Three-Phase Inclusions of Arbitrary Shape With Internal Uniform Hydrostatic Stresses in Finite Elasticity

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We study the internal stress field of a three-phase two-dimensional inclusion of arbitrary shape bonded to an unbounded matrix through an intermediate interphase layer when the matrix is subjected to remote uniform in-plane stresses. The elastic materials occupying all three phases belong to a particular class of compressible hyperelastic harmonic materials. Our analysis indicates that the internal stress field can be uniform and hydrostatic for some nonelliptical shapes of the inclusion, and all of the possible shapes of the inclusion permitting internal uniform hydrostatic stresses are identified. Three conditions are derived that ensure an internal uniform hydrostatic stress state. Our rigorous analysis indicates that for the given material and geometrical parameters of the three-phase inclusion of a nonelliptical shape, at most, eight different sets of remote uniform Piola stresses can be found, leading to internal uniform hydrostatic stresses. Finally, the analytical results are illustrated through an example. [DOI: 10.1115/1.4006240]

Keywords: Finite deformations, harmonic material, inclusion of arbitrary shape, uniform hydrostatic stress state

1 Introduction

The problem of how a remote uniform loading is disturbed by an elastic inclusion (or inhomogeneity) continues to receive investigators' attention (see, for example, Refs. [1–7]). In particular ,the internal uniform stress field inside the inclusion is especially preferred because an internal uniform stress field is optimal, in the sense that it eliminates any stress peaks within the inclusion [4,6]. Moreover, an internal uniform hydrostatic stress state will remove stress peaks on the interface because it achieves both uniform normal stress and vanishing tangential stress [5].

Most recently, within the framework of linear plane and antiplane elasticity, Wang and Gao [7] obtained the following seemingly impossible, while intriguing result, that the internal stress state within a three-phase nonelliptical inclusion can still be uniform, provided that the shape of the inclusion, the elastic properties of each phase, and the thickness of the interphase layer are appropriately designed. Thus, it is the objective of this work to extend our previous results to finite plane elasticity.

In this work, we use the complex variable formulation developed in Ref. [8] for plane strain deformations of a particular set of compressible hyperelastic materials of the harmonic type to study the finite plane deformations of a nonelliptical elastic inclusion bonded to an infinite matrix through an intermediate interphase layer. We first identify any possible nonelliptical shape of the inclusion permitting internal uniform hydrostatic stresses. Then, in each case, three conditions are found that ensure that the internal stress state is, indeed, uniform and hydrostatic.

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2 Formulation

Let the complex variable $z = x_1 + \mathrm{i}x_2$ represent the initial coordinates of a material particle in the undeformed configuration and $w(z) = y_1(z) + \mathrm{i}y_2(z)$ represents the corresponding spatial coordinates in the deformed configuration. Define the Cartesian components of the deformation gradient tensor $\mathbf{F}$ as

$$F_{ij} = \frac{\partial y_i}{\partial x_j}, \quad i,j = 1,2,3 \tag{1}$$

For a particular class of harmonic materials, the strain energy density W, defined with respect to the undeformed unit area, can be expressed by [8]

$$W = 2\mu[F(I) - J], \quad F'(I) = \frac{1}{4\alpha}\left[I + \sqrt{I^2 - 16\alpha\beta}\right] \tag{2}$$

Here, I and J are the scalar invariants of $\mathbf{FF^T}$given by

$$I = \lambda_1 + \lambda_2 = \sqrt{F_{ij}F_{ij} + 2J}, \quad J = \lambda_1\lambda_2 = \det\left[F_{ij}\right] \tag{3}$$

where we sum over repeated indices, λ_1 and λ_2 are the principal stretches, μ is the shear modulus and $1/2 \le \alpha < 1, \beta > 0$ are two material constants [8]. (We note that the function *F(I)* is a material function of I and is not to be confused with the components of the deformation gradient tensor $\mathbf{F}$). This special class of harmonic materials has attracted considerable attention recently [9–11]. For example, the preceding model has been used in Ref. [12] to investigate large deformations of a rubber sheet containing a single inhomogeneity and, more recently, in Refs. [9,13] to study the finite deformations of an annular membrane induced by the rotation of a rigid hub and the finite deformations of a crack, respectively. We adopt the same model in our own investigations.

