International Journal of Algebra, Vol. 9, 2015, no. 4, 195 - 201
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2015.5312

Toric Rings of Nonsimple Polyominoes

Akihiro Shikama

Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University, Toyonaka, Osaka 560-0043, Japan

Abstract

It is known that toric ring of a simple polyomino is ring homomorphic to a edge ring of a weakly chordal bipartite graph. In this paper we identify the attached toric rings of nonsimple polyominoes which are of the form “rectangle minus rectangle”.

Mathematics Subject Classification: 13C05, 05E40

Keywords: polyominoes, toric ideals

Introduction

Polyominoes are two dimensional objects which are originally rooted in recreational mathematics and combinatorics. They have been widely discussed in connection with tiling problems of the plane. Typically, a polyomino is plane figure obtained by joining squares of equal sizes, which are known as cells. In connection with commutative algebra, polyominoes are first discussed in [5] by assigning each polyomino the ideal of inner 2-minors or the *polyomino ideal*. The study of ideal of t-minors of an $m \times n$ matrix is a classical subject in commutative algebra. The class of polyomino ideal widely generalizes the class of ideals of 2-minors of $m \times n$ matrix as well ass the ideal of inner 2-minors attached to a two sided ladder.

Let $\mathcal{P}$ be a polyomino and K be a field. We denote by $I_{\mathcal{P}}$, the polyomino ideal attatched to $\mathcal{P}$, in a suitable polynomial ring over K. The residue class ring defined by $I_{\mathcal{P}}$ denoted by $K[\mathcal{P}]$. It is natural to investigate the algebraic properties of $I_{\mathcal{P}}$ depending on shape of $\mathcal{P}$. The classes of polyominoes whose polyomino ideal is prime has been discussed in many papers. In [5] it is shown that polyomino ideal of convex polyominoes are prime. In [3] they introduced the class of *balanced* polyominoes and proved that polyomino ideal of balanced polyominoes are prime. Later in [4] it is shown that simple polyominoes are balanced. In [6], independently from [4], it is proved that polyomino ideal of simple polyominoes are prime by identifying them with toric rings of edge rings of graphs. Very recently, in [2] it is shown that polyomino ideal of polyomino of the shape "rectangle minus convex" is prime by using localization argument on the attached rings of those polyominoses.

In this paper, we identify the toric rings of polyomino ideals for the special class of nonsimple polyominoes, "rectangle minus rectangle".

1. Toric rings of nonsimple polyominoes

First we recall some definitions and notation from [5]. Given $a = (i, j)$ and $b = (k, l)$ in $\mathbb{N}^2$ we write $a \leq b$ if $i \leq k$ and $j \leq l$. The set $[a, b] = \{c \in \mathbb{N}^2 \colon\ a \leq c \leq b\}$ is called an *interval*. If $i < k$ and $j < l$, then the elements a and b are called *diagonal* corners and (i, l) and (k, j) are called *anti-diagonal* corners of $[a, b]$. An interval of the from $C = [a, a + (1, 1)]$ is called a *cell* (with left lower corner a). The elements (corners) $a, a + (0, 1), a + (1, 0), a + (1, 1)$ of $[a, a + (1, 1)]$ are called the *vertices* of C. The sets $\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}$ and $\{a + (0, 1), a + (1, 1)\}$ are called the *edges* of C. We denote the set of edge of C by $E(C)$.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$. The the vertex set of $\mathcal{P}$, denoted by $V(\mathcal{P})$ is given by $V(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} V(C)$. For each vertex v in $V(\mathcal{P})$, we write $v = (v_1, v_2)$ where v_1 is the first and v_2 is the second coodinate of v. The edge set of $\mathcal{P}$, denoted by $E(\mathcal{P})$ is given by $E(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} E(C)$. Let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected*, if there is a sequence of cells $\mathcal{C} : C = C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m - 1$. If in addition, $C_i \neq C_j$ for all $i \neq j$, then $\mathcal{C}$ is called a *path* (connecting C and D). The collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected, see Figure 1.

