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Research Article

Synchronization Analysis of Two Coupled Complex Networks with Time Delays

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This paper studies the synchronized motions between two complex networks with time delays, which include individual inner synchronization in each network and outer synchronization between two networks. Based on the Lyapunov stability theory and the linear matrix equality (LMI), a synchronous criterion for inner synchronization inside each network is derived. Numerical examples are given which fit the theoretical analysis. In addition, the involved numerical results show that the delays between two networks have little effect on inner synchronization. It is also shown that synchronous motions within each network or between two networks are not enhanced if individual intranetwork connections are allowed.

1. Introduction

We refer to network synchronization, that is, synchronizing all the nodes inside a network, as "inner synchronization", which is widely studied in the science world based on the appearance of small-world [1] and scale-free [2] network models. Afterwards, the improved and expanded work in this respect—that is, introducing weighted connections, time varying coupling matrices, nonlinear coupling function, time delays, and so forth—can be found in the literature [3–10] and many references cited therein. The main approaches on studying synchronization inside a network is decoupling network systems, and studying the low-dimensional systems through the master stability function, or using the tool in equality (LMI toolbox).

Recently "outer synchronization" was proposed, which aimed at studying synchronization between coupled networks. In [11], the authors theoretically and numerically demonstrated the possibility of synchronization between two networks. By the open-plus-closed-loop (OPCL) method [12], synchronization between two networks can be realized with the same topological structures. In reality, if network nodes are of similar properties (same node dynamics), we regard it as one network, otherwise, as more networks. Several examples on two networks are presented in the literature [11, 13].

Actions between two networks are colorful, such as through the nonlinear signals, bipartite connection, or by special nodes. Recently, Sorrentino and Ott studied network synchronization of groups and discussed the feasibility of inner synchronization of individual network [14]. The problem of collective behavior in a network or between two networks is of broad interest. As a first example, in subway systems (e.g., Shanghai, China), when the trains arrive at the platform, the outer and inner doors simultaneously open or close, which shows that both inner synchronization and outer synchronization happen. Another example is two unmanned vehicles are assigned to accomplish independent tasks, such as cooperative searches and attacks [14], which focuses on inner synchronization. If the communicated information between two networks is not amount to lead to desynchronization, we should adopt control skills [15–17] to make synchronization happen between them, which has received increasing attention; see [18–21]. It is also found that present studies on synchronization inside a network and between two networks simultaneously are much less.

Time delays commonly exist in natural systems, since the finite speed of signal transmission over a distance gives rise to finite time delay, which plays an important role in the stability of synchronization [22]. For the two coupled networks, the signals from one network to another often create delays due to the distance. Therefore, motivated by the impact of topological structures and the delays on the dynamics of the networks, this paper mainly focuses on the effect of delays on inner synchronization inside each network and outer synchronization between two coupled networks. In Section 2, network models and synchronization analysis are presented, and numerical examples are shown in Section 3, including two examples for the same or different dimension of node dynamics. Finally, the discussions are included in the last section.

2. Model Presentation and Synchronization Analysis

Consider the following network models: the node dynamics in network $\mathbb{X}$ is $\dot{x}_i = f(x_i(t))$, $i = 1, \ldots, N_x$, and $\dot{y}_j = g(y_j(t))$, $j = 1, \ldots, N_y$, for the nodes in network $\mathbb{Y}$, where $f(x) : R^{n_x} \to R^{n_x}$, $g(y) : R^{n_y} \to R^{n_y}$ are continuously differential functions and $x_i(y_j)$ is an n_x-dimensional (n_y-dimensional) state vector. The dynamical equations of the network systems with time delays are as follows:

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) + \sum_{j=1}^{N_y} A_{ij}\Gamma_1 y_j(t-\tau_y), \quad i = 1, 2, \ldots, N_x, \\ \dot{y}_j(t) &= g(y_j(t)) + \sum_{i=1}^{N_x} B_{ji}\Gamma_2 x_i(t-\tau_x), \quad j = 1, 2, \ldots, N_y, \end{aligned} \tag{2.1}$$

where A is an $N_x \times N_y$ dimensional coupling matrix, whose entries (A_{ij}) represent the intensity of the direct interaction from i in network $\mathbb{X}$ to j in network $\mathbb{Y}$, analogously the entries of (B_{ji}). Matrix $\Gamma_1(\Gamma_2) \in R^{n_x \times n_y}(R^{n_y \times n_x})$ is the inner-coupling matrix. τ_x, τ_y are the time delays between networks. The action sketch between two networks with time delays is shown in Figure 1.