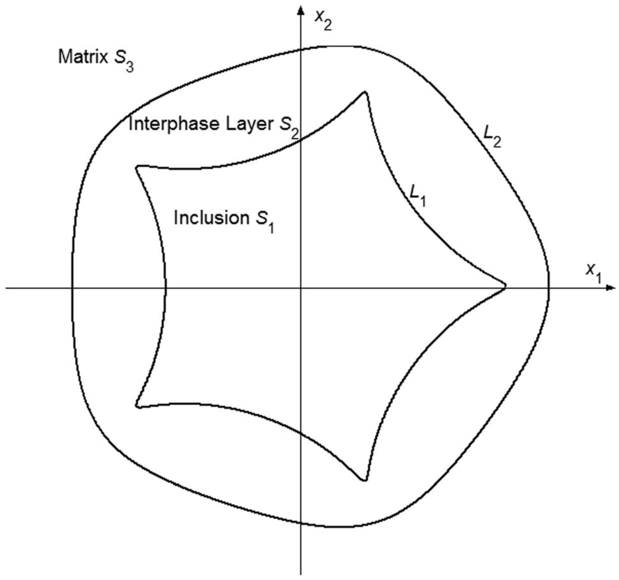

Fig. 1 Three-phase composite with a nonelliptical inclusion

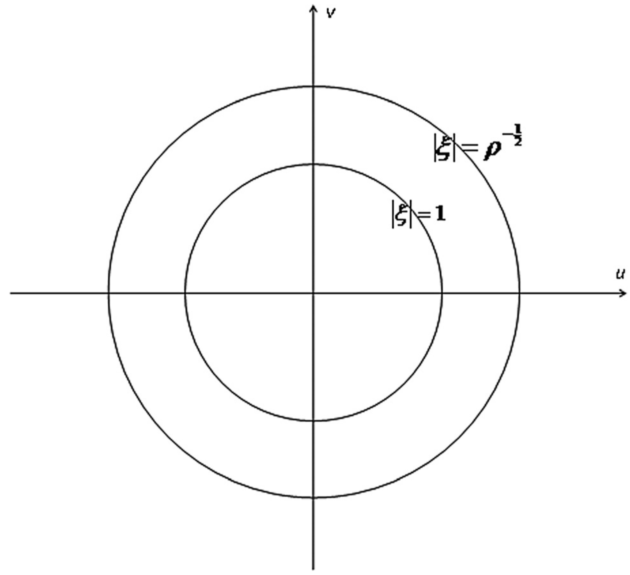

Fig. 2 The mapped ξ-plane

According to the formulation developed by Ru [8], the deformation $w(z)$ can be written in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ as

$$\mathrm{i}w(z)=\alpha\varphi(z)+\mathrm{i}\overline{\psi(z)}+\frac{\beta z}{\overline{\varphi'(z)}} \tag{4}$$

and the complex Piola stress function $\chi(z)$ is then given by

$$\chi(z)=2\mathrm{i}\mu\left[(\alpha-1)\varphi(z)+\mathrm{i}\overline{\psi(z)}+\frac{\beta z}{\overline{\varphi'(z)}}\right] \tag{5}$$

In addition, the Piola stress components can be written in terms of the Piola stress function χ as

$$-\sigma_{21}+\mathrm{i}\sigma_{11}=\chi_{,2},\quad \sigma_{22}-\mathrm{i}\sigma_{12}=\chi_{,1} \tag{6}$$

We consider a three-phase inclusion of arbitrary shape, as shown in Fig. 1. The elastic materials occupying the inclusion, the intermediate interphase layer, and the unbounded matrix belong to the class of harmonic materials characterized by Eq. (2) with the associated elastic constants (μ_1,α_1,β_1), (μ_2,α_2,β_2), and (μ_3,α_3,β_3), respectively. Let S_1, S_2, and S_3 denote the inclusion, the interphase layer, and the matrix, respectively, all of which are perfectly bonded through two sharp interfaces L_1 and L_2. Throughout the remainder of this paper, the subscripts 1, 2, and 3 [or the superscripts (1), (2), and (3)] are used to identify the associated quantities in S_1, S_2, and S_3.

The following conformal mapping is introduced [14]

$$z=\omega(\xi)=R\left(\xi+\sum_{n=1}^{N}\frac{p_n}{\xi^n}\right) \tag{7}$$

where R is a real scaling constant and p_n ($n=1,2,\ldots,N$) are complex constants. In this study, we will focus on inclusions of nonelliptical shape by assuming that $N\geq 2$ in Eq. (7). The preceding mapping function can conformally map L_1 and L_2 onto two coaxial circles with the radii 1 and $\rho^{-1/2}$, $(0\leq\rho\leq 1)$, in the ξ-plane (Fig. 2). Here, ρ can be considered as a parameter measuring the relative thickness of the interphase layer. Thus, the regions S_2 and S_3 are mapped onto $1<|\xi|<\rho^{-1/2}$ and $|\xi|>\rho^{-1/2}$, respectively. In order to make the mapping function "one-to-one" or conformal, we must have $\omega'(\xi)\neq 0$ for $|\xi|>1$. In the following analysis, for simplicity, we will adopt the notation that $\varphi_i(\xi)=\varphi_i(\omega(\xi)),\psi_i(\xi)=\psi_i(\omega(\xi)),i=1,2,3$.

