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WEAKLY NONLINEAR STABILITY ANALYSIS OF A NON-UNIFORMLY HEATED NON-NEWTONIAN FALLING FILM

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ABSTRACT

A thin liquid layer of a non-Newtonian film falling down an inclined plane that is subjected to non-uniform heating has been considered. The temperature of the inclined plane is assumed to be linearly distributed and the case when the temperature gradient is positive or negative is investigated. The film flow is influenced by gravity, mean surface-tension and thermocapillary force acting along the free surface. The coupling of thermocapillary instability and surface-wave instabilities is studied for two-dimensional disturbances. A non-linear evolution equation is derived by applying the long-wave theory and the equation governs the evolution of a power-law film flowing down an inclined plane. The linear stability analysis shows that the film flow system is stable when the plate temperature is decreasing in the downstream direction while it is less stable for increasing temperature along the plate. Weakly non-linear stability analysis using the method of multiple scales has been investigated and this leads to a secular equation of the Ginzburg-Landau type. The analysis shows that both supercritical stability and subcritical instability are possible for the film flow system. The results indicate the existence of finite-amplitude waves and the threshold amplitude and non-linear speed of these waves are influenced by thermocapillarity. The results for the dilatant as well as pseudoplastic fluids are obtained and it is observed that the result for the Newtonian model agrees with the available literature report. The influence of non-uniform heating of the film flow system on the stability of the system is compared with the stability of the corresponding uniformly heated film flow system.

INTRODUCTION

Interfacial wave behaviour and flow characteristics of falling liquid films on vertical or inclined planes have been extensively studied due to scientific interests and very wide technological applications. The interfacial waves propagating along the plane exhibit fascinating non-linear phenomena, such as solitary waves, transverse secondary instabilities and complex disordered patterns. A vast body of literature exists on the thin-film instability and has been reviewed in [1,2]. The investigations based on linear stability analysis [3] show that the basic flat-film solution on an inclined plane is unstable to long-wavelength disturbances, if the Reynolds number is greater than the critical value given by $Re_c = \frac{5}{2}\cot\beta$, where β is the angle of inclination with the horizontal. The investigations on the evolution of waves subsequent to wave inception have been made by developing approximate nonlinear stability theories. As the cut-off wave number is small for very thin layers, small wave number approximation and lubrication theory can be used to examine the nonlinear extension of the stability analysis [4]. The investigations on weakly nonlinear stability analysis of the long-wave evolution equation up to various orders of accuracy performed in [5-7] show that the solution of the Benney-type equations predict supercritical or subcritical instabilities of the waves. The flow and stability of thin, free surface nonisothermal films are of importance in coat-

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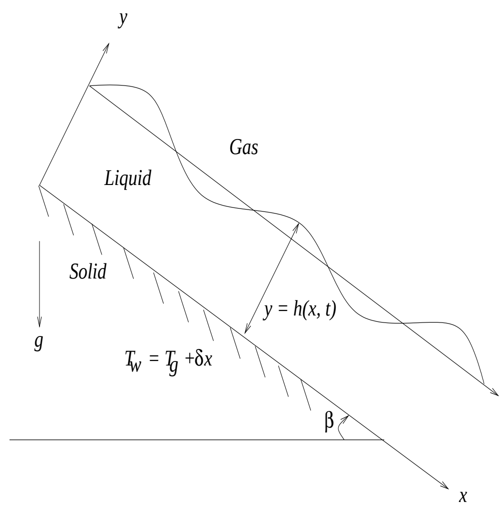

Figure 1. SCHEME OF FLUID FLOW

ing and heat transfer applications and they have been the source of both experimental and theoretical investigations for a number of years. In addition to surface wave instabilities, falling films on heated planes are susceptible to instabilities driven by shear stresses arising from the temperature dependence of the surface tension(Marangoni effect). A detailed review of interfacial thermocapillary phenomena is given by Nepomnyashchy *et al.* [8]. The results of the investigation by Kabov [9] show that non-uniform heating of falling liquid films may provide a promising solution to heat transfer enhancement, since it induces steady-state deformations of the liquid-gas interface that are beneficial to the heat transfer processes. Motivated by the need to understand the influence of non-uniform heating of a thin liquid film and to examine whether they help in improving the heat transfer from the film or hinder it by propelling the film to its rupture, a number of studies on the dynamics and stability of non-uniformly heated Newtonian liquid films down an inclined plane have been reported [10-12]. However, such investigations for non-Newtonian non-uniformly heated films have not been considered and in view of this, the present study examines the dynamics and stability of a non-Newtonian liquid film(power-law model) on a non-uniformly heated or cooled inclined plane. Such an investigation would help in understanding the influence of non-Newtonian rheology on the development of stability of inelastic films.

The nonlinear dynamic behaviour of a power-law liquid film on an inclined plane with temperature filed that increases or decreases linearly in the downstream direction has been examined in the present study. The constant temperature gradient imposed along the plane affect the basic flat-film state aswell as the surface-wave instabilities induced by gravity. The finite-amplitude long-wave instabilities of two-dimensional power-law films has been considered. The method of multiple scales has been employed to solve the nonlinear generalized kinematic equation order by order. This leads to a secular equation of Ginzurg-Landau type. The analysis shows that supercritical stability and subcritical instability are both possible for the film flow system and that the power-law index n strongly influences the stability characteristics of the non-Newtonian inelastic fluids.

