Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 406240, 12 pages
http://dx.doi.org/10.1155/2013/406240

Research Article

Approximate Solution of *n*th-Order Fuzzy Linear Differential Equations

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Received 29 January 2013; Accepted 19 March 2013

Academic Editor: Ker-Wei Yu

The approximate solution of nth-order fuzzy linear differential equations in which coefficient functions maintainthe sign is investigated by the undetermined fuzzy coefficients method. The differential equations is converted to a crisp function system of linear equations according to the operations of fuzzy numbers. The fuzzy approximate solution of the fuzzy linear differential equation is obtained by solving the crisp linear equations. Some numerical examples are given to illustrate the proposed method. It is an extension of Allahviranloo's results.

1. Introduction

Fuzzy differential equations (FDEs), which are utilized for the purpose of the modeling problems in science and engineering, have been studied by many researchers. Most practical problems require the solutions of fuzzy differential equations (FDEs) which are satisfied with fuzzy initial conditions, and therefore a fuzzy initial problem occurs and needs to be solved. However, most fuzzy initial value problems could not be solved exactly. So it is necessary to consider their approximating methods.

Prior to discussing fuzzy differential equations and their associated numerical algorithms, it is necessary to present an appropriate brief introduction to derivative of the fuzzy-valued function. The concept of a fuzzy derivative was first introduced by Chang and Zadeh [1], and it was followed up by Dubois and Prade [2] who used the extension principle in their approach. Other fuzzy derivative concepts have been proposed by Puri and Ralescu [3] and Goetschel Jr. and Voxman [4] as an extension of the Hukuhara derivative of multivalued functions. In recent years, many works have been produced in the aspects of theories and applications on fuzzy differential equations; see [5–15]. The notation of fuzzy differential equation was initially introduced by Kandel and Byatt [16, 17] who later applied the concept of fuzzy differential equation to the analysis of fuzzy dynamical problems [18, 19]. A thorough theoretical research of fuzzy Cauchy problems was given by Kaleva [20, 21], Seikkala [22], Ouyang and Wu [23], Kloeden [24], and Wu [25]. A generalization of fuzzy differential equation was given by Aubin [26, 27], Baĭdosov [6], Leland [28], and Colombo and Křivan [29]. Some numerical methods for solving fuzzy differential equations were introduced in [30–33].

For an nth-order linear differential equation

$$y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0(t)\, y = g(t) \quad (1)$$

with fuzzy initial conditions

$$y(t_0) = \tilde{b}_0,\ y'(t_0) = \tilde{b}_1, \ldots, y^{(n-1)}(t_0) = \tilde{b}_{n-1}, \quad (2)$$

where $g(t)$, $a_i(t)$, $i = 0, 1, \ldots, n-1$, $t \in [t_0, T]$ are continuous functions, Buckley and Feuring [34] presented two analytical methods for solving them. The first was to fuzzify the crisp solution and then check to see if it satisfies the fuzzy differential equations with fuzzy initial conditions. The second method was the reverse of the first one, in that they firstly solved the fuzzy initial value problem and then checked to see if it defines a fuzzy function. In 2008, Allahviranloo et al. [35] utilized the collocation method to transfer (1) and (2) into a crisp $2(n+1) \times 2(n+1)$ system of linear equations

$$S(t)\, X = Y(r) \quad (3)$$

for three specious cases; that is, all coefficient functions $a_k(t)$ are positive, negative, $a_k(t)$ $(k = 0, 1, \dots, n-m)$, are negative $a_k(t)$ $(k = n-m+1, \dots, n-1)$ are positive, respectively. By computing (3), they obtained the approximate solution of (1) and (2). However, their methods will be restricted for the general case.

In this paper, the nth-order linear differential equation with fuzzy initial conditions is further investigated. It shows that the result obtained in this paper is an extension of Allahviranloo's conclusions. In addition, considering that the case of the order of the fuzzy differential equation and the number of basic functions in assumed solution are not always equal, we obtain the approximate solution of the original equations (1) and (2) by calculating the minimal norm least squares solution of crisp system of linear equations. Three illustrating examples are given, and one of them is compared with Allahviranloo's work and is shown be more accurate. The structure of this paper is organized as follows.

In Section 2, we recall some basic definitions and results about fuzzy numbers and the undetermined fuzzy coefficients method. In Section 3, a class of nth-order fuzzy linear differential equations is investigated by converting it to a crisp system of linear equations, and some corollaries for the special cases are given. The proposed algorithms are illustrated by solving some examples in Section 4, and the conclusion is drawn in Section 5.

2. Preliminaries

2.1. Fuzzy Number

Definition 1 (see [1]). A fuzzy number is a fuzzy set like $u : R \to I = [0,1]$ which satisfies the following.

(1) u is upper semicontinuous.

(2) u is fuzzy convex; that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in R, \lambda \in [0,1]$.

(3) u is normal; that is, there exists $x_0 \in R$ such that $u(x_0) = 1$.

(4) $\operatorname{supp} u = \{x \in R \mid u(x) > 0\}$ is the support of u, and its closure $\operatorname{cl}(\operatorname{supp} u)$ is compact.

Let E^1 be the set of all fuzzy numbers on R.

Definition 2 (see [36]). A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfies the following requirements.

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function.

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function.

(3) $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

For arbitrary fuzzy number $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r)) \in E^1$, $0 \leq r \leq 1$, and real number $k \in R$,

(1) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$;

(2) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$;

(3) $x - y = (\underline{x}(r) - \overline{y}(r), \overline{x}(r) - \underline{y}(r))$;

(4)

$$kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r)), & k \geq 0, \\ (k\overline{x}(r), k\underline{x}(r)), & k < 0. \end{cases} \tag{4}$$

Definition 3 (see [23]). For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r)) \in E^1$, $0 \leq r \leq 1$, the quantity

$$D(u,v) = \left(\int_0^1 (\underline{u}(r) - \underline{v}(r))^2 dr + \int_0^1 (\overline{u}(r) - \overline{v}(r))^2 dr \right)^{1/2} \tag{5}$$

is the distance between fuzzy numbers u and v.