In the ξ-plane, the boundary value problem for the three-phase inclusion of arbitrary shape now takes the following form

$$\begin{aligned}
&\alpha_2\varphi_2(\xi)+\mathrm{i}\overline{\psi_2(\xi)}+\frac{\beta_2\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi_2'(\xi)}}\\
&\quad=\alpha_1\varphi_1(\xi)+\mathrm{i}\overline{\psi_1(\xi)}+\frac{\beta_1\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi_1'(\xi)}}\\
&(\alpha_2-1)\varphi_2(\xi)+\mathrm{i}\overline{\psi_2(\xi)}+\frac{\beta_2\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi_2'(\xi)}}\\
&\quad=\Gamma_1\left[(\alpha_1-1)\varphi_1(\xi)+\mathrm{i}\overline{\psi_1(\xi)}+\frac{\beta_1\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi_1'(\xi)}}\right]\quad\text{on}\quad|\xi|=1
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\alpha_3\varphi_3(\xi)+\mathrm{i}\overline{\psi_3(\xi)}+\frac{\beta_3\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi_3'(\xi)}}\\
&\quad=\alpha_2\varphi_2(\xi)+\mathrm{i}\overline{\psi_2(\xi)}+\frac{\beta_2\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi_2'(\xi)}}\\
&(\alpha_3-1)\varphi_3(\xi)+\mathrm{i}\overline{\psi_3(\xi)}+\frac{\beta_3\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi_3'(\xi)}}\\
&\quad=\frac{(\alpha_2-1)}{\Gamma_3}\varphi_2(\xi)+\frac{\mathrm{i}}{\Gamma_3}\overline{\psi_2(\xi)}+\frac{\beta_2\omega(\xi)\overline{\omega'(\xi)}}{\Gamma_3\overline{\varphi_2'(\xi)}}\quad\text{on}\quad|\xi|=\rho^{-1/2}
\end{aligned} \tag{9}$$

$$\varphi_3(\xi)\cong\mathrm{i}AR\xi+O(1),\quad\psi_3(\xi)\cong BR\xi+O(1),\quad\text{as}\quad|\xi|\to\infty \tag{10}$$

where $\Gamma_1=\mu_1/\mu_2$ and $\Gamma_3=\mu_3/\mu_2$ are two stiffness ratios, A and B are complex constants determined by the remote uniform Piola stresses $(\sigma_{11}^\infty,\sigma_{22}^\infty,\sigma_{12}^\infty,\sigma_{21}^\infty)$ such that

$$\begin{aligned}
(1-\alpha_3)A-\frac{\beta_3}{\bar{A}}&=\frac{\sigma_{11}^\infty+\sigma_{22}^\infty+\mathrm{i}(\sigma_{21}^\infty-\sigma_{12}^\infty)}{4\mu_3},\\
B&=\frac{\sigma_{11}^\infty-\sigma_{22}^\infty-\mathrm{i}(\sigma_{12}^\infty+\sigma_{21}^\infty)}{4\mu_3}
\end{aligned} \tag{11}$$

3 The Internal Uniform Hydrostatic Stress State

In order to achieve an internal uniform hydrostatic stress state within the inclusion of arbitrary shape, $\varphi_1(\xi)$ and $\psi_1(\xi)$ must take the following forms

$$\varphi_1(\xi)=\mathrm{i}X\left(\xi+\sum_{n=1}^{N}\frac{p_n}{\xi^n}\right),\quad\psi_1(\xi)=0 \tag{12}$$

where X is a real constant to be determined. Thus, by enforcing the interface conditions on $|\xi|=1$, we arrive at the following expressions for $\varphi_2(\xi)$ and $\psi_2(\xi)$

$$\varphi_2(\xi)=\mathrm{i}\left([\alpha_1+\Gamma_1(1-\alpha_1)]X+\frac{(1-\Gamma_1)R^2\beta_1}{X}\right)\left(\xi+\sum_{n=1}^{N}\frac{p_n}{\xi^n}\right)$$

$$\psi_2(\xi)=\begin{pmatrix}[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]X+\dfrac{R^2\beta_1[\Gamma_1\alpha_2+(1-\alpha_2)]}{X}\\ -\dfrac{R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{pmatrix}\times\left(\frac{1}{\xi}+\sum_{n=1}^{N}\bar{p}_n\xi^n\right)\quad(1<|\xi|<\rho^{-1/2}) \tag{13}$$

Similarly, by enforcing the interface conditions on $|\xi|=\rho^{-1/2}$, we can finally obtain the following expressions for $\varphi_3(\xi)$ and $\psi_3(\xi)$

$$\frac{\Gamma_3}{\Gamma_3-1}\varphi_3(\xi)=\mathrm{i}\begin{pmatrix}\dfrac{[\Gamma_3\alpha_2+(1-\alpha_2)]}{\Gamma_3-1}\left([\alpha_1+\Gamma_1(1-\alpha_1)]X+\dfrac{(1-\Gamma_1)R^2\beta_1}{X}\right)\\ +\dfrac{R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{pmatrix}\times\left(\xi+\sum_{n=1}^{N}\frac{p_n}{\xi^n}\right)$$
$$+\mathrm{i}\begin{pmatrix}[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]X+\dfrac{R^2\beta_1[\Gamma_1\alpha_2+(1-\alpha_2)]}{X}\\ -\dfrac{R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{pmatrix}\times\left(\rho\xi+\sum_{n=1}^{N}\frac{p_n}{\rho^n\xi^n}\right)\quad(|\xi|>\rho^{-1/2}) \tag{14}$$