MATHEMATICAL FORMULATION

A thin power-law liquid film flowing down a plane inclined at an angle β with the horizontal has been considered(Fig. 1). A constant temperature gradient δ has been imposed along the plane and δ can take both positive or negative values. The film is bounded above by a motionless gas at ambient temperature T_g and pressure p_g. The free surface is assumed adiabatic. The origin is located on the plane surface where the temperature is T_g and a coordinate system is chosen with x as the streamwise coordinate along the plane and y as normal to the plane. The wall temperature is given by $T_w = T_g + \delta x$, and it increases(decreases) in the streamwise direction with positive(negative) δ. The surface tension σ depends linearly on temperature and is given by $\sigma = \sigma_0 - \gamma(T - T_g)$, where σ_0 is the mean surface tension at temperature T_g and $\gamma = -\dfrac{d\sigma}{dT}$ is a positive constant for most common liquids. The governing equations in dimensionless form are the two-dimensional mass, momentum and energy equations for the power-law model given by(taking the same notation for both dimensional and non-dimensional quantities)

$$u_x + v_y = 0 \tag{1}$$
$$\varepsilon\left(u_t + uu_x + vu_y\right) = -\varepsilon p_x + Re + \varepsilon^2\left(\tau_{xx}\right)_x + \left(\tau_{xy}\right)_y \tag{2}$$
$$\varepsilon^2\left(v_t + uv_x + vv_y\right) = -p_y - Re\cot\beta + \varepsilon\left(\tau_{xy}\right)_x + \varepsilon\left(\tau_{yy}\right)_y \tag{3}$$
$$\varepsilon Pr_n\left(T_t + uT_x + vT_y\right) = \varepsilon^2 T_{xx} + T_{yy} \tag{4}$$

and the boundary conditions are

$$u = 0,\ v = 0,\ T = x \qquad \text{on } y = 0 \tag{5}$$
$$-p\left(1+\varepsilon^2 h_x^2\right) + 2\varepsilon^3 h_x^2 \eta u_x - 2\varepsilon h_x \eta\left(u_y + \varepsilon^2 v_x\right) + 2\varepsilon\eta v_y = \frac{Sh_{xx}\left(1 - CaT\right)}{\sqrt{1+\varepsilon^2 h_x^2}} \qquad \text{on } y = h \tag{6}$$
$$2\varepsilon^2 h_x \eta\left(v_y - u_x\right) + \left(1 - \varepsilon^2 h_x^2\right)\eta\left(u_y + \varepsilon^2 v_x\right) = -M_n\left(T_x + h_x T_y\right)\sqrt{1+\varepsilon^2 h_x^2} \qquad \text{on } y = h \tag{7}$$
$$T_y - \varepsilon^2 h_x T_x = 0 \qquad \text{on } y = h \tag{8}$$
$$v = h_t + uh_x \qquad \text{on } y = h \tag{9}$$

where the following scales have been used for non-dimensionalization. The coordinate y and the interface position $y=h(x,t)$ have been scaled by mean film thickness h_0, x by a length l proportional to the disturbance wavelength, t by l/U, u by U, v by Uh_0/l, $p-p_g$ by ρU^2, $T-T_g$ by δl, τ_{xx} and τ_{yy} by $\mu_n\left(\frac{U}{h_0}\right)^{n-1}\frac{U}{l}$ and τ_{xy} by $\mu_n\left(\frac{U}{h_0}\right)^{n}$, where $U\left(\frac{\mu_n}{\rho h_0^n}\right)^{\frac{1}{2-n}}$ and δl measures the temperature difference along the plate between two points separated by a wavelength. Here, u and v are velocity components along x and y directions respectively, $\tau_{xx}=2\mu_n\eta u_x$, $\tau_{xy}=\mu_n\eta\left(u_y+v_x\right)$, $\tau_{yy}=2\mu_n\eta v_y$ and $\eta=\left[2\varepsilon^2\left(u_x^2+v_y^2\right)+\left(u_y+\varepsilon^2 v_x\right)^2\right]^{\frac{n-1}{2}}$, μ_n is the consistency coefficient, n is the power-law index and $\varepsilon=\dfrac{h_0}{l}\,(\ll 1)$ is a small parameter. The flow is governed by the dimensionless parameters, the Reynolds number, $Re=G\sin\beta$, $G=\left(\dfrac{\rho^2 h_0^{n+2}}{\mu_n^2}\right)^{\frac{1}{2-n}}g$; the Marangoni number, $Ma_n=\dfrac{\gamma\,\delta\,l}{\chi}\left(\dfrac{\rho^{n-1}h_0^n}{\mu_n}\right)^{\frac{1}{2-n}}$; the capillary number, $Ca=\dfrac{\gamma\,\delta\,l}{\sigma_0}$; the surface tension parameter $S=\varepsilon^2\dfrac{\sigma_0}{\left(\frac{\mu_n^2}{\rho^n h_0^{3n-2}}\right)^{\frac{1}{2-n}}}$ and the Prandtl number, $Pr_n=\dfrac{1}{\chi}\left(\dfrac{\mu_n}{\rho h_0^{2n-2}}\right)^{\frac{1}{2-n}}$, where χ is the thermal diffusivity.

When the power-law exponent n is equal to 1, the model describes the Newtonian fluid [11]; if $n<1$, the fluid is said to be pseudoplastic or shear thinning and if $n>1$, the fluid is called dilatant or shear thickening. In practical applications, $S\simeq O(1)$ and the analysis is performed by taking $Re\simeq O(1)$, $Ca\simeq O(\varepsilon^2)$, $Pr_n\simeq O(1)$, $M_n=\varepsilon\dfrac{Ma_n}{Pr_n}\simeq O(1)$(modified Marangoni number).

It is assumed that the velocity and temperature in the liquid film vary slowly along the plane and the length of the surface waves is much larger than the mean film thickness. Since the long-wavelength modes are the most unstable ones for the film flow, the physical quantities u,v,p and T are expanded in powers of the small parameter ε. Substituting these in (1)-(9) and collecting the coefficients of like powers of ε, the zeroth and the first order equations are obtained and the solutions are determined. The generalized kinematic equation is then obtained from (9) as

$$h_t+A(h)h_x+\varepsilon\left(B(h)h_x+C(h)h_{xxx}\right)_x+O(\varepsilon^2)=0 \qquad (10)$$