2.2. The Undetermined Fuzzy Coefficients Method. The undetermined fuzzy coefficients method is to seek an approximate solution as

$$\widetilde{y}_N(t) = \sum_{k=0}^{N} \widetilde{\alpha}_k \phi_k(t), \tag{6}$$

where $\phi_k(t)$, $k = 0, 1, \dots, N$, are positive basic functions whose all differentiations are positive. We compute the fuzzy coefficients $\widetilde{\alpha}_k$ in (6) by setting the error to zero as follows:

$$\begin{aligned} E = {} & D\left(\widetilde{y}^{(n)} + a_{n-1}(t)\,\widetilde{y}^{(n-1)} + \cdots + a_1(t)\,\widetilde{y}' + a_0(t)\,\widetilde{y}, \widetilde{g}(t)\right) \\ & + D\left(\widetilde{y}(t_0), \widetilde{b}_0\right) + \cdots + D\left(\widetilde{y}^{(n-1)}(t_0), \widetilde{b}_{n-1}\right). \end{aligned} \tag{7}$$

We substitute (6) in (7) and represent them in parametric forms; then,

$$\begin{gathered} \underline{y^{(n)}(t,r) + a_{n-1}(t)\, y^{(n-1)}(t,r) + \cdots + a_1(t)\, y'(t,r) + a_0(t)\, y(t,r)} \\ = \underline{g}(t,r), \\ \underline{y}(t_0, r) = \underline{b}_0(r), \\ \vdots \end{gathered}$$

$$\begin{aligned}
&\underline{y}^{(n-1)}\left(t_0,r\right)=\underline{b}_{n-1}\left(r\right),\\
&\overline{y^{(n)}\left(t,r\right)+a_{n-1}\left(t\right)y^{(n-1)}\left(t,r\right)+\cdots+a_1\left(t\right)y'\left(t,r\right)+a_0\left(t\right)y\left(t,r\right)}\\
&\quad=\overline{g\left(t,r\right)},\\
&\overline{y}\left(t_0,r\right)=\overline{b}_0\left(r\right),\\
&\vdots\\
&\overline{y}^{(n-1)}\left(t_0,r\right)=\overline{b}_{n-1}\left(r\right).
\end{aligned}\tag{8}$$

Lemma 4 (see [35]). *Let basic functions $\phi_k(t)$, $k=0,1,\ldots,N$, and all of their differentiations be positive; without loss of generality, then $(\underline{y}_N)^{(i)}(t)=\underline{y_N^{(i)}}(t)$ and $(\overline{y}_N)^{(i)}(t)=\overline{y_N^{(i)}}(t)$, $i=0,1,\ldots,n$.*

3. Basic Results

In order to solve (1) and (2), we need to consider the system of linear equations (8). In this section, we study its general case at first then give some corollaries for some special cases.

3.1. New Models

Theorem 5. *Suppose that each $a_i(t)$, $i=0,1,\ldots,n-1$, in* (1) *is either nonnegative or negative over $[t_0,T]$; then, the nth-order fuzzy linear differential equations*

$$\begin{gathered}
y^{(n)}+a_{n-1}\left(t\right)y^{(n-1)}+\cdots+a_1\left(t\right)y'+a_0\left(t\right)y=g\left(t\right),\\
y\left(t_0\right)=\widetilde{b}_0,\ y'\left(t_0\right)=\widetilde{b}_1,\ldots,y^{(n-1)}\left(t_0\right)=\widetilde{b}_{n-1}
\end{gathered}\tag{9}$$

can be extended into a system of linear equations

$$\begin{pmatrix}
\beta_0 & \beta_1 & \cdots & \beta_N & \xi_0 & \xi_1 & \cdots & \xi_N\\
\sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
\sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} & 0 & 0 & \cdots & 0\\
\xi_0 & \xi_1 & \cdots & \xi_N & \beta_0 & \beta_1 & \cdots & \beta_N\\
0 & 0 & \cdots & 0 & \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N}\\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
0 & 0 & \cdots & 0 & \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N}
\end{pmatrix}
\times\begin{pmatrix}
\underline{\alpha}_0\left(r\right)\\
\underline{\alpha}_1\left(r\right)\\
\vdots\\
\underline{\alpha}_N\left(r\right)\\
\overline{\alpha}_0\left(r\right)\\
\overline{\alpha}_1\left(r\right)\\
\vdots\\
\overline{\alpha}_N\left(r\right)
\end{pmatrix}=\begin{pmatrix}
\underline{g}\left(t,r\right)\\
\underline{b}_0\left(r\right)\\
\vdots\\
\underline{b}_{n-1}\left(r\right)\\
\overline{g}\left(t,r\right)\\
\overline{b}_0\left(r\right)\\
\vdots\\
\overline{b}_{n-1}\left(r\right)
\end{pmatrix},\tag{10}$$

where

$$\begin{aligned}
&\beta_k=\phi_k^{(n)}\left(t\right)+\sum_{a_i(t)\geq 0}a_i\left(t\right)\phi_k^{(i)}\left(t\right),\quad i=0,1,\ldots,n-1,\\
&\xi_k=\sum_{a_i(t)<0}a_i\left(t\right)\phi_k^{(i)}\left(t\right),\quad i=0,1,\ldots,n-1,\\
&\sigma_{jk}=\phi_k^{(j)}\left(t_0\right),\quad j=0,1,\ldots,n-1,\ k=0,1,\ldots,N.
\end{aligned}\tag{11}$$

By setting $t=a$, $a\in[t_0,T]$, and by solving (10), *the fuzzy approximate solution of the original fuzzy linear differential equations is obtained as follows:*

$$\begin{aligned}
&\underline{y}\left(t,r\right)=\underline{\alpha}_0\left(r\right)\phi_0\left(t\right)+\underline{\alpha}_1\left(r\right)\phi_1\left(t\right)+\cdots+\underline{\alpha}_N\left(r\right)\phi_N\left(t\right),\\
&\overline{y}\left(t,r\right)=\overline{\alpha}_0\left(r\right)\phi_0\left(t\right)+\overline{\alpha}_1\left(r\right)\phi_1\left(t\right)+\cdots+\overline{\alpha}_N\left(r\right)\phi_N\left(t\right).
\end{aligned}\tag{12}$$

Proof. Let (5) be substituted in (1) and (2); we have

$$\begin{aligned}
&\sum_{k=0}^{N}\widetilde{\alpha}_k\left(r\right)\phi_k^{(n)}\left(t\right)+a_{n-1}\left(t\right)\sum_{k=0}^{N}\widetilde{\alpha}_k\left(r\right)\phi_k^{(n-1)}\left(t\right)+\cdots\\
&\quad+a_1\left(t\right)\sum_{k=0}^{N}\widetilde{\alpha}_k\left(r\right)\phi_k'\left(t\right)+a_0\left(t\right)\sum_{k=0}^{N}\widetilde{\alpha}_k\left(r\right)\phi_k\left(t\right)=\widetilde{g}\left(t\right),\\
&\sum_{k=0}^{N}\widetilde{\alpha}_k\left(r\right)\phi_k\left(t_0\right)=\widetilde{b}_0\left(r\right),\\
&\sum_{k=0}^{N}\widetilde{\alpha}_k\left(r\right)\phi_k'\left(t_0\right)=\widetilde{b}_1\left(r\right),\ldots,\\
&\sum_{k=0}^{N}\widetilde{\alpha}_k\left(r\right)\phi_k^{(n-1)}\left(t_0\right)=\widetilde{b}_{n-1}\left(r\right).
\end{aligned}\tag{13}$$

