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Generalized two-scale weighted Cantor set model for solar wind turbulence

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[1] In order to quantify the multifractality of solar wind turbulence, we consider a generalized weighted Cantor set with two different scales describing nonuniform compression of cascading eddies. We investigate the resulting multifractal spectrum of generalized dimensions depending on two scaling parameters and one probability measure parameter, especially for asymmetric scaling. In particular, we show that intermittent pulses are stronger for the model with two different scaling parameters and a much better agreement with the solar wind data is obtained, especially for the negative index of the generalized dimensions. Therefore we argue that there is a need to use a two-scale cascade model. Hence we propose this new more general model as a useful tool for analysis of intermittent turbulence in various environments. **Citation:** Macek, W. M., and A. Szczepaniak (2008), Generalized two-scale weighted Cantor set model for solar wind turbulence, *Geophys. Res. Lett.*, *35*, L02108, doi:10.1029/2007GL032263.

1. Introduction

[2] The question of multifractality is of great importance for space plasmas because it allows us to look at intermittent turbulence in the solar wind [e.g., *Burlaga*, 1991; *Carbone*, 1993; *Carbone and Bruno*, 1996; *Marsch et al.*, 1996; *Marsch and Tu*, 1997; *Bruno et al.*, 2001]. Starting from Richardson's scenario of turbulence, many authors try to recover the observed scaling exponents, using some simple and more advanced fractal and multifractal models of turbulence describing distribution of the energy flux between cascading eddies at various scales. In particular, the multifractal spectrum has been investigated using Voyager (magnetic field fluctuations) data in the outer heliosphere [e.g., *Burlaga*, 1991; *Burlaga et al.*, 1993; *Burlaga*, 2001] and using Helios (plasma) data in the inner heliosphere [e.g., *Marsch et al.*, 1996].

[3] In general, the spectrum of generalized dimensions D_q as a function of a continuous index, $-\infty < q < \infty$ quantifies multifractality of a given system [e.g., *Ott*, 1993]. A chaotic strange attractor has been identified in the solar wind data by *Macek* [1998] as further examined by *Macek and Redaelli* [2000]. We have also considered the D_q spectrum for the solar wind attractor using a simple multifractal model with a measure of the self-similar weighted Cantor set with one parameter describing uniform compression and another parameter for the probability measure of the attractor of the system. The spectrum is found to be consistent with the data, at least for positive index q of the generalized dimensions D_q [*Macek*, 2002, 2003, 2006; *Macek et al.*, 2005, 2006]. However, the full singularity spectrum is necessary to quantify the degree of multifractality. Notwithstanding of the well-known statistical problems with negative q [*Macek*, 2006], we have recently succeeded in estimating the entire spectrum for solar wind attractor using a generalized weighted Cantor set with two different scales describing nonuniform compression [*Macek*, 2007].

[4] Therefore here, in order to further quantify the multifractality, we consider this generalized weighted Cantor set also in the context of turbulence cascade. Even though one can find the two-scale Cantor set in many classical textbooks [e.g., *Ott*, 1993], it is still difficult to understand this strange attractor that exhibits multifractality in various complex real systems, also in case of intermittent turbulence. Hence we argue that there is, in fact, need to use a two-scale cascade model. Therefore we investigate the resulting multifractal spectrum depending on two scaling parameters and one probability measure parameter, demonstrating that intermittent pulses are stronger for asymmetric scaling and a much better agreement is obtained, especially for $q < 0$. We hope that this generalized new asymmetric multifractal model could shed light on the nature of turbulence and will be a useful tool for analysis of intermittent turbulence in various environments.

2. Theoretical Model

2.1. Generalized Two-Scale Weighted Cantor Set

[5] At each stage of construction of the weighted two-scale Cantor set we basically have two scaling parameters l_1 and l_2, where $l_1 + l_2 \leq 1$, and two different weights p_1 and p_2. In order to obtain the generalized dimensions $D_q \equiv \tau(q)/(q - 1)$ for this interesting example of multifractals we use the following partition function at the n-th level of construction [*Hentschel and Procaccia*, 1983; *Halsey et al.*, 1986]

$$\Gamma_n(l_1, l_2, p_1, p_2) = \left(\frac{p_1^q}{l_1^{\tau(q)}} + \frac{p_2^q}{l_2^{\tau(q)}} \right)^n = 1 \qquad (1)$$

