A new estimate for Hölder approximation by Bernstein operators

H. Gonska[a,*], J. Prestin[b], G. Tachev[c]

[a] *University of Duisburg-Essen, 47048 Duisburg, Germany*
[b] *University of Lübeck, 23560 Lübeck, Germany*
[c] *University of Architecture, 1046 Sofia, Bulgaria*

Abstract

In this work we discuss the rate of simultaneous approximation of Hölder continuous functions by Bernstein operators, measured by Hölder norms with different exponents. We extend the known results on this topic.

Keywords:
Lipschitz functions, Hölder approximation, Bernstein operators, degree of approximation.

1. Introduction

Let $C^m[0,1]$ be the space of all real-valued, m-times continuously differentiable functions on $[0,1]$ equipped with the sup-seminorm $\|\cdot^{(m)}\|$. The first order modulus of continuity of $f \in C[0,1] = C^0[0,1]$ is given by

$$\omega_1(f,t) := \sup\{|f(x) - f(y)| : |x - y| \le t\}, \quad 0 \le t.$$

Using this quantity, for $0 \le \alpha \le 1$ and $0 < \delta \le 1$, we also define for $f \in C[0,1]$

$$\Theta_\alpha(f,\delta) := \sup_{0<t\le\delta} \frac{\omega_1(f,t)}{t^\alpha}, \quad \Theta_\alpha(f) := \sup_{0<\delta\le 1} \Theta_\alpha(f,\delta).$$

*Corresponding author
Email address: heiner.gonska@uni-due.de (H. Gonska)

To formulate our main result we introduce the following notation: For $m \in \mathbb{N}_0, 0 \leq \alpha \leq 1$ define

$$C^{m,\alpha}[0,1] := \left\{ f \in C^m[0,1] : \|f\|_{m,\alpha} := \sum_{k=0}^{m} \|f^{(k)}\| + \Theta_\alpha(f^{(m)}) < \infty \right\}$$

and

$$\tilde{C}^{m,\alpha}[0,1] := \left\{ g \in C^{m,\alpha}[0,1] : \omega_1(g^{(m)};\delta) = o(\delta^\alpha), \delta \to 0+ \right\}.$$

In particular,

$$C^{m,0}[0,1] = \tilde{C}^{m,0}[0,1] = C^m[0,1], \quad m \geq 0.$$

It is clear that the space $C^{0,\alpha}[0,1]$ coincides with the space $\mathrm{Lip}_\alpha[0,1]$ (see [8]), which is complete under the so-called Hölder norm $\|\cdot\|_{0,\alpha}$ and its elements are said to satisfy a Lipschitz or Hölder condition of order α. Also $\tilde{C}^{0,\alpha}[0,1] \equiv \mathrm{lip}_\alpha[0,1]$. As a continuation of a result in [2] where, for the classical Bernstein operator B_n, it was proved that for all $f \in \tilde{C}^{0,\alpha}[0,1]$, the quantity $\|B_n f - f\|_{0,\alpha}$ converges to 0 as $n \to \infty$, in [4] the following quantitative variant of this convergence was established:

Theorem A *There exists a constant K such that for every $f \in \tilde{C}^{0,\alpha}[0,1]$, $0 < \alpha < 1$,*

$$\|B_n f - f\|_{0,\alpha} \leq K \cdot \Theta_\alpha(f, \frac{1}{\sqrt{n}}).$$

A similar inequality was given by Bustamante and Roldan in [3]. The aim of the present note is to generalize Theorem A for $f \in \tilde{C}^{m,\alpha}[0,1]$, $m \in \mathbb{N}_0$. Very recently in [6] the following result for simultaneous approximation by Bernstein operators in Hölder norms was proved:

Theorem B *Let $r, m \in \mathbb{N}_0, 0 \leq \alpha, \beta \leq 1, r \leq m$, $r + \beta \leq m + \alpha$. Then for $f \in C^{m,\alpha}[0,1]$ and $n > m+1$ one has*

$$||B_n f - f||_{r,\beta} \leq C_r \cdot (n-r-1)^{\max\left\{-1, \frac{r+\beta-m-\alpha}{2}\right\}} \cdot ||f||_{m,\alpha}.$$

Here C_r is a constant depending only on r.