We express the pervious equations in parametric forms; then,

$$\begin{aligned}
&\sum_{k=0}^{N}\left(\underline{\alpha}_k(r),\overline{\alpha}_k(r)\right)\phi_k^{(n)}(t)\\
&\quad+a_{n-1}(t)\sum_{k=0}^{N}\left(\underline{\alpha}_k(r),\overline{\alpha}_k(r)\right)\phi_k^{(n-1)}(t)\\
&\quad+\cdots+a_1(t)\sum_{k=0}^{N}\left(\underline{\alpha}_k(r),\overline{\alpha}_k(r)\right)\phi_k'(t)\\
&\quad+a_0(t)\sum_{k=0}^{N}\left(\underline{\alpha}_k(r),\overline{\alpha}_k(r)\right)\phi_k(t)=\left(\underline{g}(t,r),\overline{g}(t,r)\right),\\
&\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k(t_0)=\underline{b}_0(r),\ldots,\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k^{(n-1)}(t_0)=\underline{b}_{n-1}(r),\\
&\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k(t_0)=\overline{b}_0(r),\ldots,\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k^{(n-1)}(t_0)=\overline{b}_{n-1}(r).
\end{aligned}\tag{14}$$

By setting

$$\begin{aligned}
\beta_k&=\phi_k^{(n)}(t)+\sum_{a_i(t)\geq 0}a_i(t)\phi_k^{(i)}(t),\quad i=0,1,\ldots,n-1,\\
\xi_k&=\sum_{a_i(t)<0}a_i(t)\phi_k^{(i)}(t),\quad i=0,1,\ldots,n-1,\\
\sigma_{jk}&=\phi_k^{(j)}(t_0),\quad j=0,1,\ldots,n-1,\ k=0,1,\ldots,N,
\end{aligned}\tag{15}$$

thus we have the corresponding systems $S(t)X(r)=Y(r)$ as follows:

$$\begin{pmatrix}
\beta_0 & \beta_1 & \cdots & \beta_N & \xi_0 & \xi_1 & \cdots & \xi_N\\
\sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
\sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} & 0 & 0 & \cdots & 0\\
\xi_0 & \xi_1 & \cdots & \xi_N & \beta_0 & \beta_1 & \cdots & \beta_N\\
0 & 0 & \cdots & 0 & \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N}\\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
0 & 0 & \cdots & 0 & \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N}
\end{pmatrix}
\times\begin{pmatrix}
\underline{\alpha}_0(r)\\ \underline{\alpha}_1(r)\\ \vdots\\ \underline{\alpha}_N(r)\\ \overline{\alpha}_0(r)\\ \overline{\alpha}_1(r)\\ \vdots\\ \overline{\alpha}_N(r)
\end{pmatrix}
=\begin{pmatrix}
\underline{g}(t,r)\\ \underline{b}_0(r)\\ \vdots\\ \underline{b}_{n-1}(r)\\ \overline{g}(t,r)\\ \overline{b}_0(r)\\ \vdots\\ \overline{b}_{n-1}(r)
\end{pmatrix},\tag{16}$$

being $2(n+1)\times 2(N+1)$ linear systems.

By setting $t=a, a\in[t_0,T]$, and by solving (10), we obtain the values of parameters

$$\underline{\alpha}_0(r),\underline{\alpha}_1(r),\ldots,\underline{\alpha}_N(r),\overline{\alpha}_0(r),\overline{\alpha}_1(r),\ldots,\overline{\alpha}_N(r);\tag{17}$$

therefore, we get the fuzzy approximate solution of the original fuzzy equation as follows:

$$\begin{aligned}
\underline{y}(t,r)&=\underline{\alpha}_0(r)\phi_0(t)+\underline{\alpha}_1(r)\phi_1(t)+\cdots+\underline{\alpha}_N(r)\phi_N(t),\\
\overline{y}(t,r)&=\overline{\alpha}_0(r)\phi_0(t)+\overline{\alpha}_1(r)\phi_1(t)+\cdots+\overline{\alpha}_N(r)\phi_N(t).
\end{aligned}\tag{18}$$

□

Now, we consider another special case; that is, coefficient functions $a_k(t)$, $k=0,1,\ldots,n-1$, are positive and negative alternately.

Corollary 6. *Suppose that the coefficients functions $a_{n-1}(t), a_{n-3}(t),\ldots$ are positive and $a_{n-2}(t), a_{n-4}(t),\ldots$ are negative, and n is an odd number. Then, the nth-order fuzzy linear differential equations* (1) *and* (2) *can be extended into the following linear equations:*

$$\begin{aligned}
&\sum_{k=0}^{N}\underline{\alpha}_k(r)\beta_k+\sum_{k=0}^{N}\overline{\alpha}_k(r)\gamma_k=\underline{g}(t,r),\\
&\sum_{k=0}^{N}\underline{\alpha}_k(r)\sigma_{0k}=\underline{b}_0(r),\\
&\vdots\\
&\sum_{k=0}^{N}\underline{\alpha}_k(r)\sigma_{n-1,k}=\underline{b}_{n-1}(r),\\
&\sum_{k=0}^{N}\overline{\alpha}_k(r)\beta_k+\sum_{k=0}^{N}\underline{\alpha}_k(r)\gamma_k=\overline{g}(t,r),\\
&\sum_{k=0}^{N}\overline{\alpha}_k(r)\sigma_{0k}=\overline{b}_0(r),\\
&\vdots\\
&\sum_{k=0}^{N}\overline{\alpha}_k(r)\sigma_{n-1,k}=\overline{b}_{n-1}(r),
\end{aligned}\tag{19}$$

where

$$\begin{aligned}\beta_k &= \phi_k^{(n)}(t) + a_{n-1}(t)\,\phi_k^{(n-1)}(t) + a_{n-3}(t)\,\phi_k^{(n-3)}(t)\\ &\quad + \cdots + a_0(t)\,\phi_k(t),\\ \gamma_k &= a_{n-2}(t)\,\phi_k^{(n-2)}(t) + a_{n-4}(t)\,\phi_k^{(n-4)}(t)\\ &\quad + \cdots + a_1(t)\,\phi_k'(t),\\ \sigma_{jk} &= \phi_k^{(j)}(t_0), \quad j = 0,1,\ldots,n-1,\ k = 0,1,\ldots,N.\end{aligned} \tag{20}$$