where

$$A(h)=(Re\,h-M_n)^{1/n}h$$

$$\begin{aligned}
B(h)=\;&\frac{\cot\beta}{nRe^2}\Bigg[\frac{2n^3\left((-M_n)^{(2n+1)/n}-(Re\,h-M_n)^{(2n+1)/n}\right)}{(n+1)(2n+1)}\\
&+\frac{2Re\,n^2h\,(Re\,h-M_n)^{(n+1)/n}}{(n+1)}-nRe^2h^2\,(Re\,h-M_n)^{1/n}\Bigg]\\
&+\frac{M_nPr_n}{2nRe^4}\Bigg[\frac{2n^2\left((-M_n)^{(2n+1)/n}-(Re\,h-M_n)^{(2n+1)/n}\right)}{(n+1)(2n+1)}\\
&+\frac{2Re\,nh\,(Re\,h-M_n)^{(n+1)/n}}{(n+1)}+Re^2h^2\,(Re\,h-M_n)^{1/n}\Bigg]\\
&\times\Bigg[\frac{n^2\left((-M_n)^{(n+1)/n}-(Re\,h-M_n)^{(n+1)/n}\right)}{(n+1)}\\
&+nRe\,h\,(Re\,h-M_n)^{1/n}\Bigg]+\frac{n\,(Re\,h-M_n)^{2/n}}{Re^4\,(n+1)}\\
&\times\Bigg[-Re\,h+\frac{n}{n+1}\,(Re\,h-M_n)\Bigg]\Bigg[-\frac{2n\,(Re\,h-M_n)^{(2n+1)/n}}{2n+1}\\
&+Re\,h\,(Re\,h-M_n)^{(n+1)/n}+(Re\,h-M_n)\,(-M_n)^{(n+1)/n}\\
&-\frac{(-M_n)^{(2n+1)/n}}{(2n+1)}\Bigg]+\frac{n^2M_n\,(Re\,h-M_n)^{1/n}}{Re^4\,(n+1)\,(n+2)\,(2n+2)}\\
&\times\Bigg[(Re\,h-M_n)^{(2n+2)/n}-\frac{(2n+2)\,Re\,h\,(Re\,h-M_n)^{(n+2)/n}}{n}\\
&-(-M_n)^{(2n+2)/n}\Bigg]+\frac{n^2\,(Re\,h-M_n)^{1/n}}{Re^4\,(n+1)\,(2n+1)\,(2n+2)\,(3n+2)}\\
&\times\Bigg[(Re\,h-M_n)^{(3n+2)/n}-\frac{(3n+2)\,Re\,h\,(Re\,h-M_n)^{(2n+2)/n}}{n}\\
&-(-M_n)^{(3n+2)/n}\Bigg]+\frac{n\,(Re\,h-M_n)^{1/n}}{Re^4\,(n+1)}\Bigg[-\frac{2n^2\,(-M_n)^{(2n+1)/n}}{(n+1)\,(2n+1)}\\
&+Re^2h^2\,(Re\,h-M_n)^{1/n}-\frac{n\,Re\,h}{(n+1)}\,(Re\,h-M_n)^{(n+1)/n}\Bigg]\\
&\times\Bigg[(-M_n)^{(n+1)/n}+\frac{(n+1)\,Re\,h\,(Re\,h-M_n)^{1/n}}{n}\\
&-(Re\,h-M_n)^{(n+1)/n}\Bigg]
\end{aligned}$$

$$\begin{aligned}
C(h)=\;&\frac{S}{nRe^3}\Bigg[\frac{2n^3\left(-(-M_n)^{(2n+1)/n}+(Re\,h-M_n)^{(2n+1)/n}\right)}{(n+1)(2n+1)}\\
&-\frac{2Re\,n^2h\,(Re\,h-M_n)^{(n+1)/n}}{(n+1)}+nRe^2h^2\,(Re\,h-M_n)^{1/n}\Bigg]
\end{aligned}$$

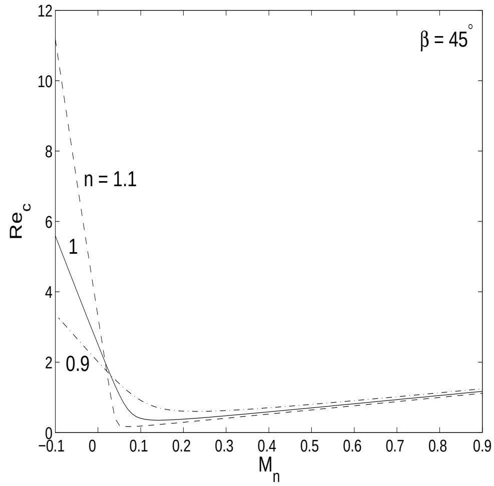

Figure 2. CRITICAL REYNOLDS NUMBER AS A FUNCTION OF M_n FOR DIFFERENT VALUES OF n

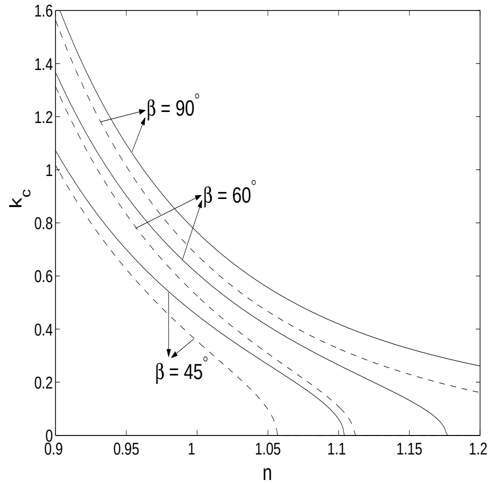

Figure 3. NEUTRAL STABILITY CURVES; TEMPERATURE OF THE PLANE DECREASING LINEARLY DOWNSTREAM(- - - -); TEMPERATURE OF THE PLANE INCREASING(——-) LINEARLY DOWNSTREAM

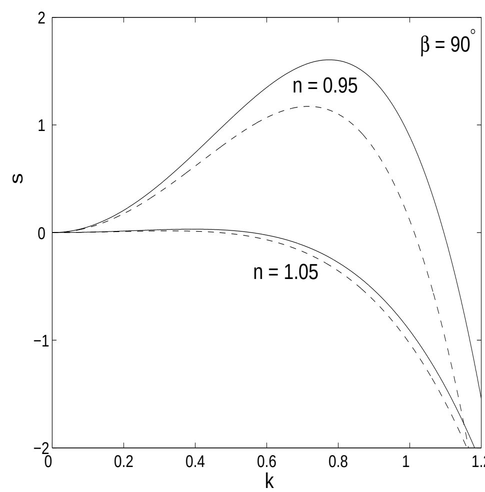

Figure 4. GROWTH RATE OF DISTURBED WAVES FOR A VERTICAL PLANE WITH TEMPERATURE DISTRIBUTION EITHER DECREASING(- - - -) OR INCREASING(——-) LINEARLY DOWNSTREAM

Eqn.(10) describes the evolution of a non-Newtonian falling film on a non-uniformly heated inclined plane and when $n = 1$, it reduces to that obtained by Miladinova *et al.* [11].