In the sequel we consider the case $r = m$ and $0 \le \beta \le \alpha \le 1$. Our main result is

Theorem 1 *Let $r \in \mathbb{N}_0, 0 \le \beta \le \alpha \le 1$. Then for $f \in C^{r,\alpha}[0,1]$ and $n \ge \max\{r+2, r(r+1)\}$ one has*

$$||B_n f - f||_{r,\beta} \le n^{\frac{\beta-\alpha}{2}} \cdot \Big[5 \cdot \Theta_\alpha(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{3}{2} r^2 \cdot n^{\frac{\alpha-\beta-1}{2}} \cdot \sum_{i=0}^{r} \|f^{(i)}\|\Big].$$

2. Proof of Theorem 1

Proof: For $r = 0$ from the well-known inequality of Popoviciu we obtain – using a slightly improved version given in Lorentz' book [8] –

$$\|B_n f - f\| \le \frac{5}{4} \cdot \omega_1(f, \frac{1}{\sqrt{n}}) \le \frac{5}{4} \cdot n^{-\frac{\alpha}{2}} \cdot \Theta_\alpha(f, \frac{1}{\sqrt{n}}). \tag{1}$$

For $r \ge 1$ we use Popoviciu's inequality again to obtain

$$\|B_n f - f\| \le \frac{5}{4} \cdot \frac{1}{\sqrt{n}} \cdot ||f'||,\ f \in C^1[0,1]. \tag{1'}$$

For $1 \le i \le r-1$ and $n \ge \max\{r+2, r(r+1)\}$ we apply an estimate for simultaneous approximation due to Knoop and Pottinger [7] (see also [1]):

$$\|(B_n f - f)^{(i)}\| \le \frac{5}{4} \cdot \omega_1(f^{(i)}, \frac{1}{\sqrt{n}}) + \frac{i(i-1)}{2n} \cdot \|f^{(i)}\|$$

$$\le \frac{5}{4} \cdot \frac{1}{\sqrt{n}} \cdot \|f^{(i+1)}\| + \frac{i(i-1)}{2n} \cdot \|f^{(i)}\|. \tag{2}$$

For $i = r$ also the result in [7] yields, for the same values of n,

$$\|(B_n f - f)^{(r)}\| \le \frac{5}{4} \cdot n^{-\frac{\alpha}{2}} \cdot \Theta_\alpha(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{r(r-1)}{2n} \cdot \|f^{(r)}\|. \tag{3}$$

From (1'), (3) and after summing up (2) for $1 \le i \le r-1$ we get

$$\sum_{i=0}^{r} \|(B_n f - f)^{(i)}\| \le \frac{5}{4} \cdot n^{-\frac{\alpha}{2}} \cdot \Theta_\alpha(f^{(r)}, \frac{1}{\sqrt{n}}) + \Big[\frac{r(r-1)}{2n} + \frac{5}{4\sqrt{n}}\Big] \cdot \sum_{i=0}^{r} \|f^{(i)}\|$$

$$\leq \frac{5}{4} \cdot n^{-\frac{\alpha}{2}} \cdot \Theta_\alpha(f^{(r)}, \frac{1}{\sqrt{n}}) + C_r \frac{1}{\sqrt{n}} \cdot \sum_{i=0}^{r} \|f^{(i)}\|, \tag{4}$$

with $C_r = \frac{r(r-1)}{2} + \frac{5}{4} = O(r^2)$.

To estimate the Hölder term $\Theta_\beta((B_n f - f)^{(r)})$, $r \geq 0$, we consider two cases according to the value of h. First we verify that

$$\begin{aligned}
&\sup_{0<h\leq\frac{1}{\sqrt{n}}} h^{-\beta} \cdot \omega_1((B_n f - f)^{(r)}, h) \\
&\leq \sup_{0<h\leq\frac{1}{\sqrt{n}}} h^{\alpha-\beta} \cdot h^{-\alpha} \cdot \Big[\omega_1(f^{(r)}, h) + \omega_1((B_n f)^{(r)}, h)\Big] \\
&\leq \Big(\frac{1}{\sqrt{n}}\Big)^{\alpha-\beta} \cdot \sup_{0<h\leq\frac{1}{\sqrt{n}}} 3h^{-\alpha} \cdot \omega_1(f^{(r)}, h) \leq 3n^{\frac{\beta-\alpha}{2}} \cdot \Theta_\alpha(f^{(r)}, \frac{1}{\sqrt{n}}),
\end{aligned} \tag{5}$$

where in the last inequality we have used Proposition 3.2 in [5]:

$$\omega_1((B_n f)^{(r)}, h) \leq 2 \cdot \omega_1(f^{(r)}, h).$$

Further for $\frac{1}{\sqrt{n}} \leq h \leq 1$ we apply again the estimate in [1] and obtain