$$\Gamma_3\psi_3(\xi)=\begin{pmatrix}[\Gamma_3\alpha_2(1-\alpha_3)-\alpha_3(1-\alpha_2)]\left([\alpha_1+\Gamma_1(1-\alpha_1)]X+\dfrac{(1-\Gamma_1)R^2\beta_1}{X}\right)\\ +\dfrac{[\alpha_3+\Gamma_3(1-\alpha_3)]R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{pmatrix}\left(\frac{1}{\rho\xi}+\sum_{n=1}^{N}\bar{p}_n\rho^n\xi^n\right)$$
$$+[\alpha_3+\Gamma_3(1-\alpha_3)]\begin{pmatrix}[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]X+\dfrac{R^2\beta_1[\Gamma_1\alpha_2+(1-\alpha_2)]}{X}\\ -\dfrac{R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{pmatrix}\left(\frac{1}{\xi}+\sum_{n=1}^{N}\bar{p}_n\xi^n\right)$$
$$-\frac{R^2\Gamma_3^2\beta_3\left(\dfrac{1}{\rho\xi}+\displaystyle\sum_{n=1}^{N}\bar{p}_n\rho^n\xi^n\right)\left(\xi-\displaystyle\sum_{n=1}^{N}\frac{np_n}{\xi^n}\right)}{\begin{pmatrix}\left[[\Gamma_3\alpha_2+(1-\alpha_2)]\left([\alpha_1+\Gamma_1(1-\alpha_1)]X+\dfrac{(1-\Gamma_1)R^2\beta_1}{X}\right)+\dfrac{(\Gamma_3-1)R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\right]\\ \times\left(\xi-\displaystyle\sum_{n=1}^{N}\frac{np_n}{\xi^n}\right)+(\Gamma_3-1)\begin{pmatrix}[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]X+\dfrac{R^2\beta_1[\Gamma_1\alpha_2+(1-\alpha_2)]}{X}\\ -\dfrac{R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{pmatrix}\left(\rho\xi-\displaystyle\sum_{n=1}^{N}\frac{np_n}{\rho^n\xi^n}\right)\end{pmatrix}}\quad(|\xi|>\rho^{-1/2}) \tag{15}$$

At this point, only the remote boundary conditions (10) remain to be satisfied. A careful inspection of Eq. (15) reveals that the conformal mapping function (7) can take only one of the following three forms

(a) When $N=2$

$$z=\omega(\xi)=R\left(\xi+\frac{p_1}{\xi}+\frac{p_2}{\xi^2}\right) \tag{16}$$

(b) When $N=3$

$$z=\omega(\xi)=R\left(\xi+\frac{p_1}{\xi}+\frac{p_3}{\xi^3}\right) \tag{17}$$

(c) When $N\geq 4$

$$z=\omega(\xi)=R\left(\xi+\frac{p_N}{\xi^N}\right),\quad N\geq 4 \tag{18}$$

For each possible value of the integer N (≥ 2), three conditions should be met in order to satisfy the remote uniform loading conditions (10). The first and second conditions for any value of N (≥ 2) can be written as

$$
\begin{aligned}
&X^2\{[\alpha_3+\Gamma_3(1-\alpha_3)][\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]+\rho^N[\alpha_1+\Gamma_1(1-\alpha_1)][\Gamma_3\alpha_2(1-\alpha_3)-\alpha_3(1-\alpha_2)]\}\\
&\quad+R^2\beta_1\{[\Gamma_1\alpha_2+(1-\alpha_2)][\alpha_3+\Gamma_3(1-\alpha_3)]+\rho^N(1-\Gamma_1)[\Gamma_3\alpha_2(1-\alpha_3)-\alpha_3(1-\alpha_2)]\}\\
&\quad-\frac{X^2R^2\beta_2(1-\rho^N)[\alpha_3+\Gamma_3(1-\alpha_3)]}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\\
&\quad-\frac{X^2\Gamma_3^2R^2\beta_3\rho^N}{\left(\begin{array}{l}X^2\{[\alpha_1+\Gamma_1(1-\alpha_1)][\Gamma_3\alpha_2+(1-\alpha_2)]+\rho(\Gamma_3-1)[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]\}\\+R^2\beta_1\{(1-\Gamma_1)[\Gamma_3\alpha_2+(1-\alpha_2)]+\rho[\Gamma_1\alpha_2+(1-\alpha_2)](\Gamma_3-1)\}+\dfrac{X^2R^2\beta_2(1-\rho)(\Gamma_3-1)}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{array}\right)}=0\\
&\qquad (N\geq 2)
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\frac{\Gamma_3A}{\Gamma_3-1}&=\left[\frac{[\alpha_1+\Gamma_1(1-\alpha_1)][\Gamma_3\alpha_2+(1-\alpha_2)]}{\Gamma_3-1}+\rho[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]\right]\frac{X}{R}\\
&\quad+\left[\frac{(1-\Gamma_1)[\Gamma_3\alpha_2+(1-\alpha_2)]}{\Gamma_3-1}+\rho[\Gamma_1\alpha_2+(1-\alpha_2)]\right]\frac{R\beta_1}{X}+\frac{(1-\rho)R\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}
\end{aligned}
\tag{20}
$$

It is observed from condition (20) that the loading parameter A must be a real number, which implies that the remote uniform Piola stresses should be symmetric, i.e., $\sigma_{12}^{\infty}=\sigma_{21}^{\infty}$.

