth functions are uniform: $G_1(R,t) = G_2(R,t) = G(t)$. In this case, it is straightforward to verify that the solution is given by

$$r(R,t) = RG(t), \qquad I_1 = I_2 = 3, \qquad T_1 = T_2 = 0.$$

That is, this uniform isotropic growth gives rise to a pure dilatation of the shell and produces no residual stress.

For general growth functions, it is not possible to go any further than (33) and (34) in the determination of the residual stress without specifying the form of the constitutive functions f_1 and f_2. However, it is possible to derive explicit expressions for the time derivatives of the components of the residual stress in the initial state. This provides useful information on the residual stress distribution for configurations in which the shell has grown slightly.

First, observe from (22), (23) and (29) that

$$v_1(R,0) = v_2(R,0) = 1. \tag{35}$$

This implies, by (33) and (34), that

$$T_1(R,0) = T_2(R,0) = 0,$$

which simply states that initially the material is in its natural, or stress-free, state.

We shall use the following notation for the time derivative of a function $F(R,t)$ in the initial state:

$$\dot{F}(R) \equiv \left.\frac{\partial F(R,t)}{\partial t}\right|_{t=0}.$$

By differentiating (25) and applying (22) and (23), we find that

$$\begin{aligned} \dot{r}(R) &= \frac{A^2\dot{a}}{R^2} + \frac{1}{R^2}\int_A^R \xi^2\big[\dot{G}_1(\xi) + 2\dot{G}_2(\xi)\big]\mathrm{d}\xi, \\ \dot{r}'(R) &= \dot{G}_1(R) + 2\dot{G}_2(R) - \frac{2A^2\dot{a}}{R^3} - \frac{2}{R^3}\int_A^R \xi^2\big[\dot{G}_1(\xi) + 2\dot{G}_2(\xi)\big]\mathrm{d}\xi. \end{aligned}$$

Differentiation of (29), together with (22) and (23), gives

$$\begin{aligned} \dot{v}_1(R) &= \dot{r}'(R) - \dot{G}_1(R) \\ &= 2\dot{G}_2(R) - \frac{2A^2\dot{a}}{R^3} - \frac{2}{R^3}\int_A^R \xi^2\big[\dot{G}_1(\xi) + 2\dot{G}_2(\xi)\big]\mathrm{d}\xi, \end{aligned} \tag{36}$$

and

$$\begin{aligned} \dot{v}_2(R) &= \frac{\dot{r}(R)}{R} - \dot{G}_2(R) \\ &= -\dot{G}_2(R) + \frac{A^2\dot{a}}{R^3} + \frac{1}{R^3}\int_A^R \xi^2\big[\dot{G}_1(\xi) + 2\dot{G}_2(\xi)\big]\mathrm{d}\xi. \end{aligned} \tag{37}$$

Further, by (22), (23), (27), (33) and (35), we have

$$\dot{T}_1(R) = 2(f_1 + 2f_2)\int_A^R \frac{\dot{v}_2(\xi) - \dot{v}_1(\xi)}{\xi}\,\mathrm{d}\xi. \tag{38}$$

Because we are evaluating the derivatives of the stress in the initial configuration, the functions f_1 and f_2 are evaluated at $(I_1, I_2) = (3, 3)$. By substituting (36) and (37) into (38) and evaluating one integral after changing the order of integration, we find that

$$\begin{aligned}\dot{T}_1(R) \;=\; & \frac{2(f_1 + 2f_2)}{R^3}\left\{\frac{R^3 - A^3}{A}\dot{a}\right.\\ & \left. + \int_A^R \left[\frac{R^3 - \xi^3}{\xi}\dot{G}_1(\xi) - \frac{R^3 + 2\xi^3}{\xi}\dot{G}_2(\xi)\right]\mathrm{d}\xi\right\}.\end{aligned} \tag{39}$$

The initial velocity $\dot{a}$ of the inner surface appearing in (39) can be determined by using the remaining traction free boundary condition $(32)_2$ on the outer surface. Indeed, evaluation of (39) at $R = B$ and use of $(32)_2$ give

$$\dot{a} = -\frac{A}{B^3 - A^3}\int_A^B \left[\frac{B^3 - \xi^3}{\xi}\dot{G}_1(\xi) - \frac{B^3 + 2\xi^3}{\xi}\dot{G}_2(\xi)\right]\mathrm{d}\xi. \tag{40}$$