STABILITY ANALYSIS

The base state is represented by

$$\begin{aligned} u_0 &= \frac{n}{Re\,(n+1)}\left[(Re\,h - M_n)^{(n+1)/n} - [Re\,(h-y) - M_n]^{(n+1)/n}\right] \\ v_0 &= \frac{nh_x}{Re\,(n+1)}\left[(Re\,h - M_n)^{(n+1)/n} - [Re\,(h-y) - M_n]^{(n+1)/n}\right] \\ &\quad - (Re\,h - M_n)^{1/n}\, y h_x \\ p_0 &= Re \cot\beta\,(h-y) - Sh_{xx} \\ T_0 &= x \end{aligned} \tag{11}$$

with $h = 1$. For a parallel shear flow, Eqn. (10) admits normal-mode solutions of the form

$$h(x,t) = 1 + H(x,\ t) \tag{12}$$

where $H(x,\ t)$, the unsteady part of the film thickness representing the disturbance component is given by

$$H(x,\ t) = \Gamma e^{i(kx-ct)+st} + \bar{\Gamma} e^{-i(kx-ct)-st} \tag{13}$$

where Γ, k, c and s are real and represent the amplitude, the wave number, the linear phase speed and the linear growth rate of the disturbance respectively. Inserting the normal mode representation, Eqn. (13) into Eqn.(10) and retaining terms up to the order of H^3, the unsteady equation is obtained as

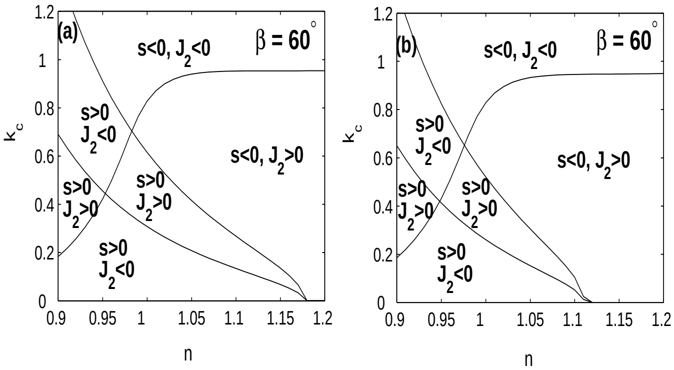

Figure 5. NEUTRAL STABILITY CURVES (a) TEMPERATURE OF THE PLANE INCREASING LINEARLY DOWNSTREAM; (b) TEMPERATURE OF THE PLANE DECREASING LINEARLY DOWNSTREAM

$$H_t + A_1 H_x + \varepsilon B_1 H_{xx} + \varepsilon C_1 H_{xxxx} = -\left[A_1' H + \frac{A_1''}{2} H^2\right] H_x$$
$$-\varepsilon \left[B_1' H + \frac{B_1''}{2} H^2\right] H_{xx} - \varepsilon \left(C_1' H + \frac{C_1''}{2} H^2\right) H_{xxxx}$$
$$-\varepsilon \left[B_1' + B_1'' H\right] H_x^2 - \varepsilon \left(C_1' + C_1'' H\right) H_x \frac{\partial^3 H}{\partial x^3} + O(H^4). \quad (14)$$

where

$A_1 = A(h=1), A_1' = A'(h=1), A_1'' = A''(h=1),$
$B_1 = B(h=1), B_1' = B'(h=1), B_1'' = B''(h=1),$
$C_1 = C(h=1), C_1' = C'(h=1), C_1'' = C''(h=1).$

and a prime denotes the derivative with respect to h.
It is important to note that this constant film thickness approximation with long-wave perturbations are reasonable approximations only for certain segments of flow. Further, the parallel flow approximation used in the derivation of Eqn. (14) implies that the unsteady equation is only locally valid. Eqn. (14) describes the behaviour of finite-amplitude disturbances on the film and it is used to predict the timewise behaviour of an initially sinusoidal disturbance on the non-uniformly heated power-law fluid film.

Linear Stability Analysis

Inserting the normal mode representation given by Eqn. (13) into the linearized equation obtained from Eqn. (14) by neglecting nonlinear terms, the complex wave celerity is obtained as

$$c + i\, s = kA_1 + i\, \varepsilon k^2 \left(B_1 - k^2 C_1\right) \quad (15)$$

where c is the linear wave speed and s is the linear growth rate. The flow is in a linear unstable supercritical condition for $s > 0$ and in a linear stable subcritical condition for $s < 0$. For $s = 0$, the flow is neutrally stable and the amplitude of the disturbance is neither amplified nor damped. The surface waves will grow for disturbance wave numbers smaller than the critical(cut-off) wave number given by $k_c = (B_1/C_1)^{1/2}$. This surface wave instability is called primary instability and corresponds to a Hopf bifurcation from the flat film solution. The emerging branch of solutions is called supercritical(subcritical) if it bifurcates towards the region where $k < k_c (k > k_c)$.