$$\begin{aligned}
&\sup_{\frac{1}{\sqrt{n}}\leq h\leq 1} h^{-\beta} \cdot \omega_1((B_n f - f)^{(r)}, h) \leq 2n^{\frac{\beta}{2}} ||f^{(r)} - B_n^{(r)} f|| \\
&\leq 2n^{\frac{\beta}{2}} \cdot \Big[\frac{5}{4} \cdot \omega_1(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{r(r-1)}{2n} \cdot \|f^{(r)}\|\Big] \\
&\leq 2n^{\frac{\beta-\alpha}{2}} \cdot \Big[\frac{5}{4} \cdot \Theta_\alpha(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{r(r-1)}{2} \cdot n^{\frac{\alpha}{2}-1} \cdot \|f^{(r)}\|\Big].
\end{aligned} \tag{6}$$

Therefore with (4), (5) and (6) we complete the proof of our theorem with upper bound for C_r being $\frac{3}{2} \cdot r(r-1) + 1.25, r \geq 1$. Also (1) and (6) yield that $C_0 = 0$, which finally gives $C_r \leq \frac{3}{2} r^2$ for all $r \geq 0$. □

3. Some Consequences

If we set $\alpha = \beta$, $r = 0$ in Theorem 1, then we get

Corollary 1 *Let $n \geq 2, 0 \leq \alpha \leq 1$. Then for $f \in C^{0,\alpha}[0,1]$ one has*

$$\|B_n f - f\|_{0,\alpha} \leq 5 \cdot \Theta_\alpha(f, \frac{1}{\sqrt{n}}).$$

The last estimate is an extension of the result in Theorem A, where a similar estimate is proved only for functions f from the class $\tilde{C}^{0,\alpha}[0,1]$.

Corollary 2 *For* $f \in C^{r,\alpha}[0,1]$, $n > \max\{r+2, r(r+1)\}$, $0 \leq \alpha = \beta \leq 1$ *one has*

$$\|B_n f - f\|_{r,\alpha} \leq 5 \cdot \Theta_\alpha(f^{(r)}, \frac{1}{\sqrt{n}}) + \frac{3}{2} r^2 \cdot n^{-\frac{1}{2}} \cdot \sum_{i=0}^{r} \|f^{(i)}\|.$$

The last estimate is a generalization of Theorem A for simultaneous approximation by Bernstein operators in Hölder norms. It also extends the result in [2], showing that for all $f \in \tilde{C}^{r,\alpha}[0,1]$, $\|B_n f - f\|_{r,\alpha}$ converges to 0 as $n \to \infty$.

Corollary 3 *For* $f \in \tilde{C}^{r,\alpha}[0,1]$, $0 \leq \beta \leq \alpha < 1$, $r \in \mathbb{N}_0$ *one has*

$$||B_n f - f||_{r,\beta} = o\Big(n^{\frac{\beta-\alpha}{2}}\Big),\ n \to \infty.$$

References

[1] C. Badea, I. Badea, H.H. Gonska, Improved estimates on simultaneous approximation by Bernstein operators, Rev. Anal. Numér. Théor. Approx. 22 (1993) 1-21.

[2] J. Bustamante, M.A. Jiménez, The degree of best approximation in the Lipschitz norm by trigonometric polynomials, Aportaciones Mat. Comun. 2 (1999) 23-30.

[3] J. Bustamante, C. Castañeda Roldan, Direct and inverse results in Hölder norms. J. Approx. Theory 138 (2006) 112123.

[4] D. Cárdenas-Morales, M.A. Jiménez-Pozo, F.J. Muñoz-Delgado, Some remarks on Hölder approximation by Bernstein polynomials, Appl. Math. Letters 19 (2006) 1118-1121.

[5] C. Cottin, H.H. Gonska, Simultaneous approximation and global smoothness preservation, Suppl. ai Rend. del Cirolo Mat. di Palermo 33 (1993) 259-279.

[6] H. Gonska, J. Prestin, G. Tachev, D.X. Zhou, Simultaneous approximation by Bernstein operators in Hölder norms, manuscript (2011) submitted.

[7] H.-B. Knoop, P. Pottinger, Ein Satz vom Korovkin-Typ für C^k-Räume, Math. Z. 148 (1976), 23-32.

[8] G.G. Lorentz, Bernstein Polynomials, Toronto: University of Toronto Press 1953.