[6] The resulting strange attractor (of 2^n closed intervals for $n \to \infty$) is the generalized weighted two-scale Cantor set of narrow segments of various widths and probabilities. The singularity spectrum $f(\alpha) = q\alpha - \tau(q)$ as a function of

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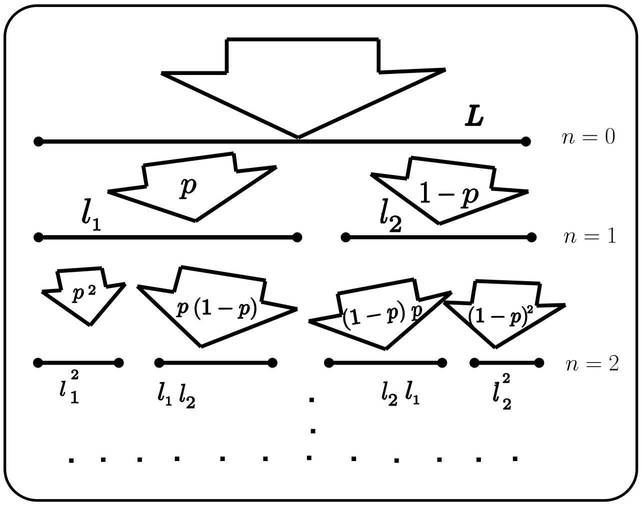

Figure 1. Generalized two-scale weighted Cantor set model for solar wind turbulence [*Macek*, 2007].

$\alpha = \tau'(q)$ could also easily be obtained by using Legendre transformation [*Ott*, 1993; *Macek*, 2006].

[7] Here we consider a standard scenario of cascading eddies, each breaking down into two new ones, but not necessarily equal and twice smaller. In particular, space filling turbulence could be recovered for $l_1 + l_2 = 1$ [*Burlaga et al.*, 1993]. Naturally, in the inertial region of the system of size L, $\eta \ll l \ll L$, we do not allow the energy to be dissipated directly, assuming $p_1 + p_2 = 1$, until the Kolmogorov scale η is reached. However, in this range at each n-th step of the binomial multiplicative process, the flux of kinetic energy density ε transferred to smaller eddies (energy transfer rate) could be divided into nonequal fractions p and $1 - p$, as schematically shown in Figure 1 [cf. *Meneveau and Sreenivasan*, 1987].

2.2. Comparison With the *P*-Model

[8] The multifractal measure [*Mandelbrot*, 1989] $\mu = \varepsilon/\langle\varepsilon_L\rangle$ (normalized) on the unit interval for (a) the usual one-scale p-model [*Meneveau and Sreenivasan*, 1987] and (b) the generalized two-scale cascade model is shown in Figure 2 ($n = 7$). It is worth noting that intermittent pulses are much stronger for the model with two different scaling parameters. In particular, for non space-filling turbulence, $l_1 + l_2 < 1$ one still could have a multifractal cascade, even for unweighted (equal) energy transfer, $p = 0.5$. Only for $l_1 = l_2 = 0.5$ and $p = 0.5$ there is no multifractality.

3. Solar Wind Data

[9] For illustration, we analyze the Helios 2 data using plasma parameters measured in situ in the inner heliosphere [*Schwenn*, 1990]. The X-velocity (mainly radial) component of the plasma flow, υ_x, has been already investigated by *Macek* [1998, 2002, 2003] and *Macek and Redaelli* [2000]. The Alfvénic fluctuations with longer (two-days) samples have been studied by *Macek* [2006, 2007] and *Macek et al.* [2005, 2006]. Now we have selected even longer (four-days) time intervals of υ_x samples in 1976 (each of 8531 data points, interpolated with sampling time of 40.5 s) for both slow and fast solar wind streams measured at various distances from the Sun.

4. Methods of Data Analysis

[10] In the inertial range the standard q-order ($q > 0$) structure function $S_u^q(l) = \langle|u(x + l) - u(x)|^q\rangle$, where $u(x + l)$ is a velocity component parallel to the longitudinal direction separated from a position x by a distance l, is scaling as $l^{\xi(q)}$. As is usual, the temporal scales can be interpreted as the spatial scales, $x = \upsilon_x t$ (Taylor's hypothesis). The transfer rate of the energy flux, ε_l, is widely estimated by $\varepsilon_l \sim S_u^3(l)/l$. Recently, limitations of this approximation are discussed by *Vasquez et al.* [2007] using power spectra, and its hydromagnetic generalization for the Alfvénic fluctuations is considered by *Sorriso-Valvo et al.* [2007].