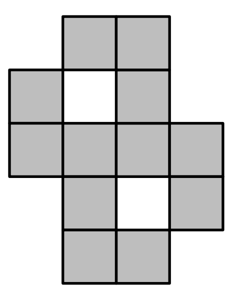

Figure 1. polyomino

Let $\mathcal{P}$ be a polyomino, and let K be a field. We denote by S the polynomial ring over K with variables x_v with $v \in V(\mathcal{P})$. We write for $v = (i, j)$, we may write $x_v = x_{ij}$ if needed. Following [5] a 2-minor $x_{ij}x_{kl} - x_{il}x_{kj} \in S$ is called an *inner minor* of $\mathcal{P}$ if all the cells $[(r,s),(r+1,s+1)]$ with $i \le r \le k-1$ and $j \le s \le l-1$ belong to $\mathcal{P}$. In that case the interval $[(i,j),(k,l)]$ is called an *inner interval* of $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset S$ generated by all inner minors of $\mathcal{P}$ is called the *polyomino ideal* of $\mathcal{P}$. We also set $K[\mathcal{P}] = S/I_{\mathcal{P}}$.

For each interval $[a,b]$ we regard the polyomino $\mathcal{P}_{(a,b)}$ in the obvious way. We give Theorem 1.2 that identifies a toric ring for some specified cases of nonsimple polyominoes which is obtained by subtracting a rectangle from a bigger rectangle.

Hereafter let $\mathcal{P}$ be a polyomino with $\mathcal{P} = \mathcal{P}_{(a,b)} \backslash \mathcal{P}_{(a'b')}$ with $(a < a' < b' < b)$. We define a map $\mu : V(\mathcal{P}) \to \mathbb{Z}$ as follows. For $v = (v_1, v_2)$ in $\mathcal{P}$,

$$\mu(v) = \begin{cases} 1 \ (for\ a_1 \le v_1 \le a'_1, a_2 \le v_2 \le a'_2) \\ 1 \ (for\ b'_1 \le v_1 \le b_1, b'_2 \le v_2 \le b_2) \\ 2 \ (for\ a_1 \le v_1 \le a'_1, b'_2 \le v_2 \le b_2) \\ 2 \ (for\ b'_1 \le v_1 \le b_1, a_2 \le v_2 \le a'_2) \\ v_1 - a'_1 + 2 \ (for\ a'_1 < v_1 < b'_1, a_2 \le v_2 \le b'_2) \\ v_2 - a'_2 + (b'_1 - a'_1) + 1 \\ \quad (for\ b'_1 \le v_1 \le b_1, a'_2 < v_2 < b'_2) \\ b'_1 - v_1 + (b'_1 - a'_1) + (b'_2 - a'_2) \\ \quad (for\ a'_1 < v_1 < b'_1, b'_2 \le v_2 \le b_2) \\ b'_2 - v_2 + 2(b'_1 - a'_1) + (b'_2 - a'_2) - 1 \\ \quad (for\ a_1 \le v_1 \le a'_1, a'_2 < v_2 < b'_2). \end{cases}$$

We call $\mu(v)$ the *labelling* of v.

Example 1.1. for $a = (1,1)$ $b = (7,5)$, $a' = (2,2)$ and $b' = (5,4)$, the labelling of vertices are given as follows.

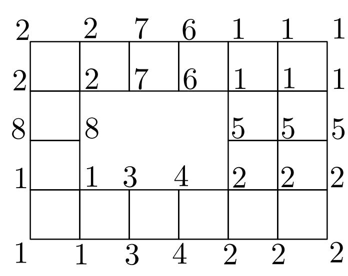

FIGURE 2. labelling of $\mathcal{P}$

We put 1 or 2 for each corners and $3, 4, \ldots$ for the other vertices anti-clockwisely.