The expressions of the third condition are quite different for different values of N (=2, 3, ≥4). When $N=2$, the third condition can be written as

$$
\begin{aligned}
\frac{\Gamma_3\left(\dfrac{B}{\bar{p}_1}+\dfrac{\rho\beta_3}{A}\right)}{\alpha_3+\Gamma_3(1-\alpha_3)}&=\left[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)+\frac{\rho[\alpha_1+\Gamma_1(1-\alpha_1)][\Gamma_3\alpha_2(1-\alpha_3)-\alpha_3(1-\alpha_2)]}{\alpha_3+\Gamma_3(1-\alpha_3)}\right]\frac{X}{R}\\
&\quad+\left[\Gamma_1\alpha_2+(1-\alpha_2)+\frac{\rho(1-\Gamma_1)[\Gamma_3\alpha_2(1-\alpha_3)-\alpha_3(1-\alpha_2)]}{\alpha_3+\Gamma_3(1-\alpha_3)}\right]\frac{R\beta_1}{X}-\frac{(1-\rho)R\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}
\end{aligned}
\tag{21}
$$

When $N=3$, the third condition becomes

$$
\begin{aligned}
\frac{\Gamma_3B}{\bar{p}_1}&=\frac{\rho[\alpha_3+\Gamma_3(1-\alpha_3)]R\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}-\frac{\rho\Gamma_3\beta_3}{A}\\
&\quad+\rho[\Gamma_3\alpha_2(1-\alpha_3)-\alpha_3(1-\alpha_2)]\left[[\alpha_1+\Gamma_1(1-\alpha_1)]\frac{X}{R}+(1-\Gamma_1)\frac{R\beta_1}{X}\right]\\
&\quad+\left(\alpha_3+\Gamma_3(1-\alpha_3)+\frac{\beta_3p_1\bar{p}_3\rho^2(1-\rho^2)(1-\Gamma_3)}{\bar{p}_1A^2}\right)\left(\begin{array}{l}[\alpha_1(1-\alpha_2)-\Gamma_1\alpha_2(1-\alpha_1)]\dfrac{X}{R}+\dfrac{R\beta_1[\Gamma_1\alpha_2+(1-\alpha_2)]}{X}\\-\dfrac{R\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}\end{array}\right)
\end{aligned}
\tag{22}
$$

It is of interest to note that when $p_3=0$, Eq. (22) reduces to Eq. (21). When $N\geq 4$, the third condition is $B=0$, which implies that now the remote uniform stresses must be hydrostatic, i.e., $\sigma_{11}^{\infty}=\sigma_{22}^{\infty},\sigma_{12}^{\infty}=\sigma_{21}^{\infty}=0$.

For given material parameters Γ_1, Γ_3, α_i, β_i (i = 1,2,3) and the given thickness parameter ρ, we can first determine the real constant X/R by solving Eq. (19), which is, in fact, a quartic equation in $(X/R)^2$. Then the two loading parameters A and B can be obtained by using either: (i) Eqs. (20) and (21) when $N=2$ (with p_1 as a variable), or (ii) Eqs. (20) and (22) when $N=3$ (with p_1 and p_3 as variables), or (iii) Eq. (20) and $B=0$ when $N\geq 4$ for a given shape of the inclusion (i.e., the mapping function $\omega(\xi)$ is given). Finally, the remote uniform Piola stresses $\sigma_{11}^{\infty},\sigma_{22}^{\infty}$, $\sigma_{12}^{\infty}=\sigma_{21}^{\infty}$ can be further determined by using Eq. (11). Apparently, at most, eight different sets of the remote uniform Piola stresses can be found, leading to internal uniform hydrostatic stresses. In addition, the parameters appearing in the mapping functions (16)–(18) should satisfy the two restrictions that $\omega'(\xi)\neq 0,(|\xi|>1)$ and $\varphi_3'(\xi)\neq 0,(|\xi|\geq\rho^{-1/2})$. The argument for the second restriction can be found in [8]. More specifically, when $N=2$, the two parameters p_1 and p_2 should satisfy the following two inequalities

$$
\xi^3-p_1\xi-2p_2\neq 0,\quad \text{for}\quad |\xi|>1 \tag{23}
$$

$$
f(\rho)\xi^3-p_1\rho f(\rho^{-1})\xi-2p_2\rho^{3/2}f(\rho^{-2})\neq 0,\quad \text{for}\quad |\xi|\geq 1 \tag{24}
$$

where

$$
\begin{aligned}
f(\rho)&=X\left(\frac{[\alpha_1+\Gamma_1(1-\alpha_1)][\Gamma_3\alpha_2+(1-\alpha_2)]}{\Gamma_3-1}+\rho[\alpha_1(1-\alpha_2)\right.\\
&\quad\left.-\Gamma_1\alpha_2(1-\alpha_1)]\right)+\frac{R^2\beta_1}{X}\\
&\quad\times\left(\frac{(1-\Gamma_1)[\Gamma_3\alpha_2+(1-\alpha_2)]}{\Gamma_3-1}+\rho[\Gamma_1\alpha_2+(1-\alpha_2)]\right)\\
&\quad+\frac{(1-\rho)R^2\beta_2X}{[\alpha_1+\Gamma_1(1-\alpha_1)]X^2+(1-\Gamma_1)R^2\beta_1}
\end{aligned}
\tag{25}
$$