By setting

$$\begin{aligned}X(r) &= \left(\underline{\alpha}_0(r), \underline{\alpha}_1(r), \ldots, \underline{\alpha}_N(r), \overline{\alpha}_0(r),\right.\\ &\qquad \left.\overline{\alpha}_1(r), \ldots, \overline{\alpha}_N(r)\right)^{\top},\\ Y(r) &= \left(\underline{g}(t,r), \underline{b}_0(r), \ldots, \underline{b}_{n-1}(r), \overline{g}(t,r),\right.\\ &\qquad \left.\overline{b}_0(r), \ldots, \overline{b}_{n-1}(r)\right)^{\top},\end{aligned} \tag{21}$$

(19) *is a system of linear equations* $S(t)X(r) = Y(r)$ *such that*

$$S(t) = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned}S_1 &= \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_N \\ \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} \end{pmatrix},\\ S_2 &= \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.\end{aligned} \tag{23}$$

When n is an even number, then the nth-order fuzzy linear differential equations (1) *and* (2) *can be extended into a system of linear equations*

$$\begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_N & \gamma_0 & \gamma_1 & \cdots & \gamma_N \\ \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} & 0 & 0 & \cdots & 0 \\ \gamma_0 & \gamma_1 & \cdots & \gamma_N & \beta_0 & \beta_1 & \cdots & \beta_N \\ 0 & 0 & \cdots & 0 & \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} \end{pmatrix} \times \begin{pmatrix} \underline{\alpha}_0(r) \\ \underline{\alpha}_1(r) \\ \vdots \\ \underline{\alpha}_N(r) \\ \overline{\alpha}_0(r) \\ \overline{\alpha}_1(r) \\ \vdots \\ \overline{\alpha}_N(r) \end{pmatrix} = \begin{pmatrix} \underline{g}(t,r) \\ \underline{b}_0(r) \\ \vdots \\ \underline{b}_{n-1}(r) \\ \overline{g}(t,r) \\ \overline{b}_0(r) \\ \vdots \\ \overline{b}_{n-1}(r) \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned}\beta_k &= \phi_k^{(n)}(t) + a_{n-1}(t)\,\phi_k^{(n-1)}(t) + a_{n-3}(t)\,\phi_k^{(n-3)}(t)\\ &\quad + \cdots + a_1(t)\,\phi_k'(t),\\ \gamma_k &= a_{n-2}(t)\,\phi_k^{(n-2)}(t) + a_{n-4}(t)\,\phi_k^{(n-4)}(t)\\ &\quad + \cdots + a_0(t)\,\phi_k(t),\\ \sigma_{jk} &= \phi_k^{(j)}(t_0), \quad j = 0,1,\ldots,n-1,\ k = 0,1,\ldots,N.\end{aligned} \tag{25}$$

Three special cases in Allahviranloo et al.'s paper [35] are viewed as corollaries from Theorem 5.

Corollary 7. *Suppose that coefficients functions* $a_k(t)$, $k = 0,1,2,\ldots,n-1$, *are nonnegative; then, the nth-order fuzzy linear differential equations* (1) *and* (2) *can be extended into the following linear equations:*

$$\begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_N & 0 & 0 & \cdots & 0 \\ \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \beta_0 & \beta_1 & \cdots & \beta_N \\ 0 & 0 & \cdots & 0 & \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} \end{pmatrix}$$

Table 1: Comparisons between the exact solution and the approximate solution.

r	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0	2.00400200066667	2.00300328843835	0.00099871222831
0.1	2.10430240100020	2.10320316558466	0.00109923541554
0.2	2.20460280133373	2.20340304273096	0.00119975860278
0.3	2.30490320166727	2.30360291987726	0.00130028179001
0.4	2.40520360200080	2.40380279702356	0.00140080497724
0.5	2.50550400233433	2.50400267416986	0.00150132816447
0.6	2.60580440266787	2.60420255131616	0.00160185135171
0.7	2.70610480300140	2.70440242846246	0.00170237453894
0.8	2.80640520333493	2.80460230560876	0.00180289772617
0.9	2.90670560366847	2.90480218275507	0.00190342091340
1	3.00700600400200	3.00500205990137	0.00200394410063

Table 2: Comparisons between the exact solution and the approximate solution.

r	$\overline{Y}(t,r)$	$\overline{y}(t,r)$	Error
0	4.01001000733734	4.00899990996165	0.00101009737569
0.1	3.90970960700380	3.90880003281535	0.00090957418846
0.2	3.80940920667027	3.80860015566904	0.00080905100122
0.3	3.70910880633673	3.70840027852274	0.00070852781399
0.4	3.60880840600320	3.60820040137644	0.00060800462676
0.5	3.50850800566967	3.50800052423014	0.00050748143953
0.6	3.40820760533613	3.40780064708384	0.00040695825230
0.7	3.30790720500260	3.30760076993754	0.00030643506506
0.8	3.20760680466907	3.20740089279124	0.00020591187783
0.9	3.10730640433553	3.10720101564493	0.00010538869060
1	3.00700600400200	3.00700113849863	0.00000486550337

$$\times \begin{pmatrix} \underline{\alpha}_0(r)\\ \underline{\alpha}_1(r)\\ \vdots \\ \underline{\alpha}_N(r)\\ \overline{\alpha}_0(r)\\ \overline{\alpha}_1(r)\\ \vdots\\ \overline{\alpha}_N(r)\end{pmatrix} = \begin{pmatrix} \underline{g}(t,r)\\ \underline{b}_0(r)\\ \vdots\\ \underline{b}_{n-1}(r)\\ \overline{g}(t,r)\\ \overline{b}_0(r)\\ \vdots\\ \overline{b}_{n-1}(r)\end{pmatrix}, \tag{26}$$

where

$$\begin{gathered} \beta_k = \phi_k^{(n)}(t) + a_{n-1}(t)\,\phi_k^{(n-1)}(t) \\ + \cdots + a_1(t)\,\phi_k'(t) + a_0(t)\,\phi_k(t), \end{gathered} \tag{27}$$

$$\sigma_{jk} = \phi_k^{(j)}(t_0), \quad j = 0,1,\ldots,n-1,\ k = 0,1,\ldots,N.$$