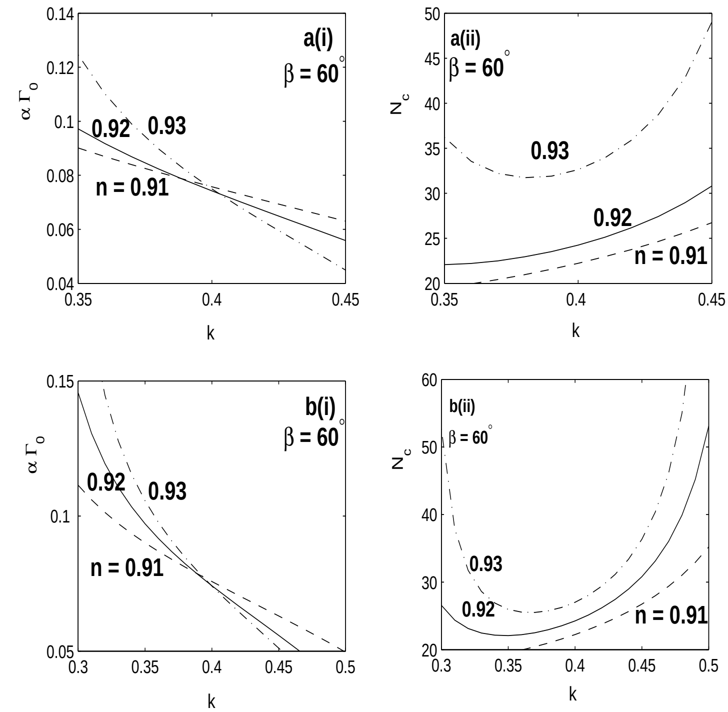

Figure 6. THRESHOLD AMPLITUDE AND NONLINEAR WAVE SPEED FOR A PSEUDOPLASTIC FLUID IN SUPERCRITICAL STABLE REGION (a) TEMPERATURE OF THE PLANE INCREASING LINEARLY DOWNSTREAM; (b) TEMPERATURE OF THE PLANE DECREASING LINEARLY DOWNSTREAM

Nonlinear Stability Analysis

As the perturbed wave grows to a finite-amplitude, linear theory cannot be used to predict the flow behaviour accurately. Therefore in order to examine whether the finite-amplitude disturbances in the linearly stable region causes instability(subcritical instability) and to investigate whether the subsequent nonlinear evolution of disturbances in the linearly unstable region develops into a new equilibrium state with a finite-amplitude(supercritical instability) or grows to be unstable, the weakly nonlinear stability analysis is employed.

The nonlinear stability analysis of Eqn. (14) by the method of multiple scales [13,14] yields

$$\left(L_0+\alpha L_1+\alpha^2 L_2\right)\left(\alpha H_1+\alpha^2 H_2+\alpha^3 H_3\right) = -\alpha^2 N_2-\alpha^3 N_3 \tag{16}$$

where α measures the distance from criticality and H is expanded in powers of α as $H\left(\alpha,x,x_1,t,t_1,t_2\right)=\alpha H_1+\alpha^2 H_2+\alpha^3 H_3$

$$\begin{aligned}
\frac{\partial}{\partial t} &\rightarrow \frac{\partial}{\partial t}+\alpha\frac{\partial}{\partial t_1}+\alpha^2\frac{\partial}{\partial t_2}\\
\frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x}+\alpha\frac{\partial}{\partial x_1};\ t_1=\alpha t,\ t_2=\alpha^2 t,\ x_1=\alpha x\\
L_0 &= \frac{\partial}{\partial t}+A_1\frac{\partial}{\partial x}+\varepsilon B_1\frac{\partial^2}{\partial x^2}+\varepsilon C_1\frac{\partial^4}{\partial x^4}\\
L_1 &= \frac{\partial}{\partial t_1}+A_1\frac{\partial}{\partial x_1}+2\varepsilon B_1\frac{\partial^2}{\partial x\partial x_1}+4\varepsilon C_1\frac{\partial^4}{\partial x^3\partial x_1}\\
L_2 &= \frac{\partial}{\partial t_2}+\varepsilon B_1\frac{\partial^2}{\partial x_1^2}+6\varepsilon C_1\frac{\partial^4}{\partial x^2\partial x_1^2}\\
N_2 &= A_1' H_1 H_{1x}+\varepsilon B_1' H_1 H_{1xx}+\varepsilon C_1' H_1 H_{1xxxx}\\
&\quad +\varepsilon B_1' H_{1x}^2+\varepsilon C_1' H_{1x} H_{1xxx}\\
N_3 &= A_1{}'\left(H_1 H_{2x}+H_1 H_{1x_1}+H_2 H_{1x}\right)\\
&\quad +\ \varepsilon B_1{}'\left(H_1 H_{2xx}+2H_1 H_{1xx_1}+H_2 H_{1xx}\right)\\
&\quad +\ \varepsilon C_1{}'\left(H_1 H_{2xxxx}+4H_1 H_{1xxxx_1}+H_2 H_{1xxxx}\right)\\
&\quad +\ \varepsilon B_1{}'\left(2H_{1x} H_{2x}+2H_{1x} H_{1x_1}\right)\\
&\quad +\ \varepsilon C_1{}'\left(H_{2xxx} H_{1x}+3H_{1xxx_1} H_{1x}+H_{1xxx} H_{2x}\right.\\
&\quad +\ \left. H_{1xxx} H_{1x_1}\right)+\frac{A_1{}''}{2} H_1{}^2 H_{1x}\\
&\quad +\ \varepsilon\frac{B_1{}''}{2} H_1{}^2 H_{1xx}+\varepsilon\frac{C_1{}''}{2} H_1{}^2 H_{1xxxx}\\
&\quad +\ \varepsilon B_1{}'' H_1 H_{1x}^2+\varepsilon C_1{}'' H_1 H_{1x} H_{1xxx}
\end{aligned}$$

The solution of Eqn. (16) at $O(\alpha)$ is obtained by solving $L_0 H_1=0$ and is in the form

$$H_1=\Gamma\, e^{i\,(kx-c\,t)}+\bar{\Gamma}\, e^{-i\,(kx-c\,t)} \tag{17}$$

where $\Gamma(x_1,t_1,t_2)$ is the nonlinear amplitude function and $\bar{\Gamma}(x_1,t_1,t_2)$ is its complex conjugate. The solution of the equation $L_0 H_2+L_1 H_1=-N_2$ at the $O(\alpha^2)$ is in the form

$$H_2=e\,\Gamma^2\, e^{2i\,(kx-c\,t)}+\bar{e}\,\bar{\Gamma}^2\, e^{-2i\,(kx-c\,t)}. \tag{18}$$