[11] It can be argued that in some region the total probability measure should scale with the exponent $\tau(q) \equiv (q - 1)D_q$ as

$$\sum_i \mu_i^q \sim l^{\tau(q)} \qquad (2)$$

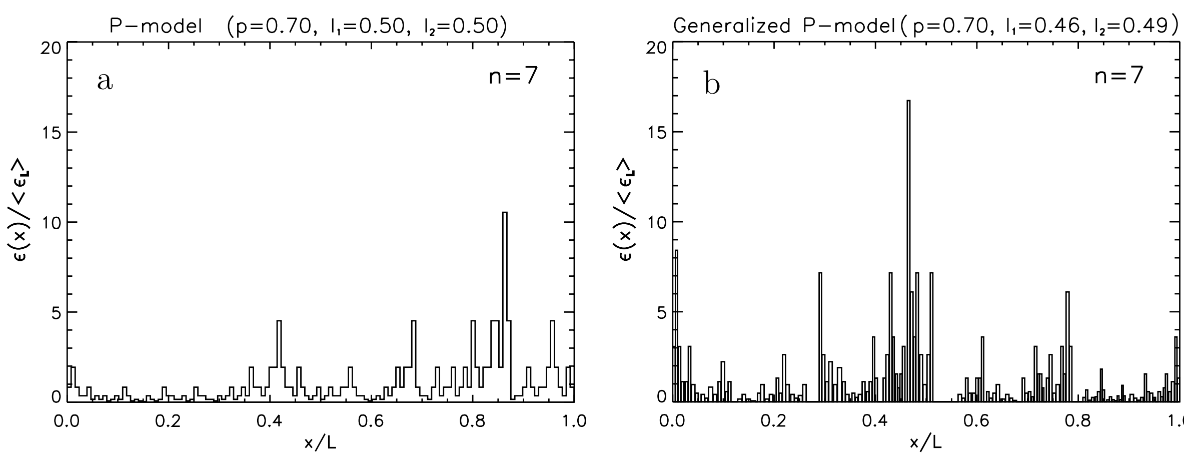

Figure 2. The multifractal measure $\mu = \varepsilon/\langle\varepsilon_L\rangle$ on the unit interval for (a) the usual one-scale p-model and (b) the generalized two-scale cascade model. Intermittent pulses are stronger for the model with two different scaling parameters.