Corollary 8. *Suppose that coefficients functions $a_k(t), k = 0,1,2,\ldots,n-1$, are negative; then, the nth-order fuzzy linear differential equations* (1) *and* (2) *can be extended into a system of linear equations*

$$\begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_N & \gamma_0 & \gamma_1 & \cdots & \gamma_N \\ \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} & 0 & 0 & \cdots & 0 \\ \gamma_0 & \gamma_1 & \cdots & \gamma_N & \beta_0 & \beta_1 & \cdots & \beta_N \\ 0 & 0 & \cdots & 0 & \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} \end{pmatrix} \times \begin{pmatrix} \underline{\alpha}_0(r)\\ \underline{\alpha}_1(r)\\ \vdots \\ \underline{\alpha}_N(r)\\ \overline{\alpha}_0(r)\\ \overline{\alpha}_1(r)\\ \vdots\\ \overline{\alpha}_N(r)\end{pmatrix} = \begin{pmatrix} \underline{g}(t,r)\\ \underline{b}_0(r)\\ \vdots\\ \underline{b}_{n-1}(r)\\ \overline{g}(t,r)\\ \overline{b}_0(r)\\ \vdots\\ \overline{b}_{n-1}(r)\end{pmatrix}, \tag{28}$$

where

$$\begin{gathered} \beta_k = \phi_k^{(n)}(t), \\ \gamma_k = a_{n-1}(t)\,\phi_k^{(n-1)}(t) + a_{n-2}(t)\,\phi_k^{(n-2)}(t) \\ + \cdots + a_0(t)\,\phi_k(t), \\ \sigma_{jk} = \phi_k^{(j)}(t_0), \quad j = 0,1,\ldots,n-1,\ k = 0,1,\ldots,N. \end{gathered} \tag{29}$$

Corollary 9. *Suppose that coefficients functions $a_{n-1}(t), a_{n-3}(t),\ldots,a_{n-m}(t)$ are nonnegative and $a_{n-m-1}(t), a_{n-m-2}(t),\ldots,a_0(t)$ are negative; then, the nth-order fuzzy linear differential equations* (1) *and* (2) *can be extended into a system of linear equations*

$$\begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_N & \gamma_0 & \gamma_1 & \cdots & \gamma_N \\ \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} & 0 & 0 & \cdots & 0 \\ \gamma_0 & \gamma_1 & \cdots & \gamma_N & \beta_0 & \beta_1 & \cdots & \beta_N \\ 0 & 0 & \cdots & 0 & \sigma_{00} & \sigma_{01} & \cdots & \sigma_{0N} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \sigma_{n-10} & \sigma_{n-11} & \cdots & \sigma_{n-1N} \end{pmatrix} \times \begin{pmatrix} \underline{\alpha}_0(r)\\ \underline{\alpha}_1(r)\\ \vdots \\ \underline{\alpha}_N(r)\\ \overline{\alpha}_0(r)\\ \overline{\alpha}_1(r)\\ \vdots\\ \overline{\alpha}_N(r)\end{pmatrix} = \begin{pmatrix} \underline{g}(t,r)\\ \underline{b}_0(r)\\ \vdots\\ \underline{b}_{n-1}(r)\\ \overline{g}(t,r)\\ \overline{b}_0(r)\\ \vdots\\ \overline{b}_{n-1}(r)\end{pmatrix}, \tag{30}$$

Table 3: Comparisons between the exact solution and the approximate solution ($t = 1.01$).

r	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0	−0.00989800000000	−0.00989582400000	0.21760000006310$e-5$
0.1	0.09111210000000	0.09111405840000	0.19583999990580$e-5$
0.2	0.19212220000000	0.19212394080000	0.17408000001495$e-5$
0.3	0.29313230000000	0.29313382320000	0.15232000003529$e-5$
0.4	0.39414240000000	0.39414370560000	0.13055999996681$e-5$
0.5	0.49515250000000	0.49515358800000	0.10879999998714$e-5$
0.6	0.59616260000000	0.59616347040000	0.08704000005189$e-5$
0.7	0.69717270000000	0.69717335280000	0.06527999985018$e-5$
0.8	0.79818280000000	0.79818323520000	0.04352000000374$e-5$
0.9	0.89919290000000	0.89919311760000	0.02176000011289$e-5$
1	1.00020300000000	1.00020300000000	0

Table 4: Comparisons between the exact solution and the approximate solution ($t = 1.01$).

r	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0	2.01030400000000	2.01030182400000	0.21760000015192$e-5$
0.1	1.90929390000000	1.90929194160000	0.19583999986139$e-5$
0.2	1.80828380000000	1.80828205920000	0.17407999997054$e-5$
0.3	1.70727370000000	1.70727217680000	0.15232000007970$e-5$
0.4	1.60626360000000	1.60626229440000	0.13056000001122$e-5$
0.5	1.50525350000000	1.50525241200000	0.10880000007596$e-5$
0.6	1.40424340000000	1.40424252960000	0.08704000005189$e-5$
0.7	1.30323330000000	1.30323264720000	0.06528000002781$e-5$
0.8	1.20222320000000	1.20222276480000	0.04352000004815$e-5$
0.9	1.10121310000000	1.10121288240000	0.02176000002407$e-5$
1	1.00020300000000	1.00020300000000	0

where

$$\begin{aligned}\beta_k &= \phi_k^{(n)}(t) + a_{n-1}(t)\,\phi_k^{(n-1)}(t) + \cdots + a_{n-m}(t)\,\phi_k^{(n-m)}(t),\\ \gamma_k &= a_{n-m-1}(t)\,\phi_k^{(n-m-1)}(t) + a_{n-m-2}(t)\,\phi_k^{(n-m-2)}(t)\\ &\quad + \cdots + a_0(t)\,\phi_k(t),\\ \sigma_{jk} &= \phi_k^{(j)}(t_0), \quad j = 0,1,\ldots,n-1,\ k = 0,1,\ldots,N.\end{aligned} \tag{31}$$

3.2. Method to Solve Linear Equations. The previous crisp linear equations (10), (19), (24), (26), (28), and (30) are all $2(n+1) \times 2(N+1)$ linear systems, and they have the same form as follows:

$$S(t)\,X(r) = Y(r). \tag{32}$$

In the process of solving (32) by setting $t = a$, $a \in [t_0, T]$, whether it is consistent or inconsistent, we obtain the minimal norm least squares solution [37] by using the generalized inverse of the coefficient matrix $S(a)$; that is,

$$X(r) = S^{\dagger}(a)\,Y(r). \tag{33}$$

Thus, we get

$$\underline{\alpha}_0(r), \underline{\alpha}_1(r), \ldots, \underline{\alpha}_N(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \ldots, \overline{\alpha}_N(r). \tag{34}$$

Therefore, we obtain the fuzzy approximate solution of the original fuzzy equation as follows:

$$\begin{aligned}\underline{y}(t,r) &= \underline{\alpha}_0(r)\,\phi_0(t) + \underline{\alpha}_1(r)\,\phi_1(t) + \cdots + \underline{\alpha}_N(r)\,\phi_N(t),\\ \overline{y}(t,r) &= \overline{\alpha}_0(r)\,\phi_0(t) + \overline{\alpha}_1(r)\,\phi_1(t) + \cdots + \overline{\alpha}_N(r)\,\phi_N(t).\end{aligned} \tag{35}$$