Using the solution H_1 and H_2 in the $O(\alpha^3)$ equation given by $L_0 H_3+L_1 H_2+L_2 H_1=-N_3$, the equation for the perturbation amplitude $\Gamma(x_1,t_1,t_2)$ is obtained as

$$\frac{\partial\Gamma}{\partial t_2}+\varepsilon\left(B_1-6k^2 C_1\right)\frac{\partial^2\Gamma}{\partial x_1^2}-\alpha^{-2}s\Gamma+\left(J_2+iJ_4\right)\Gamma^2\bar{\Gamma}=0 \tag{19}$$

where

$$\begin{aligned}
J_2 &= \varepsilon\left(7k^4 e_r C_1'-k^2 e_r B_1'-\frac{k^2}{2}B_1''+\frac{k^4}{2}C_1''\right)-k e_i A_1'\\
J_4 &= \varepsilon\left(7k^4 e_i C_1'-k^2 e_i B_1'\right)+k e_r A_1'+k\frac{A_1''}{2}\\
e_r &= \frac{2\left(B_1'-k^2 C_1'\right)}{\left(-4B_1+16k^2 C_1\right)}\\
e_i &= \frac{-A_1'}{\varepsilon\left(-4kB_1+16k^3 C_1\right)}.
\end{aligned}$$

The weakly nonlinear behaviour of the film is investigated using Eqn. (19). It is important to note that such an expansion is only valid for wave numbers close to neutral and not near-critical when α approaches zero. The solution of Eqn. (19) for a filtered wave in which spatial modulation does not exist and the diffusion terms in Eqn. (19) vanishes is obtained by taking $\Gamma=\Gamma_0 e^{-b(t_2)t_2}$. This leads to the Ginzburg - Landau equation given by

$$\begin{aligned}
\frac{\partial\Gamma_0}{\partial t_2} &= \left(\alpha^{-2}s-J_2\Gamma_0^2\right)\Gamma_0 &\quad (20)\\
\frac{\partial b\left(t_2\right)t_2}{\partial t_2} &= J_4\Gamma_0^2. &\quad (21)
\end{aligned}$$

The second term in Eqn. (20) induced by the effect of nonlinearity can either accelerate or decelarate the exponential growth of the linear disturbance depending upon the signs of s and J_2. The perturbed wave speed caused by the infinitesimal disturbances appearing in the nonlinear system can be modified using Eqn. (21). The threshold amplitude $\alpha\Gamma_0$ is given by

$$\alpha\Gamma_0=\sqrt{\frac{s}{J_2}} \tag{22}$$

and the nonlinear wave speed is given as

$$N_c=c+s\frac{J_4}{J_2}. \tag{23}$$

It is observed from Eqn. (22) that in the linear unstable region ($s>0$), the condition for existence of a supercritical stable region is $J_2>0$ and $\alpha\Gamma_0$ is the threshold amplitude. In the

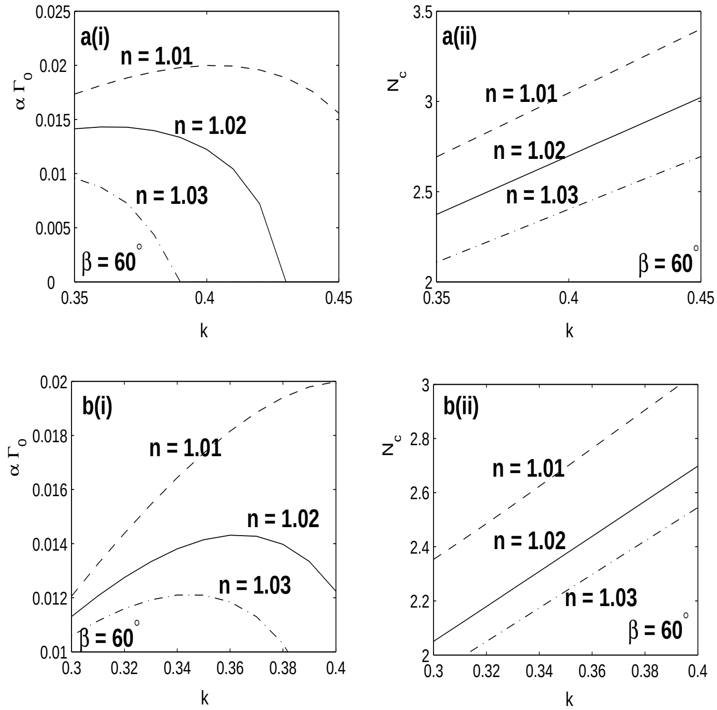

Figure 7. THRESHOLD AMPLITUDE AND NONLINEAR WAVE SPEED FOR A DILATANT FLUID IN SUPERCRITICAL STABLE REGION (a) TEMPERATURE OF THE PLANE INCREASING LINEARLY DOWNSTREAM; (b) TEMPERATURE OF THE PLANE DECREASING LINEARLY DOWNSTREAM