By substituting (40) back to (39) and simplifying the result, we arrive at

$$\begin{aligned}\dot{T}_1(R) \;=\; & -\frac{2(f_1 + 2f_2)}{(B^3 - A^3)R^3}\left\{\left(B^3 - R^3\right)\right.\\ & \times \int_A^R \left[\frac{\xi^3 - A^3}{\xi}\dot{G}_1(\xi) + \frac{A^3 + 2\xi^3}{\xi}\dot{G}_2(\xi)\right]\mathrm{d}\xi\\ & \left. + \left(R^3 - A^3\right)\int_R^B \left[\frac{B^3 - \xi^3}{\xi}\dot{G}_1(\xi) - \frac{B^3 + 2\xi^3}{\xi}\dot{G}_2(\xi)\right]\mathrm{d}\xi\right\}.\end{aligned} \tag{41}$$

Finally, differentiation of (34), together with (22), (23), (27), (35)–(37), (40) and (41), yields

$$\begin{aligned}\dot{T}_2(R) \;=\; & \dot{T}_1(R) + (f_1 + 2f_2)\left[\dot{v}_2(R) - \dot{v}_1(R)\right]\\ =\; & \frac{f_1 + 2f_2}{(B^3 - A^3)R^3}\left\{\left(B^3 + 2R^3\right)\right.\\ & \times \int_A^R \left[\frac{\xi^3 - A^3}{\xi}\dot{G}_1(\xi) + \frac{A^3 + 2\xi^3}{\xi}\dot{G}_2(\xi)\right]\mathrm{d}\xi\\ & \left. - \left(A^3 + 2R^3\right)\int_R^B \left[\frac{B^3 - \xi^3}{\xi}\dot{G}_1(\xi) - \frac{B^3 + 2\xi^3}{\xi}\dot{G}_2(\xi)\right]\mathrm{d}\xi\right\}\\ & -3(f_1 + 2f_2)\dot{G}_2(R).\end{aligned} \tag{42}$$

Equations (41) and (42) give the residual stress rates in the initial state in terms of the growth rates. Since the shell is stress free in the initial state, these equations also give the first order approximations of the residual stress components that result from the growth. Some interesting observations can be made based on these equations.

Firstly, it is straight forward to verify that the condition that $\dot{G}_1$ and $\dot{G}_2$ are equal to a same constant implies that $\dot{T}_1 = \dot{T}_2 = 0$. This confirms the observation made earlier that a uniform growth results in no residual stress.

Secondly, the radial residual stress rate $\dot{T}_1$ depends on the growth rate "globally" in the sense that the values of $\dot{T}_1$ depends on the values of $\dot{G}_1$ and $\dot{G}_2$ in the entire thickness of the shell. On the other hand, for the circumferential stress rate $\dot{T}_2$, besides a global dependence, there is also a "local" dependence on $\dot{G}_2$, as indicated by the last term in (42). This local dependence is due to the circumferential constraint provided by the continuity condition.

Thirdly, we observe from (28) and (35) that the factor f_1+2f_2 appearing in (41) and (42) is positive. It then follows from (41) that any radial fiber growth (in which $\dot{G}_1 > 0$, $\dot{G}_2 = 0$) will produce a compressive radial residual stress component in the entire shell, while a radial fiber resorption will result in a tensile radial residual stress component. This observation seems to be consistent with our intuition that growth leads to compressive residual stresses when the material is constrained in the growth direction in some way. On the other hand, it is observed from (42) that a fiber growth can lead to either tensile or compressive circumferential residual stresses, depending on the distribution of $\dot{G}_1$.

Finally, the effects of area growths on the residual stress are somewhat different, and perhaps more interesting. As can be seen from (41), an area growth perpendicular to the radius (in which $\dot{G}_2 > 0$, $\dot{G}_1 = 0$) can lead to either tensile or compressive radial residual stresses, depending again on the distribution of $\dot{G}_2$. On the other hand, equation (42) shows that the local contribution of an area growth to the circumferential residual stress is compressive, while the global contribution is tensile. This also is consistent with our intuition: consider a given spherical surface in the spherical shell. If an area growth takes place in this surface but nowhere else, a compressive circumferential residual stress will develop in the surface, while the rest of the shell will be in tension circumferentially. If, on the other hand, an area growth takes place everywhere but this spherical surface, the surface will be in tension while the rest of the shell in compression.

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