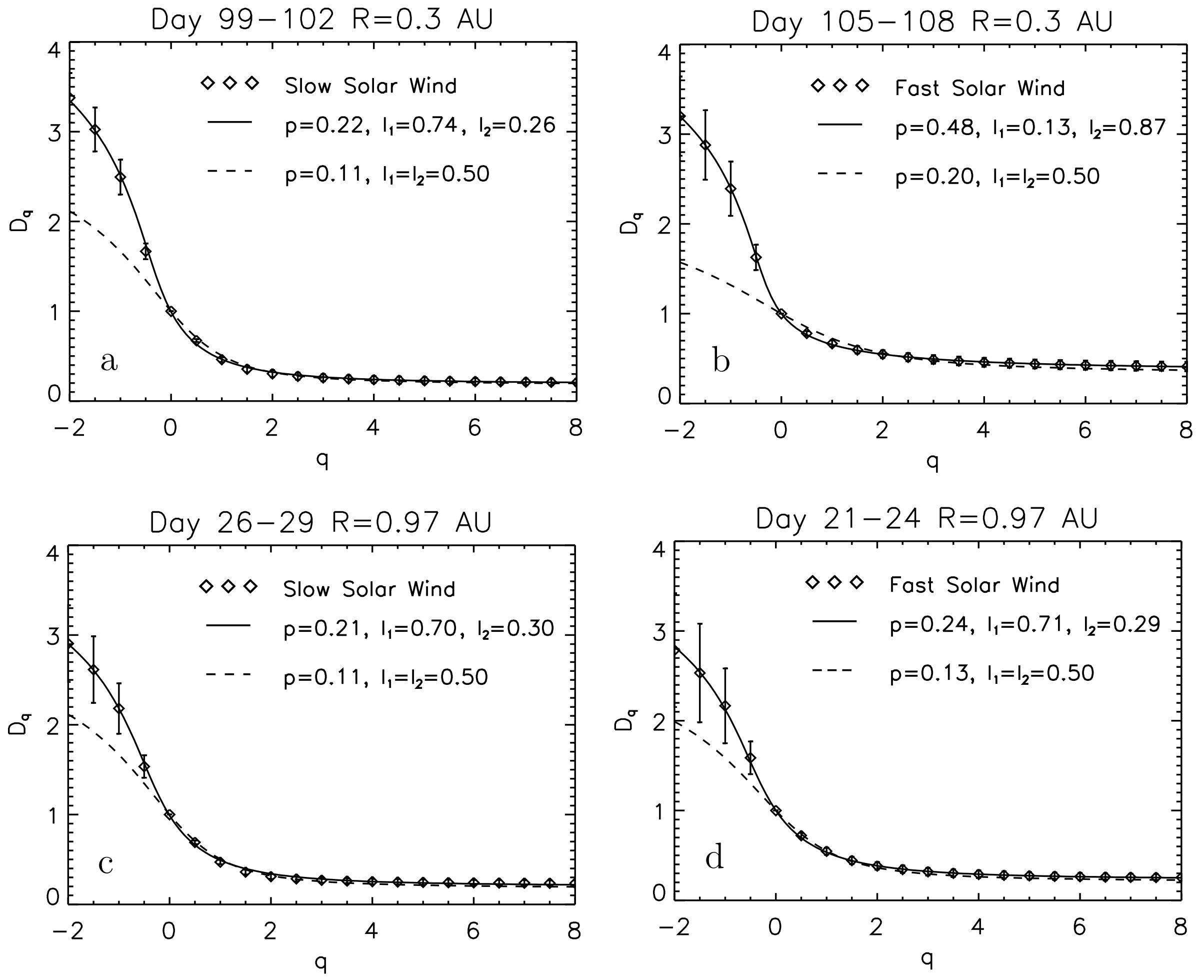

Figure 3. The generalized dimensions D_q as a function of q. The values of D_q for one-dimensional turbulence are calculated for the generalized two-scale (continuous lines) model and the usual one-scale (dashed lines) p-model and fitted using the υ_x velocity components (diamonds) for the slow (a) and (c) and fast (b) and (d) solar wind streams at distances of 0.3 AU and 0.97 AU, correspondingly.

where $\mu_i = \varepsilon_l/\langle\epsilon_L\rangle$ is the probability measure of ith eddy in the d-dimensional physical space. Here, for simplicity the third moment of structure function of velocity fluctuations is used for estimation of this measure [*Marsch et al.*, 1996]. Admittedly, the structure function scaling exponent $\xi(q)$ is easier to measure experimentally than the spectrum of dimensions $D_q \equiv \tau(q)/(q-1)$ in equation (2), which is easier to interpret theoretically, see equation (1). Surely, both have the same information about multifractality, at least for $q > 0$ [*Tsang et al.*, 2005]. However, because we are also interested in negative q, it is more convenient to use dimensions instead of structure functions.

5. Results

[12] The results for the generalized dimensions D_q as a function of q are shown in Figure 3. The values of D_q given in equation (2), for one-dimensional turbulence, $d = 1$, are calculated using the radial velocity components $u = \upsilon_x$ (in time domain) [cf. *Macek et al.*, 2005, Figure 3]. We have verified that the slopes in the scaling region are not sensitive to the number of points used [e.g., *Eckmann and Ruelle*, 1992; *Macek*, 1998, 2006, 2007]. It is well known that for $q < 0$ we have some basic statistical problems [*Macek*, 2006, 2007]. Nevertheless, in spite of large statistical errors in Figures 3a, 3b, 3c, and 3d, especially for $q < 0$, the multifractal character of the measure can still clearly be discerned. Therefore one can confirm that the spectrum of dimensions still exhibits the multifractal structure of the solar wind in the inner heliosphere.