4. Numerical Examples

Example 1. Consider the following second-order fuzzy linear differential equation:

$$\begin{aligned}y'' - 4y' + 4y &= 4t - 4, \quad t \geq 0,\\ \tilde{y}(0) &= (2+r, 4-r),\\ \tilde{y}'(0) &= (3+2r, 9-2r).\end{aligned} \tag{36}$$

The exact solution of equation is

$$\begin{aligned}\underline{Y}(t,r) &= (2+r)\,e^{2t} + (-1+r)\,te^{2t} + t,\\ \overline{Y}(t,r) &= (4-r)\,e^{2t} + (1-r)\,te^{2t} + t.\end{aligned} \tag{37}$$

Table 5: Comparisons between the exact solution and the approximate solution ($t = 1.001$).

r	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0	−0.00099899800000	−0.00099899582400	$0.21760011570393e-8$
0.1	0.09910110210000	0.09910110405840	$0.19584001087480e-8$
0.2	0.19920120220000	0.19920120394080	$0.17407990604568e-8$
0.3	0.29930130230000	0.29930130382320	$0.15232002326115e-8$
0.4	0.39940140240000	0.39940140370560	$0.13056005165879e-8$
0.5	0.49950150250000	0.49950150358800	$0.10879999123858e-8$
0.6	0.59960160260000	0.59960160347040	$0.08703997522730e-8$
0.7	0.69970170270000	0.69970170335280	$0.06527991480709e-8$
0.8	0.79980180280000	0.79980180323520	$0.04352012084041e-8$
0.9	0.89990190290000	0.89990190311760	$0.02176006042021e-8$
1	1.00000200300000	1.00000200300000	0

Table 6: Comparisons between the exact solution and the approximate solution ($t = 1.001$).

r	$\overline{Y}(t,r)$	$\overline{y}(t,r)$	Error
0	2.00100300400000	2.00100300182400	$0.21760007129501e-8$
0.1	1.90090290390000	1.90090290194160	$0.19583985544358e-8$
0.2	1.80080280380000	1.80080280205920	$0.17407990604568e-8$
0.3	1.70070270370000	1.70070270217680	$0.15232011207900e-8$
0.4	1.60060260360000	1.60060260229440	$0.13056018488555e-8$
0.5	1.50050250350000	1.50050250241200	$0.10880008005643e-8$
0.6	1.40040240340000	1.40040240252960	$0.08703997522730e-8$
0.7	1.30030230330000	1.30030230264720	$0.06528000362493e-8$
0.8	1.20020220320000	1.20020220276480	$0.04352007643149e-8$
0.9	1.10010210310000	1.10010210288240	$0.02176001601129e-8$
1	1.00000200300000	1.00000200300000	0

Let $\phi_k(t) = t^k, k = 0, 1, 2, 3$; then,

$$\begin{aligned}\underline{y}(t,r) &= \underline{\alpha}_0 + \underline{\alpha}_1 t + \underline{\alpha}_2 t^2 + \underline{\alpha}_3 t^3,\\ \overline{y}(t,r) &= \overline{\alpha}_0 + \overline{\alpha}_1 t + \overline{\alpha}_2 t^2 + \overline{\alpha}_3 t^3.\end{aligned} \tag{38}$$

From (10), the extended linear equation $S(t)X(r) = Y(r)$ is as follows:

$$\begin{pmatrix} 4 & 4t & 2+4t^2 & 6t+4t^3 & 0 & -4 & -8t & -12t^2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -8t & -12t^2 & 4 & 4t & 2+4t^2 & 6t+4t^3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \underline{\alpha}_0(r) \\ \underline{\alpha}_1(r) \\ \underline{\alpha}_2(r) \\ \underline{\alpha}_3(r) \\ \overline{\alpha}_0(r) \\ \overline{\alpha}_1(r) \\ \overline{\alpha}_2(r) \\ \overline{\alpha}_3(r) \end{pmatrix} = \begin{pmatrix} 4t-4 \\ 2+r \\ 3+2r \\ 4t-4 \\ 4-r \\ 9-2r \end{pmatrix}. \tag{39}$$

By setting $t = 1/2$, the parameters $\underline{\alpha}_0(r), \underline{\alpha}_1(r), \underline{\alpha}_2(r), \underline{\alpha}_3(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \overline{\alpha}_2(r), \overline{\alpha}_3(r)$ are obtained, and by putting them into (38), we have

$$\begin{aligned}\underline{y}(t,r) = {} & (2+r) + (3+2r)t \\ & + (3.28767123 - 1.22739726r)t^2 \\ & + (0.76712328 - 1.13972602r)t^3, \\ \overline{y}(t,r) = {} & (4-r) + (9-2r)t \\ & + (-0.08767123 + 1.22739726r)t^2 \\ & + (-2.36712328 + 1.13972602r)t^3.\end{aligned} \tag{40}$$

Tables 1 and 2 show comparisons between the exact solution and the approximate solution at $t = 0.001$ for some $r \in [0, 1]$, and all data were calculated by MATLAB 6.x.

Example 2. Consider the following three-order fuzzy linear differential equation:

$$\begin{gathered} t^3 y''' - 3t^2 y'' + 6ty' - 6y = 0, \quad t \geq 1, \\ \tilde{y}(1) = (r, 2-r), \end{gathered}$$

Table 7: Comparisons between the exact solution and the approximate solution ($t = 0.01$).

r	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0	−0.09911518137409	−0.09912683067200	$0.11649297906799e-4$
0.1	−0.08901568303658	−0.08902706143300	$0.11378396415129e-4$
0.2	−0.07891618469908	−0.07892729219400	$0.11107494923460e-4$
0.3	−0.06881668636157	−0.06882752295500	$0.10836593431776e-4$
0.4	−0.05871718802406	−0.05872775371600	$0.10565691940079e-4$
0.5	−0.04861768968655	−0.04862798447700	$0.10294790448395e-4$
0.6	−0.03851819134904	−0.03852821523800	$0.10023888956719e-4$
0.7	−0.02841869301153	−0.02842844599900	$0.09752987465039e-4$
0.8	−0.01831919467403	−0.01832867676000	$0.09482085973359e-4$
0.9	−0.00821969633652	−0.00822890752100	$0.09211184481680e-4$
1	0.00187980200099	0.00187086171800	$0.08940282990000e-4$

Table 8: Comparisons between the exact solution and the approximate solution ($t = 0.01$).