When $N=3$, the two parameters p_1 and p_3 should satisfy the two inequalities

$$\left| p_1 \pm \sqrt{p_1^2 + 12p_3} \right| \le 2$$
$$\text{and} \quad \rho \left| p_1 f(\rho^{-1}) \pm \sqrt{p_1^2 [f(\rho^{-1})]^2 + 12 p_3 f(\rho) f(\rho^{-3})} \right| < 2|f(\rho)| \tag{26}$$

Finally, when $N \ge 4$, the parameter p_N should satisfy the two inequalities

$$|p_N| \le \frac{1}{N} \quad \text{and} \quad |p_N| < \frac{\rho^{-\frac{N+1}{2}} |f(\rho)|}{N |f(\rho^{-N})|} \tag{27}$$

In particular, if we choose $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$ for the situation in which $F'(I)/I$ approaches unity as I tends to infinity [8,15], and assume that $\beta_1 = \beta_2 = \beta_3$, the roots of Eq. (19) can be explicitly given by

$$\frac{X}{R\sqrt{2\beta_1}} = \pm 1, \pm 1, \pm \sqrt{1 - \frac{k_1 - \sqrt{k_1^2 - k_0 k_2}}{k_2}}, \pm \sqrt{1 - \frac{k_1 + \sqrt{k_1^2 - k_0 k_2}}{k_2}} \tag{28}$$

where

$$\begin{aligned} k_2 &= \left[(1+\Gamma_1)^2 (1+\Gamma_3) - \rho (1-\Gamma_1^2)(1-\Gamma_3) \right] \\ &\quad \times \left[(1-\Gamma_1^2)(1+\Gamma_3) - \rho^N (1+\Gamma_1)^2 (1-\Gamma_3) \right] \\ k_1 &= 2 \Big[(1-\Gamma_1^2)(1+\Gamma_3)(\Gamma_1 + 2\Gamma_3 + \Gamma_1\Gamma_3) - 2\rho^N \Gamma_1 (1+\Gamma_1)^2 \\ &\quad \times (1-\Gamma_3^2) - \rho^{N+1} (1-\Gamma_1^2)(1-\Gamma_3)(-\Gamma_1 + 2\Gamma_3 + \Gamma_1\Gamma_3) \Big] \\ k_0 &= 8 \Big[\Gamma_3 (1-\Gamma_1^2)(1+\Gamma_3) - 2\rho^N \Gamma_1^2 (1-\Gamma_3^2) \\ &\quad - \rho^{N+1} \Gamma_3 (1-\Gamma_1^2)(1-\Gamma_3) \Big] \end{aligned} \tag{29}$$

The roots $X/R = \pm\sqrt{2\beta_1}$ will lead to the trivial loading $\sigma_{11}^\infty = \sigma_{22}^\infty = \sigma_{12}^\infty = \sigma_{21}^\infty = 0$ and so can be ignored. In the following, we will illustrate the preceding results through an example.

Remark. Once X/R is known, the internal uniform hydrostatic stress field can be determined by $\sigma_{11}^{(1)} = \sigma_{22}^{(1)} = 2\mu_1[(1-\alpha_1)X/R - \beta_1 R/X], \sigma_{12}^{(1)} = \sigma_{21}^{(1)} = 0$.

4 Example

Letting $\Gamma_1 = \Gamma_3 = 10$, $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$, $\beta_1 = \beta_2 = \beta_3 = 0.6$, and $\rho = 0.8$, we can identify the following remote uniform Piola stresses leading to internal uniform hydrostatic stresses within the inclusion.

- When $N = 2$, the remote Piola stresses can be determined as

(i)

$$\begin{aligned} \frac{\pm\sigma_{11}^\infty}{\mu_3} &= 2.4757 + 0.6506\mathrm{Re}\{p_1\}, \\ \frac{\pm\sigma_{22}^\infty}{\mu_3} &= 2.4757 - 0.6506\mathrm{Re}\{p_1\}, \\ \frac{\pm\sigma_{12}^\infty}{\mu_3} &= \frac{\pm\sigma_{21}^\infty}{\mu_3} = 0.6506\mathrm{Im}\{p_1\} \end{aligned} \tag{30}$$

which will lead to the internal uniform hydrostatic stresses $\pm\sigma_{11}^{(1)} = \pm\sigma_{22}^{(1)} = 0.5859\mu_1$.