[13] For $q \geq 0$ these results agree with the usual one-scale p-model fitted to the generalized dimensions as obtained analytically using $l_1 = l_2 = 0.5$ in equation (1) and the corresponding value of the parameter p = 0.11, 0.20, 0.11, and 0.13 for the slow (a) and (c) and fast (b) and (d) solar wind streams at distances of 0.3 AU and 0.97 AU, correspondingly, as shown by dashed lines. On the contrary, for $q < 0$ the p-model cannot describe the observational results, as noted by *Marsch et al.* [1996]. Here we show that the experimental values are consistent with the generalized dimensions obtained numerically from equation (1) for the weighted two-scale Cantor set using an asymmetric scaling, i.e., using unequal scales $l_1 \neq l_2$, as is shown in Figures 3a, 3b, 3c, and 3d by continuous lines.

[14] We see that the multifractal spectrum of the solar wind is only roughly consistent with that for the multifractal

measure of the self-similar weighted symmetric one-scale weighted Cantor set only for $q \geq 0$, as also seen from the standard structure function analysis. On the other hand, this spectrum is in a very good agreement with two-scale asymmetric weighted Cantor set schematically shown in Figure 1 for both positive and negative q. Obviously, taking two different scales for eddies in the cascade, one obtains a more general situation than in the usual p-model of *Meneveau and Sreenivasan* [1987] for fully developed turbulence, especially for an asymmetric scaling, $l_1 \neq l_2$. Hence we hope that this generalized model will be a useful tool for analysis of intermittent turbulence in space plasmas.

[15] The value of parameter p (within some factor) is related to the usual models, which are based on the p-model of turbulence [e.g., *Meneveau and Sreenivasan*, 1987]. The values of p obtained here are roughly consistent with the fitted value in the literature both for laboratory and the solar wind turbulence, which is in the range $0.13 \leq p \leq 0.3$ [e.g., *Burlaga*, 1991; *Carbone*, 1993; *Carbone and Bruno*, 1996; *Marsch et al.*, 1996].

6. Conclusions

[16] We have studied the inhomogeneous rate of the transfer of the energy flux indicating multifractal and intermittent behavior of solar wind turbulence in the inner heliosphere. In particular, we have demonstrated that intermittent pulses are stronger for the model with two different scaling parameters and a much better agreement with the real data is obtained, especially for $q < 0$. We confirm that the degree of multifractality of the solar wind in the inner heliosphere is different for slow and fast streams. Also as the heliocentric distance increases the solar wind becomes more multifractal in agreement with other studies. In particular, we observe radial evolution of multifractality (intermittency) as noticed, e.g., by *Bruno et al.* [2003].

[17] Basically, the generalized dimensions for solar wind are consistent with the generalized p-model for both positive and negative q, but rather with different scaling parameters for sizes of eddies, while the usual p-model can only reproduce the spectrum for $q \geq 0$. Thus, we also confirm the utility of the model introduced by *Burlaga et al.* [1993], using a different data set. In general, the proposed generalized two-scale weighted Cantor set model should also be valid for non space filling turbulence. Therefore we propose this cascade model describing intermittent energy transfer for analysis of turbulence in various environments.

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References

Bruno, R., V. Carbone, P. Veltri, E. Pietropaolo, and B. Bavassano (2001), Identifying intermittency events in the solar wind, *Planet. Space Sci.*, *49*, 1201–1210.

Bruno, R., V. Carbone, L. Sorriso-Valvo, and B. Bavassano (2003), Radial evolution of solar wind intermittency in the inner heliosphere, *J. Geophys. Res.*, *108*(A3), 1130, doi:10.1029/2002JA009615.

Burlaga, L. F. (1991), Multifractal structure of the interplanetary magnetic field: Voyager 2 observations near 25 AU, 1987–1988, *Geophys. Res. Lett.*, *18*, 69–72.

Burlaga, L. F. (2001), Lognormal and multifractal distributions of the heliospheric magnetic field, *J. Geophys. Res.*, *106*, 15,917–15,927.

Burlaga, L. F., J. Perko, and J. Pirraglia (1993), Cosmic-ray modulation, merged interaction regions, and multifractals, *Astrophys. J.*, *407*, 347–358.

Carbone, V. (1993), Cascade model for intermittency in fully developed magnetohydrodynamic turbulence, *Phys. Rev. Lett.*, *71*, 1546–1548.