Let $T = K[r_{v_1} s_{v_2} t_{\mu(v)} \mid v \in V(\mathcal{P})]$ be the subring of a polynomial ring $K[r_{a_1}, \ldots r_{b_1}, s_{a_2}, \ldots s_{b_2}, t_1 \ldots t_M]$ where $M = \max\{\mu(v) \mid v \in V(\mathcal{P})\}$. We define the surjective ring homomorphism $\varphi : K[\mathcal{P}] \to T$ by setting $\varphi(x_v) =$

$r_{v_1}s_{v_2}t_{\mu(v)}$. For the sake of convenience, we write $\varphi(v) = r_{v_1}s_{v_2}t_{\mu(v)} = r_v s_v t_v$ in case it does not make readers confused. We call the *toric ideal* $J_{\mathcal{P}} = \ker\varphi$

Theorem 1.2. *Let $\mathcal{P} = \mathcal{P}_{(a,b)} \setminus \mathcal{P}_{(a'b')}$ with $(a < a' < b' < b)$. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

Proof. First it is easy to see that $I_{\mathcal{P}} \subset \ker\varphi$. We show that every quadratic binomials belong to $J_{\mathcal{P}}$ is a generator of $I_{\mathcal{P}}$, namely an inner 2-minor in $\mathcal{P}$ and every binomials of higher degree belongs to $I_{\mathcal{P}}$.

First, suppose that $x_{v_1}x_{v_2} - x_{v_3}x_{v_4}$ is a quadratic binomial belongs to $J_{\mathcal{P}}$ with $x_{v_1}x_{v_2} \neq x_{v_3}x_{v_4}$. Then either $(r_{v_1}, r_{v_2}, s_{v_1}s_{v_2}) = (r_{v_3}, r_{v_4}, s_{v_4}s_{v_3})$ or $(r_{v_1}, r_{v_2}, s_{v_1}s_{v_2}) = (r_{v_4}, r_{v_3}, s_{v_3}s_{v_4})$ holds. Thus, the quadratic binomial $x_{v_1}x_{v_2} - x_{v_3}x_{v_4}$ is associated with an interval. Now we show that the interval is an inner interval in $\mathcal{P}$. Assume that the interval spanned by $x_{v_1}x_{v_2} - x_{v_3}x_{v_4}$ has v_1 and v_2 as diagonal corners. Also, we may assume that$r_{v_1} = r_{v_3}$ and $s_{v_1} = s_{v_4}$ hold.

Suppose $[v_1, v_2]$ is not an inner interval in $\mathcal{P}$. Then $\mathcal{P}_{[a',b']} \subset \mathcal{P}_{[v_1,v_2]}$ can not be happened because otherwise, from our definition of μ, one has $t_1^2 = t_{v_1}t_{v_2} \neq t_{v_3}t_{v_4} = t_2^2$. Therefore we may assume that one of $[v_1, v_3]$, $[v_1, v_4]$, $[v_3, v_2]$ or $[v_4, v_2]$ is not an inner interval in $\mathcal{P}$. Then from our definition of labelling we can not have $t_{v_1}t_{v_2} = t_{v_3}t_{v_4}$. Hence every degree 2 binomial in $J_{\mathcal{P}}$ is a generator of $I_{\mathcal{P}}$.

Now we show that each binomial of higher degree belongs to $I_{\mathcal{P}}$. Let $f = f^{(+)} - f^{(-)} \in \ker\varphi$ be a homogeneous binomial. Let $V_+ = v_1 \ldots v_s$ be the set of vertices such that x_{v_i} appear in $f^{(+)}$ and $V_- = u_1 \ldots x_{u_t}$ the vertices appear in $f^{(-)}$.

We may assume that $V_+ \cap V_- = \emptyset$, otherwise, say if $v_1 = u_1$ holds, we have $f = x_{v_1}(f^{(+)}/x_{v_1} - f^{(-)}/x_{u_1})$ and we can check $f' = f^{(+)}/x_{v_1} - f^{(-)}/x_{u_1}$ instead of f to complete the proof.