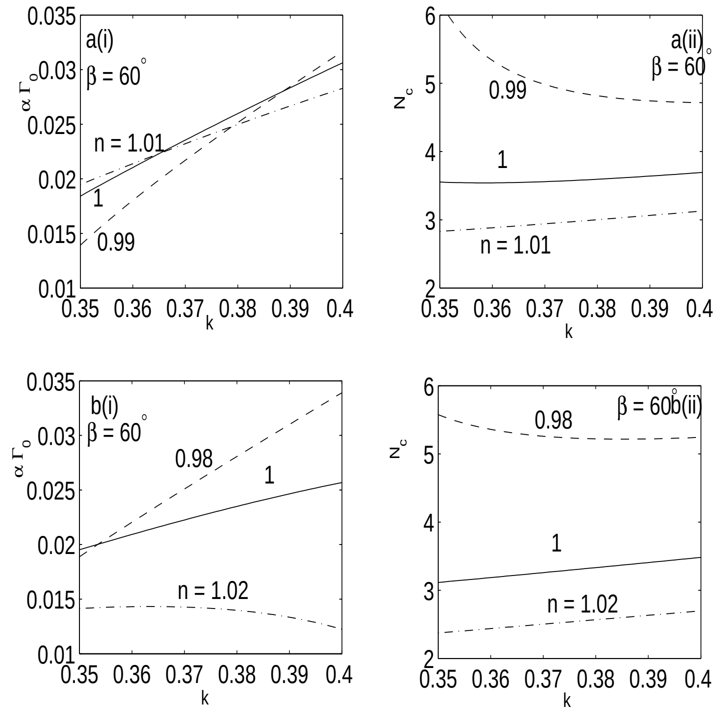

Figure 8. THRESHOLD AMPLITUDE AND NONLINEAR WAVE SPEED OF NONLINEAR WAVES IN A POWER-LAW FLUID IN SUPERCRITICAL STABLE REGION (a) TEMPERATURE OF THE PLANE INCREASING; (b) DECREASING LINEARLY DOWNSTREAM

linear stable region ($s < 0$), if $J_2 < 0$, then the flow has the behaviour of subcritical instability and $\alpha\Gamma_0$ is the threshold amplitude. The condition for the existence of a subcritical stable region is $s < 0, J_2 > 0$ and $J_2 = 0$ gives the condition of existence of a neutral stability curve. It is also observed from Eqn. (22) that that a negative value of J_2 can make the system unstable. In this case, if $s > 0$ then the amplitude of the disturbance may become larger than the threshold amplitude and cause the system to reach an explosive state.

NUMERICAL RESULTS AND DISCUSSION

The conditions obtained for linear and nonlinear stability of the non-uniformly heated power-law fluid film on an inclined plane have been numerically evaluated for Reynolds number Re, Modified Marangoni number M_n, the Prandtl number Pr_n and the surface tension parameter S which are defined in terms of power-law index n. The values of dimensional quantities are taken as([15,16]) $\rho = 998 kg/m^3$, density; $\mu_n = 1.002 \times 10^{-3} mPas^n$, dynamic viscosity; $h_0 = 10^{-4} m$, mean film thickness; $\sigma_0 = 0.0727 N/m$, mean surface tension; $\chi = 1.169 \times 10^{-7} m^2/s$, thermal diffusivity; $c_p = 4258.290735 Ws/Kg\ K$,specific heat capacity; $\gamma = 0.1103 \times 10^{-3} N/mK$, surface tension variation with temperature, a positive constant.

In what follows, attention is focussed on the investigation of the influence of non-uniform heating on the stability of the flow of power-law fluid film down an incline.

Linear Stability Solutions

The influence of gravity and thermocapillarity on the film instability is shown in Fig. 2, displaying the critical Reynolds number Re_c versus M_n for $S = 0$ for different values of power-law index n when $\beta = 45°$. It is observed that there exists a stable region below each curve, due to thermocapillarity and hydrostatic pressure. The Marangoni number for which the critical Reynolds number is the smallest depends on n, which in turn depends on Pr_n. Further, Re_c is more(less) for a dilatant fluid than for a pseudoplastic fluid when the temperature of the plane decreases(increases) downstream. In other words, the pseudoplastic fluid is less(more) stable than the dilatant fluid, when the temperature of the plane decreases(increases) downstream. Further, the dilatant(pseudoplastic) fluid is more(less) stable than the Newtonian film when the temperature of the plane decreases and

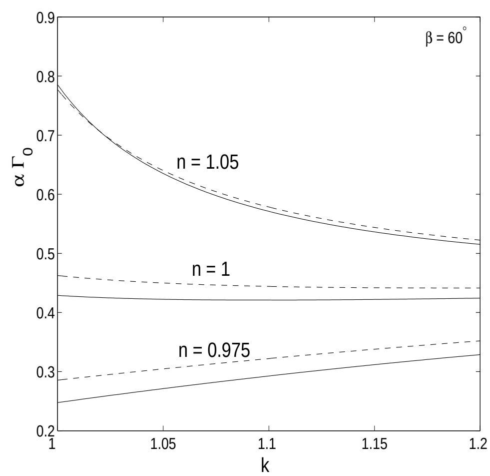

Figure 9. THRESHOLD AMPLITUDE IN SUBCRITICAL UNSTABLE REGION FOR TEMPERATURE OF THE PLANE EITHER DECREASING(- - - -) OR INCREASING(——-) LINEARLY DOWNSTREAM

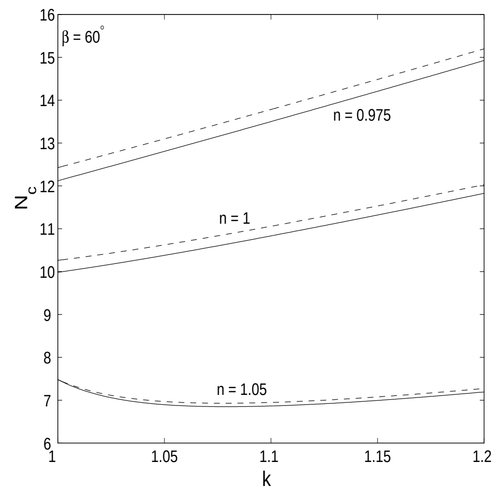

Figure 10. NONLINEAR WAVE SPEED IN SUBCRITICAL UNSTABLE REGION FOR TEMPERATURE OF THE PLANE EITHER DECREASING(- - - -) OR INCREASING(——-) LINEARLY DOWNSTREAM

a reverse trend is observed when the temperature of the plane increases.

The linear stability analysis yields the neutral stability curve, which separates the $k_c - n$ plane into two regions. Figure 3 shows the neutral stability curves when the temperature of the plane increases or decreases in the downstream direction for different angles of inclination. It is observed that, for a particular angle of inclination, the linearly unstable region is larger when the temperature of the plane increases than when the temperature of the plane decreases linearly downstream. In both cases, the unstable region becomes larger as the angle of inclination increases.