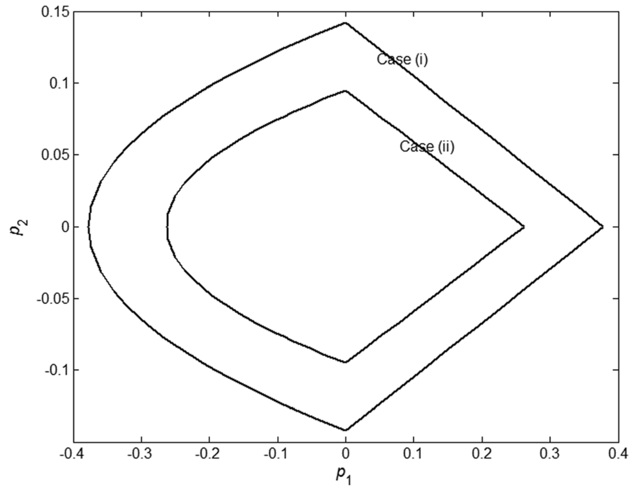

Fig. 3 The permissible real values of p_1 and p_2 appearing in Eqs. (30) and (31)

(ii)

$$\begin{aligned} \frac{\pm\sigma_{11}^\infty}{\mu_3} &= 2.9567 + 0.9223\mathrm{Re}\{p_1\}, \\ \frac{\pm\sigma_{22}^\infty}{\mu_3} &= 2.9567 - 0.9223\mathrm{Re}\{p_1\}, \\ \frac{\pm\sigma_{12}^\infty}{\mu_3} &= \frac{\pm\sigma_{21}^\infty}{\mu_3} = 0.9223\mathrm{Im}\{p_1\} \end{aligned} \tag{31}$$

which will lead to the internal uniform hydrostatic stresses $\pm\sigma_{11}^{(1)} = \pm\sigma_{22}^{(1)} = 0.2778\mu_1$.

The parameters p_1 and p_2 should satisfy the inequalities (23) and (24). When these two parameters are real, their permissible values are shown in Fig. 3. The pair (p_1, p_2) should lie within the region enclosed by the curve for each case.

- When $N = 3$, the remote Piola stresses can be determined as

(i)

$$\begin{aligned} \frac{\pm\sigma_{11}^\infty}{\mu_3} &= 4.2078 + 1.7199\mathrm{Re}\{p_1\} + 9.6226\mathrm{Re}\{\bar{p}_1 p_3\} \\ \frac{\pm\sigma_{22}^\infty}{\mu_3} &= 4.2078 - 1.7199\mathrm{Re}\{p_1\} - 9.6226\mathrm{Re}\{\bar{p}_1 p_3\} \\ \frac{\pm\sigma_{12}^\infty}{\mu_3} &= \frac{\pm\sigma_{21}^\infty}{\mu_3} = 1.7199\mathrm{Im}\{p_1\} + 9.6226\mathrm{Im}\{\bar{p}_1 p_3\} \end{aligned} \tag{32}$$

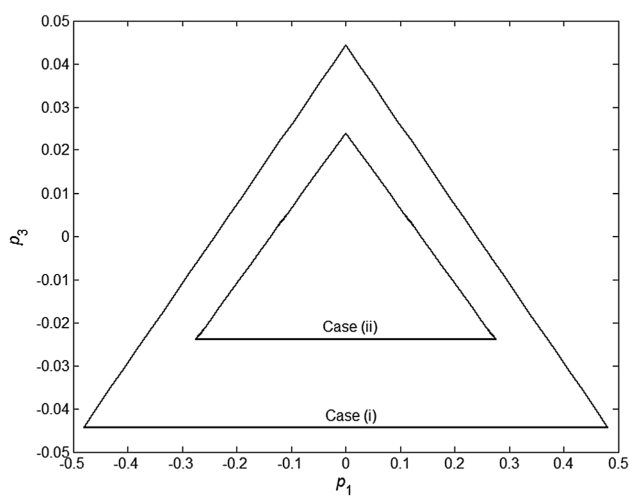

Fig. 4 The permissible real values of p_1 and p_3 appearing in Eqs. (32) and (33)

Table 1 The values of the remote hydrostatic stresses, the induced internal uniform hydrostatic stresses, and the range of $|p_N|$ for different values of $N \geq 4$

	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$	$N=15$
$\pm\sigma_{11}^{\infty}/\mu_3 = \pm\sigma_{22}^{\infty}/\mu_3$	6.2665 8.4504	8.7822 12.1423	11.8930 16.7304	15.7598 22.4498	20.5786 29.5889	26.5918 38.5059	34.1008 49.6472	111.0442 163.8871
$\pm\sigma_{11}^{(1)}/\mu_1 = \pm\sigma_{22}^{(1)}/\mu_1$	0.9057 0.2553	0.9762 0.2530	1.0238 0.2517	1.0575 0.2509	1.0819 0.2503	1.1002 0.2499	1.1140 0.2496	1.1480 0.2489
$\|p_N\| <$	0.0174 0.0087	0.0079 0.0038	0.0040 0.0019	0.0022 0.0010	0.0012 0.0006	0.0007 0.0003	0.0004 0.0002	4.768×10^{-5} 2.170×10^{-5}

which will lead to the internal uniform hydrostatic stresses $\pm\sigma_{11}^{(1)} = \pm\sigma_{22}^{(1)} = 0.7932\mu_1$.