Now we show that if we have $v \in V_+ \cup V_-$ such that $\mu(v) \notin \{1, 2\}$, then we are done in this case. Assume that we have a vertex u_1 in V_- such that $u_{1_1} \leq a'_1$ and $\mu(v) \neq 1, 2$. Since f belongs to the kernel, we have 2 vertices, say v_1 and v_2 in V_+ such that $s_{v_1}t_{v_1}|\varphi(x_{u_1})$ and $r_{v_1}|\varphi(x_{u_1})$. Let c be the rest corner of the interval spanned by v_1, v_2 and u_1. Our rule of labelling show that the interval is an inner interval in $\mathcal{P}$ See Figure 3 to see the situation. We obtain

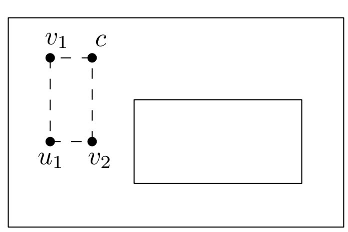

FIGURE 3

$$
\begin{aligned}
f &= f^{(+)} - f^{(-)} \\
&= x_{v_1}x_{v_2}\frac{f^{(+)}}{x_{v_1}x_{v_2}} - x_c\frac{f^{(-)}}{x_c} \\
&= (x_{v_1}x_{v_2} - x_{u_1}x_c)\frac{f^{(+)}}{x_{v_1}x_{v_2}} + x_{u_1}x_c\frac{f^{(+)}}{x_{v_1}x_{v_2}} - x_c\frac{f^{(-)}}{x_c} \\
&= (x_{v_1}x_{v_2} - x_{u_1}x_c)\frac{f^{(+)}}{x_{v_1}x_{v_2}} + x_c\left(x_{u_1}\frac{f^{(+)}}{x_{v_1}x_{v_2}} - \frac{f^{(-)}}{x_c}\right)
\end{aligned}
$$

Since $x_{v_1}x_{v_2} - x_{u_1}x_c$ is an inner minor of $\mathcal{P}$, we have $x_{u_1}(f^{(+)}/x_{v_1}x_{v_2}) - (f^{(-)}/x_c) \in J_{\mathcal{P}}$, we can apply induction on the degree of f to complete the proof.

Now we assume every vertex in $V_{(+)}$ and $V_{(-)}$ are labelled 1 or 2. For $u_1 \in V_{(-)}$, we can find a vertex v_1 and v_2 in V_+ where $r_{u_1} = r_{v_1}$ and $s_{u_1} = s_{v_2}$. If $t_{u_1} = t_{v_1}$ or $t_{u_1} = t_{v_2}$, then using the same formula as above, we can apply induction. Now we assume that $t_{u_1} = t_1$ and $t_{v_1} = t_{v_2} = t_2$. By using similar argument, one has $u_2 \in V_-$ such that $s_{u_2} = s_{v_1}$. Since $\varphi(f^{(+)}) = \varphi(f^{(-)})$ holds, we have a vertex $u_2 \in V_-$ such that $t_{u_2} = t_1$. See Figure 4 for this situation. Then the interval spanned by u_1 and u_3 is an inner minor of $\mathcal{P}$. Let c, d be the

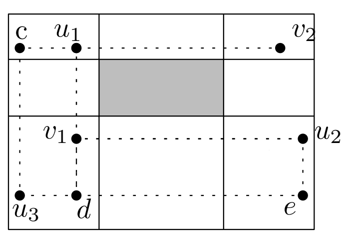

FIGURE 4

other corners of this interval. Then,

$$
\begin{aligned}
f &= f^{(+)} - f^{(-)} \\
&= f^{(+)} - x_{u_1}x_{u_3}\frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
&= f^{(+)} - x_c x_d\frac{f^{(-)}}{x_{u_1}x_{u_3}} - (x_{u_1}x_{u_3} - x_c x_d)\frac{f^{(-)}}{x_{u_1}x_{u_3}}
\end{aligned}
$$