Figure 4 shows the temporal growth rate as a function of wave number k for a vertical plane. It is observed that, growth rate decreases with increase in k for dilatant fluid, while, for pseudoplastic fluids, there is a critical value of k, below(above) which the growth rate increases(decreases). The growth rate of pseudoplastic fluid is more when the temperature of the plane increases than when it decreases linearly downstream.

Nonlinear Stability Solutions

Figure 5 shows the region of subcritical instability($s < 0, J_2 < 0$) in the linearly stable region and supercritical explosive state($s > 0, J_2 < 0$) in the linearly unstable region. It is evident from Fig. 5 that both these states of the film flow system exist for power-law film down an incline, where the temperature of the plane increases or decreases linearly downstream. The pseudoplastic fluid is supercritically stable in two regions whereas, there is only one region where the dilatant fluid is supercritically stable. Further, the critical value of n below which the film flow system is unstable is larger when the temperature of the plane increases than when it decreases downstream.

An infinitesimal disturbance in the linearly unstable region($s > 0$) will attain a finite equilibrium amplitude, when the nonlinear amplification rate J_2 in the region is positive. Figure 6 shows the threshold amplitude and nonlinear wave speed in the supercritical stable region as a function of the wave number for a pseudoplastic fluid film. It is observed that the threshold amplitude decreases with increase in the wave number k irrespective of whether the temperature along the plane increases or decreases linearly in the downstream direction. Also, in each case, there exist critical wave numbers below which the threshold amplitude increases with increase in n for pseudoplastic fluids and a reverse trend is observed for values of k above the critical value. Further, the nonlinear wave speed of the pseudoplastic fluid increases with increase in n for a fixed k. The nonlinear wave speed is more for the case when the temperature of the inclined plane increases than when it decreases linearly downstream. In addition, for each n ($n < 1$), nonlinear wave speed decreases up to a certain wave number and then it begins to increase.

From Fig. 7, it is observed that the threshold amplitude increases up to some k for any n and then begins to decrease for a dilatant fluid when the temperature of the plane increases linearly. However, this tendency is not exhibited by dilatant fluids for large n, when the temperature of the plane increases linearly downstream. The nonlinear wave speed for dilatant fluids increases with increase in k. However, for a fixed k, the nonlinear wave speed decreases with increase in n. In the supercritical stable region, the nonlinear wave speed is higher for pseudoplastic fluid than for dilatant fluids(Fig. 8).

Figure 9 shows the threshold amplitude as a function of the wave number k in the subcritical unstable region. It is observed that for any n, threshold amplitude is more for the case when the temperature of the plane decreases than when it increases. The threshold amplitude of a dilatant fluid decreases with increase in k; it is almost linear for a Newtonian film; and for a pseudoplastic film, threshold amplitude increases with increase in k. For a fixed k, threshold amplitude increases with increase in n.

The nonlinear wave speed in subcritical unstable region is presented in Fig. 10 and it is observed that nonlinear wave speed is more for the case when the temperature of the plane decreases than when it increases linearly downstream. For a fixed k, nonlinear wave speed decreases with increase in n.

CONCLUSION

Weakly nonlinear stability analysis of a power-law film on an inclined plane whose temperature either increases or decreases linearly downstream has been considered. The free surface has been assumed adiabatic. The analysis has been performed under the assumption that the liquid film is very thin; the heat flux is weak and the induced gravity-driven flow is relatively slow. That is, $|kh_x| \ll 1$ and this approximation gives qualitative results for the constant film thickness assumption at the zeroth order. It is important to note that this constant film thickness assumption with long-wave perturbations are reasonable approximations only for certain segments of a power-law fluid film on an inclined plane with a temperature field decreasing or increasing linearly downstream along the surface.

The generalized nonlinear evolution equation for the film thickness has been obtained and the stability of the film flow system has been investigated. The linear stability analysis of the equation yields critical values of the Reynolds number and the dimensionless linearized phase speed for different values of the modified Marangoni number. The critical Reynolds number grows linearly with the absolute value of the modified Marangoni number if the temperature of the plane is decreased in the downstream direction($M_n < 0$). In this case, linear stability threshold increases with n. Increasing the temperature of the plane results in a decrease in the critical Reynolds number. This reduction is important for dilatant fluids. Therefore, decreasing the temperature of the plane in the downstream direction has a considerable stabilizing effect on the Newtonian film and this effect is enhanced on the dilatant fluid film but is less on the pseudoplastic fluid film. On the other hand, temperature increase plays a destabilizing role on the power-law film and the effect is more pronounced on pseudoplastic film than on Newtonian film, so that the pseudoplastic film is more unstable than the Newtonian film; however, in this case, the dilatant fluid is less unstable than the Newtonian film.

The results of the weakly nonlinear stability analysis using the method of multiple scales show that both supercritical stability and subcritical instability are possible for the film flow system.

The analysis indicates the existence of finite-amplitude waves and the threshold amplitude and nonlinear speed of these waves are influenced by thermocapillarity. Depending on the direction of the imposed temperature gradient, the thermocapillary effect can be either stabilizing or destabilizing. If the temperature of the inclined plane is decreasing in the down stream direction, the thermocapillary force acts in the same direction. As a result, the disturbances converge to finite-amplitude waves with small amplitudes. In the supercritical stable region such waves in a dilatant fluid film have less threshold amplitude than in a Newtonian film. While, the threshold amplitude in a pseudoplastic film is more than that in a Newtonian film. It is important to note that for an isothermal film(Newtonian), the disturbances do not converge to finite-amplitude waves. When the temperature of the inclined plane is increasing in the down stream direction, the thermocapillary force acts in the opposite direction and therefore promotes the growth rate of the wave amplitude with respect to the isothermal case. Beyond a certain wave number, the threshold amplitude of finite-amplitude waves in the supercritical stable region is more for a pseudoplastic fluid film than for Newtonian and dilatant films. Thus, the non-uniform heating of the film influences the film instability through the thermocapillary force acting along the free surface and the non-Newtonian rheology plays a vital role in either increasing or decreasing the threshold amplitude of waves with respect to Newtonian film.

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