r	$\overline{Y}(t,r)$	$\overline{y}(t,r)$	Error
0	0.10287478537607	0.10286855421000	$0.62311660731923e-5$
0.1	0.09277528703856	0.09276878497120	$0.65020673648691e-5$
0.2	0.08267578870106	0.08266901573240	$0.67729686565599e-5$
0.3	0.07257629036355	0.07256924649360	$0.70438699482367e-5$
0.4	0.06247679202604	0.06246947725480	$0.73147712399066e-5$
0.5	0.05237729368853	0.05236970801600	$0.75856725315904e-5$
0.6	0.04227779535102	0.04226993877720	$0.78565738232741e-5$
0.7	0.03217829701351	0.03217016953840	$0.81274751149579e-5$
0.8	0.02207879867601	0.02207040029960	$0.83983764066348e-5$
0.9	0.01197930033850	0.01197063106080	$0.86692776983185e-5$
1	0.00187980200099	0.00187086182200	$0.89401789899965e-5$

$$\overline{y}'(1) = (-1+r, 1-r),$$
$$\overline{y}''(1) = (2+2r, 6-2r). \tag{41}$$

The exact solution of equation is

$$\underline{Y}(t,r) = (3+2r)\,t + (-5-2r)\,t^2 + (2+r)\,t^3,$$
$$\overline{Y}(t,r) = (7-2r)\,t + (-9+2r)\,t^2 + (4-r)\,t^3. \tag{42}$$

Let $\phi_k(t) = t^k$, $k = 0, 1, 2, 3$; then,

$$\underline{y}(t,r) = \underline{\alpha}_0 + \underline{\alpha}_1 t + \underline{\alpha}_2 t^2 + \underline{\alpha}_3 t^3,$$
$$\overline{y}(t,r) = \overline{\alpha}_0 + \overline{\alpha}_1 t + \overline{\alpha}_2 t^2 + \overline{\alpha}_3 t^3. \tag{43}$$

From (19), the extended linear equation $S(t)X(r) = Y(r)$ is as follows:

$$\begin{pmatrix} 0 & \frac{6}{t^2} & \frac{12}{t} & 24 & \frac{-6}{t^3} & \frac{-6}{t^2} & -\frac{12}{t} & -24 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 & 0 & 0 & 0 \\ \frac{-6}{t^3} & \frac{-6}{t^2} & -\frac{12}{t} & -24 & 0 & \frac{6}{t^2} & \frac{12}{t} & 24 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \end{pmatrix}$$
$$\times \begin{pmatrix} \underline{\alpha}_0(r) \\ \underline{\alpha}_1(r) \\ \underline{\alpha}_2(r) \\ \underline{\alpha}_3(r) \\ \overline{\alpha}_0(r) \\ \overline{\alpha}_1(r) \\ \overline{\alpha}_2(r) \\ \overline{\alpha}_3(r) \end{pmatrix} = \begin{pmatrix} 0 \\ r \\ -1+r \\ 2+2r \\ 0 \\ 2-r \\ 1-r \\ 6-2r \end{pmatrix}. \tag{44}$$

Table 9: Comparisons between the exact solution and the approximate solution ($t = 0.001$).

r	$\underline{Y}(t,r)$	$\underline{y}(t,r)$	Error
0	−0.09991195018134	−0.09991206746457	$0.11728323451310e-6$
0.1	−0.08990195518300	−0.08990206974373	$0.11456072925020e-6$
0.2	−0.07989196018467	−0.07989207202289	$0.11183822400118e-6$
0.3	−0.06988196518634	−0.06988207430206	$0.10911571875216e-6$
0.4	−0.05987197018800	−0.05987207658122	$0.10639321350314e-6$
0.5	−0.04986197518967	−0.04986207886038	$0.10367070826106e-6$
0.6	−0.03985198019133	−0.03985208113954	$0.10094820300510e-6$
0.7	−0.02984198519300	−0.02984208341870	$0.09822569775955e-6$
0.8	−0.01983199019467	−0.01983208569786	$0.09550319251053e-6$
0.9	−0.00982199519633	−0.00982208797702	$0.09278068726151e-6$
1	0.00018799980200	0.00018790974382	$0.09005818201011e-6$

Table 10: Comparisons between the exact solution and the approximate solution ($t = 0.001$).

r	$\overline{Y}(t,r)$	$\overline{y}(t,r)$	Error
0	0.10028794978534	0.10028788695321	$0.62832127500911e-7$
0.1	0.09027795478700	0.09027788923237	$0.65554632569520e-7$
0.2	0.08026795978867	0.08026789151153	$0.68277137610373e-7$
0.3	0.07025796479034	0.07025789379069	$0.70999642678982e-7$
0.4	0.06024796979200	0.06024789606985	$0.73722147712896e-7$
0.5	0.05023797479367	0.05023789834902	$0.76444652760688e-7$
0.6	0.04022797979534	0.04022790062818	$0.79167157815418e-7$
0.7	0.03021798479700	0.03021790290734	$0.81889662859741e-7$
0.8	0.02020798979867	0.02020790518650	$0.84612167911002e-7$
0.9	0.01019799480033	0.01019790746566	$0.87334672960529e-7$
1	0.00018799980200	0.00018790974482	$0.90057178009920e-7$

By setting $t = 4/3$, the parameters $\underline{\alpha}_0(r), \underline{\alpha}_1(r), \underline{\alpha}_2(r), \underline{\alpha}_3(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \overline{\alpha}_2(r), \overline{\alpha}_3(r)$ are obtained, and by putting them into (43), we have

$$\begin{aligned}\underline{y}(t,r) = {} & (-2.1760 + 2.1760r) + (9.5280 - 4.5280r)\,t \\ & + (-11.5280 + 4.5280r)\,t^2 + (4.1760 - 1.1760r)\,t^3, \\ \overline{y}(t,r) = {} & (2.1760 - 2.1760r) + (0.4720 + 4.5280r)\,t \\ & + (-2.4720 - 4.5280r)\,t^2 + (1.8240 + 1.1760r)\,t^3.\end{aligned} \tag{45}$$

Tables 3, 4, 5, and 6 show comparisons between the exact solution and the approximate solution at $t = 1.01$ and $t = 1.001$ for some $r \in [0, 1]$.