Notice that one of $t_c = t_1$ or $t_d = t_1$ holds. Assume $t_d = t_1$. Then the interval spanned by u_2 and d has v_1 as a corner vertex. Suppose that e is the rest corner vertex of this interval. Then, we have

$$\begin{aligned}
f =& f^{(+)} - x_c x_d \frac{f^{(-)}}{x_{u_1}x_{u_3}} - (x_{u_1}x_{u_3} - x_c x_d)\frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
=& x_{v_1}\frac{f^{(+)}}{x_{v_1}} - x_c x_d x_{u_2}\frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}} - (x_{u_1}x_{u_3} - x_c x_d)\frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
=& x_{v_1}\frac{f^{(+)}}{x_{v_1}} - x_{v_1}x_c x_e \frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}} - (x_d x_{u_2} - x_{v_1}x_e)\left(x_c\frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}}\right) \\
& - (x_{u_1}x_{u_3} - x_c x_d)\frac{f^{(-)}}{x_{u_1}x_{u_3}} \\
=& x_{v_1}\left(\frac{f^{(+)}}{x_{v_1}} - x_c x_e\frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}}\right) - (x_d x_{u_2} - x_{v_1}x_e)\left(x_c\frac{f^{(-)}}{x_{u_1}x_{u_2}x_{u_3}}\right) \\
& - (x_{u_1}x_{u_3} - x_c x_d)\frac{f^{(-)}}{x_{u_1}x_{u_3}}.
\end{aligned}$$

Since $(x_d x_{u_2} - x_{v_1}x_e)$ and $(x_{u_1}x_{u_3} - x_c x_d)$ are inner minors of $\mathcal{P}$ and since degree of $f^{(+)}/x_{v_1} - x_c x_e f^{(-)}/x_{u_1}x_{u_2}x_{u_3}$ is less than degree of f, we can apply induction on degree of f to complete the proof. □

Example 1.3. A toric ring of the polyomino given in Example 1.1 is identified by Theorem 1.2 as follows.

$$S/I_{\mathcal{P}} \cong K\begin{bmatrix}
r_1s_5t_2 & r_2s_5t_2 & r_3s_5t_7 & r_4s_5t_6 & r_5s_5t_1 & r_6s_5t_1 & r_7s_5t_1 \\
r_1s_4t_2 & r_2s_4t_2 & r_3s_4t_7 & r_4s_4t_6 & r_5s_4t_1 & r_6s_4t_1 & r_7s_4t_1 \\
r_1s_3t_8 & r_2s_3t_8 & & & r_5s_3t_5 & r_6s_3t_5 & r_7s_3t_5 \\
r_1s_2t_1 & r_2s_2t_1 & r_3s_2t_3 & r_4s_2t_4 & r_5s_2t_2 & r_6s_3t_2 & r_7s_2t_2 \\
r_1s_1t_1 & r_2s_1t_1 & r_3s_1t_3 & r_4s_1t_4 & r_5s_1t_2 & r_6s_2t_2 & r_7s_1t_2
\end{bmatrix}.$$

In this paper we identify toric rings of polyominoes which are of the form "rectangle minus rectangle". In general it is interesting but not so easy to identify toric ring for a given binomial prime ideal. The wider class of polyominoes which can be discussed about its toric ring is "rectangle minus convex", because it is proved in [2] that their polyomino ideals are prime.

References

[1] V. Ene, J. Herzog and T. Hibi, Linearly related polyominoes, J. Alg Comb. **41**, 949-968, (2014). http://dx.doi.org/10.1007/s10801-014-0560-3

[2] T. Hibi and A. A. Qureshi, Nonsimple polyominoes and prime ideals, arXiv: 1502.03669[math.AC].

[3] J. Herzog, A. A. Qureshi and A. Shikama, Gröbner basis of balanced polyominoes, Math. Nachr. **288**, 775-783, (2015). http://dx.doi.org/10.1002/mana.201400153

[4] J. Herzog and S. S. Madani, The coordinate ring of a simple polyomino, arxiv:1408.4275v1

[5] A. A. Qureshi, Ideals generated by 2-minors, collections of cells and stack polyominoes, J. Algebra, **357**, 279-303, (2012). http://dx.doi.org/10.1016/j.jalgebra.2012.01.032

[6] A. A. Qureshi, T. Shibuta and A. Shikama, Simple polyominoes are prime, arXiv: 1501.15107[math.AC]

Received: March 17, 2015; Published: May 15, 2015