(ii)

$$\frac{\pm\sigma_{11}^{\infty}}{\mu_3} = 5.4557 + 2.6169\mathrm{Re}\{p_1\} + 23.4742\mathrm{Re}\{\bar{p}_1 p_3\}$$
$$\frac{\pm\sigma_{22}^{\infty}}{\mu_3} = 5.4557 - 2.6169\mathrm{Re}\{p_1\} - 23.4742\mathrm{Re}\{\bar{p}_1 p_3\} \tag{33}$$
$$\frac{\pm\sigma_{12}^{\infty}}{\mu_3} = \frac{\pm\sigma_{21}^{\infty}}{\mu_3} = 2.6169\mathrm{Im}\{p_1\} + 23.4742\mathrm{Im}\{\bar{p}_1 p_3\}$$

which will lead to the internal uniform hydrostatic stresses $\pm\sigma_{11}^{(1)} = \pm\sigma_{22}^{(1)} = 0.2603\mu_1$.

The parameters p_1 and p_3 appearing in Eqs. (32) and (33) should satisfy the two inequalities in Eq. (26). When these two parameters are real, their permissible values are shown in Fig. 4. It is observed from Fig. 4 that the pair (p_1, p_3) should lie within the triangle for each case.

- When $N \geq 4$, the values of the remote hydrostatic stresses, the induced internal uniform hydrostatic stresses, and the range of $|p_N|$ are listed in Table 1. Interestingly $\left|\sigma_{11}^{(1)}\right| = \left|\sigma_{22}^{(1)}\right| \approx \mu_1$ or $\mu_1/4$ for any value of $N \geq 4$.

5 Conclusions

We have found that the internal Piola stress state inside a three-phase nonelliptical inclusion of particular compressible hyperelastic harmonic materials can be uniform and hydrostatic. Three conditions leading to internal uniform hydrostatic stresses were derived. The first two conditions were given by Eqs. (19) and (20), while the third one was given by either (i) Eq. (21) for $N=2$, or (ii) Eq. (22) for $N=3$, or (iii) $B=0$ for $N \geq 4$. The results illustrate that, at most, eight different sets of the remote uniform Piola stresses can be found, leading to internal uniform hydrostatic stresses for the given geometrical and material parameters of the three-phase composite. It is expected that if we increase the number of interphase layers, more unexpected results can be obtained. Our preliminary analysis indicates that, at most, 2^N different sets of the remote uniform Piola stresses can be found, leading to internal uniform hydrostatic stresses within an N-phase inclusion of arbitrary shape. In addition, the case of $N=2$ (in the absence of the interphase layer) has been discussed by Ru et al. [10].

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References

[1] Hardiman, N. J., 1954, "Elliptic Elastic Inclusion in an Infinite Elastic Plate," Q. J. Mech. Appl. Math., **7**(2), pp. 226–230.
[2] Eshelby, J. D., 1957, "The Determination of the Elastic Field of an Ellipsoidal Inclusion, and Related Problems," Proc. R. Soc. London, **A241**, pp. 376–396.
[3] Mura, T., 1987, *Micromechanics of Defects in Solids*, Martinus Nijhoff, Dordrecht.
[4] Ru, C. Q., Schiavone, P., and Mioduchowski, A., 1999, "Uniformity of the Stresses Within a Three-Phase Elliptical Inclusion in Anti-Plane Shear," J. Elast., **52**, pp. 121–128.
[5] Ru, C. Q., 1999, "Three-Phase Elliptical Inclusions With Internal Uniform Hydrostatic Stresses," J. Mech. Phys. Solids, **47**, pp. 259–273.
[6] Antipov, Y. A., and Schiavone, P., 2003, "On the Uniformity of Stresses Inside an Inhomogeneity of Arbitrary Shape," IMA J. Appl. Math., **68**, pp. 299–311.
[7] Wang, X., and Gao, X. L., 2011, "On the Uniform Stress State Inside an Inclusion of Arbitrary Shape in a Three-Phase Composite," Z. Angew. Math. Phys. (in press).
[8] Ru, C. Q., 2002, "On Complex-Variable Formulation for Finite Plane Elastostatics of Harmonic Materials," Acta Mech., **156**, pp. 219–234.
[9] Li, X., and Steigmann, D. J., 1993, "Finite Plane Twist of an Annular Membrane," Q. J. Mech. Appl. Math., **46**, pp. 601–625.
[10] Ru, C. Q., Schiavone, P., Sudak, L. J., and Mioduchowski, A., 2005, "Uniformity of Stresses Inside an Elliptical Inclusion in Finite Elastostatics," Int. J. Nonlinear Mech., **40**, pp. 281–287.
[11] Wang, X., 2011, "Three-Phase Elliptical Inclusions With Internal Uniform Hydrostatic Stresses in Finite Plane Elastostatics," Acta Mech., **219**, pp. 77–90.
[12] Varley, E., and Cumberbatch, E., 1980, "Finite Deformation of Elastic Materials Surrounding Cylindrical Holes," J. Elast., **10**, pp. 341–405.
[13] Abeyaratne, R., 1983, "Some Finite Elasticity Problems Involving Crack Tips," *Modelling Problems in Crack Tip Mechanics*, J. T. Pindera, ed., University of Waterloo, Waterloo, pp. 3–24.
[14] England, A. H., 1971, *Complex Variable Methods in Elasticity*, Wiley, London.
[15] Knowles, J. K., and Sternberg, E., 1975, "On the Singularity Induced by Certain Mixed Boundary Conditions in Linearized and Nonlinear Elastostatics," Int. J. Solids Struct., **11**, pp. 1173–1201.