Example 3 (see [35]). Consider the following fuzzy linear differential equation:

$$\begin{aligned} & y'' + y = -t, \quad t \in [0,1], \\ & \tilde{y}(0) = (-0.1 + 0.1r, 0.1 - 0.1r), \\ & \tilde{y}'(0) = (0.088 + 0.1r, 0.288 - 0.1r). \end{aligned} \tag{46}$$

The exact solution of equation is

$$\begin{aligned} \underline{Y}(t,r) &= (-0.1 + 0.1r)\cos t + (1.088 + 0.1r)\sin t - t, \\ \overline{Y}(t,r) &= (0.1 - 0.1r)\cos t + (1.288 - 0.1r)\sin t - t. \end{aligned} \tag{47}$$

Let $\phi_k(t) = t^k, k = 0, 1, 2, 3$; then,

$$\begin{aligned} \underline{y}(t,r) &= \underline{\alpha}_0 + \underline{\alpha}_1 t + \underline{\alpha}_2 t^2 + \underline{\alpha}_3 t^3, \\ \overline{y}(t,r) &= \overline{\alpha}_0 + \overline{\alpha}_1 t + \overline{\alpha}_2 t^2 + \overline{\alpha}_3 t^3. \end{aligned} \tag{48}$$

From (26), the extended linear equation $S(t)X(r) = Y(r)$ is as follows:

$$\begin{pmatrix} 1 & t & 2+t^2 & 6t+t^3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & 2+t^2 & 6t+t^3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \underline{\alpha}_0(r) \\ \underline{\alpha}_1(r) \\ \underline{\alpha}_2(r) \\ \underline{\alpha}_3(r) \\ \overline{\alpha}_0(r) \\ \overline{\alpha}_1(r) \\ \overline{\alpha}_2(r) \\ \overline{\alpha}_3(r) \end{pmatrix} = \begin{pmatrix} -t \\ -0.1+0.1r \\ 0.088+0.1r \\ -t \\ 0.1-0.1r \\ 0.288-0.1r \end{pmatrix}. \tag{49}$$

By setting $t = 1/2$, the parameters $\underline{\alpha}_0(r), \underline{\alpha}_1(r), \underline{\alpha}_2(r), \underline{\alpha}_3(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \overline{\alpha}_2(r), \overline{\alpha}_3(r)$ are obtained, and by putting them into (48), we have

$$\begin{aligned} \underline{y}(t,r) = {} & (-0.100+0.100r)+(0.088+0.100r)\,t \\ & +(-0.067371-0.02276r)\,t^2 \\ & +(-0.093572-0.03161r)\,t^3, \\ \overline{y}(t,r) = {} & (0.100-0.100r)+(0.288-0.100r)\,t \\ & +(-0.11289+0.022760r)\,t^2 \\ & +(-0.15679+0.031612r)\,t^3. \end{aligned} \tag{50}$$

Tables 7, 8, 9, and 10 show comparisons between the exact solution and the approximate solution at $t = 0.01$ and $t = 0.001$ for some $r \in [0,1]$.

From Tables 7, 8, 9, and 10, we can see that our results are more accurate than those of Example 3.1 [35] in Allahviranloo et al.'s work.

5. Conclusion

In this paper, an approximate method similar to the undetermined fuzzy coefficients method, based on a positive basis for solving fuzzy differential equations, was further discussed. The more general case was considered, and the model system of linear equation was set up according to different case of coefficient functions. Fuzzy approximate solution was obtained by solving the model system. Our work enriched the theory of fuzzy linear differential equations.

Acknowledgments

The work is supported by the Natural Scientific Funds of PR China (71061013) and the Youth Research Ability Project of Northwest Normal University (NWNU-LKQN-11-20).

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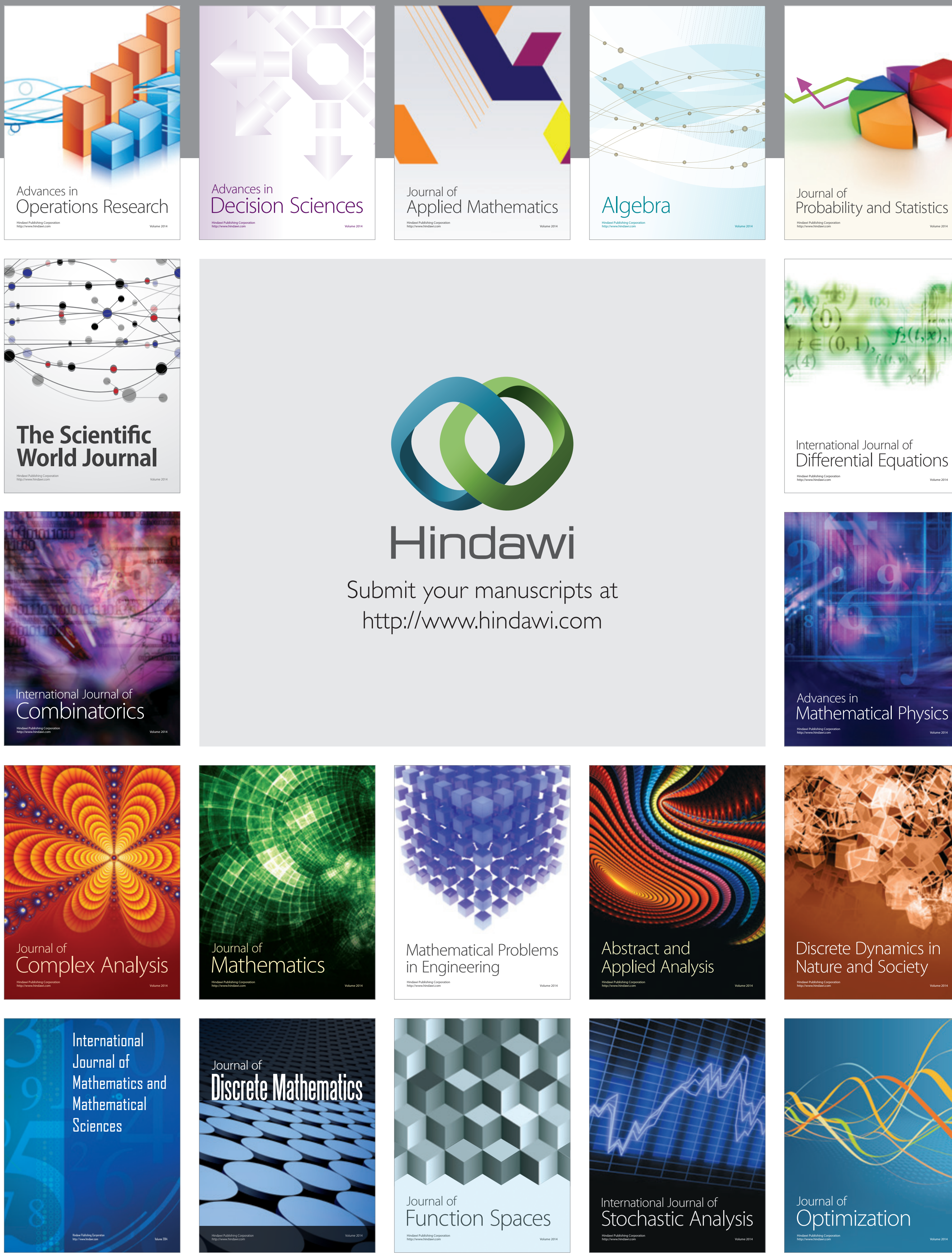
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