Carbone, V., and R. Bruno (1996), Cancellation exponents and multifractal scaling laws in the solar wind magnetohydrodynamic turbulence, *Ann. Geophys.*, *14*, 777–785.

Eckmann, J.-P., and D. Ruelle (1992), Fundamental limitations for estimating dimensions and Lyapunov exponents in dynamical systems, *Physica D*, *56*, 185–187, doi:10.1016/0167-2789(92)90023-G.

Halsey, T. C., M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman (1986), Fractal measures and their singularities: The characterization of strange sets, *Phys. Rev. A*, *33*, 1141–1151.

Hentschel, H. G. E., and I. Procaccia (1983), The infinite number of generalized dimensions of fractals and strange attractors, *Physica D*, *8*, 435–444.

Macek, W. M. (1998), Testing for an attractor in the solar wind flow, *Physica D*, *122*, 254–264.

Macek, W. M. (2002), Multifractality and chaos in the solar wind, in *Experimental Chaos*, edited by S. Boccaletti et al., *AIP Conf. Proc.*, *622*, 74–79.

Macek, W. M. (2003), The multifractal spectrum for the solar wind flow, in *Solar Wind 10*, edited by R. Velli, R. Bruno, and F. Malara, *AIP Conf. Proc.*, *679*, 530–533.

Macek, W. M. (2006), Modeling multifractality of the solar wind, *Space Sci. Rev.*, *122*, 329–337, doi:10.1007/s11214-006-8185-z.

Macek, W. M. (2007), Multifractality and intermittency in the solar wind, *Nonlinear Proc. Geophys.*, *14*, 695–700.

Macek, W. M., and S. Redaelli (2000), Estimation of the entropy of the solar wind flow, *Phys. Rev. E*, *62*, 6496–6504.

Macek, W. M., R. Bruno, and G. Consolini (2005), Generalized dimensions for fluctuations in the solar wind, *Phys. Rev. E*, *72*, 017202, doi:10.1103/PhysRevE.72.017202.

Macek, W. M., R. Bruno, and G. Consolini (2006), Testing for multifractality of the slow solar wind, *Adv. Space Res.*, *37*, 461–466, doi:10.1016/j.asr.2005.06.057.

Mandelbrot, B. B. (1989), Multifractal measures, especially for the geophysicist, *Pure Appl. Geophys.*, *131*, 5–42.

Marsch, E., and C.-Y. Tu (1997), Intermittency, non-Gaussian statistics and fractal scaling of MHD fluctuations in the solar wind, *Nonlinear Proc. Geophys.*, *4*, 101–124.

Marsch, E., C.-Y. Tu, and H. Rosenbauer (1996), Multifractal scaling of the kinetic energy flux in solar wind turbulence, *Ann. Geophys.*, *14*, 259–269.

Meneveau, C., and K. R. Sreenivasan (1987), Simple multifractal cascade model for fully developed turbulence, *Phys. Rev. Lett.*, *59*, 1424–1427.

Ott, E. (1993), *Chaos in Dynamical Systems*, Cambridge Univ. Press, Cambridge, U. K.

Schwenn, R. (1990), Large-scale structure of the interplanetary medium, in *Physics of the Inner Heliosphere*, *Phys. and Chem. in Space*, vol. 20, edited by R. Schwenn and E. Marsch, pp. 99–181, Springer, Berlin.

Sorriso-Valvo, L., R. Marino, V. Carbone, F. Lepreti, P. Veltri, A. Noullez, R. Bruno, B. Bavassano, and E. Pietropaolo (2007), Observation of inertial energy cascade in interplanetary space plasma, *Phys. Rev. Lett.*, *99*, 115001, doi:10.1103/PhysRevLett.99.115001.

Tsang, Y.-K., E. Ott, T. M. Antonsen Jr., and P. N. Guzdar (2005), Intermittency in two-dimensional turbulence with drag, *Phys. Rev. E*, *71*, 066313, doi:10.1103/PhysRevE.71.066313.

Vasquez, B. J., C. W. Smith, K. Hamilton, B. T. MacBride, and R. J. Leamon (2007), Evaluation of the turbulent energy cascade rates from the upper inertial range in the solar wind at 1 AU, *J. Geophys. Res.*, *112*, A07101, doi:10.1029/2007JA012305.

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