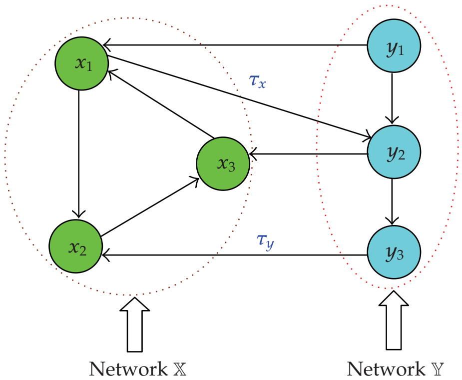

Figure 1: A simple action sketch between two coupled networks with time delays.

For the more general models, we allow connections in each network, the networked system then reads as

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N_y} A_{ij}\Gamma_1 y_j(t-\tau_y) + \sum_{m=1}^{N_x} C_{im}\Gamma_3 x_m(t), \quad i = 1,2,\ldots,N_x,$$
$$\dot{y}_j(t) = g(y_j(t)) + \sum_{i=1}^{N_x} B_{ji}\Gamma_2 x_i(t-\tau_x) + \sum_{k=1}^{N_y} D_{jk}\Gamma_4 y_k(t), \quad j = 1,2,\ldots,N_y, \tag{2.2}$$

where $C = (C_{im}) \in R^{N_x \times N_x}$, $D = (D_{jk}) \in R^{N_y \times N_y}$ are the coupling matrices, satisfying the sum of every row being zero. $\Gamma_3 \in R^{n_x \times n_x}$, $\Gamma_4 \in R^{n_y \times n_y}$ are also inner-coupling matrices.

Let us now consider the possibility that the individual networks achieve synchronization; that is, $x_1(t) = \cdots = x_{N_x}(t) = x_s(t)$ and $y_1(t) = \cdots = y_{N_y}(t) = y_s(t)$. If there exist such synchronous states, satisfying

$$\sum_{j=1}^{N_y} A_{ij} = \mu_1, \quad \forall i \in \mathbb{X}, \quad \sum_{i=1}^{N_x} B_{ji} = \mu_2, \quad \forall j \in \mathbb{Y}, \tag{2.3}$$

without loss of generality, we assume that $\mu_1 = \mu_2 = 1$. Thus, the synchronized state equations are

$$\dot{x}_s(t) = f(x_s(t)) + \Gamma_1 y_s(t-\tau_y),$$
$$\dot{y}_s(t) = g(y_s(t)) + \Gamma_2 x_s(t-\tau_x). \tag{2.4}$$

Linearizing the synchronous states around x_s and y_s, we get

$$\delta\dot{x}_i(t) = J(t)\delta x_i(t) + \sum_{j=1}^{N_y} A_{ij}\Gamma_1 \delta y_j(t-\tau_y), \quad i = 1,2,\ldots,N_x,$$
$$\delta\dot{y}_j(t) = W(t)\delta y_j(t) + \sum_{i=1}^{N_x} B_{ji}\Gamma_2 \delta x_i(t-\tau_x), \quad j = 1,2,\ldots,N_y, \tag{2.5}$$

where $J(t) = Df(x_s(t))$ and $W(t) = Dg(y_s(t))$ are the Jacobians of $f(x(t))$ and $g(y(t))$ at x_s and y_s, respectively.

We assume that the $(N_x + N_y)$-independent solutions of (2.5) can be expressed in the form $\delta x_i = \Phi_{x_i}\delta\overline{x}$, $i = 1,2,\ldots,N_x$, $\delta y_j = \Phi_{y_j}\delta\overline{y}$, $j = 1,2,\ldots,N_y$, where $\{\Phi_{x_i}\}, \{\Phi_{y_j}\}$ are the suitable time-independent scalars and $\delta\overline{x}, \delta\overline{y}$ are the appropriate variables. If the dimension of the space vectors given by the values of Φ_{x_i}, Φ_{y_j} $(i = 1,2,\ldots,N_x,\ j = 1,2,\ldots,N_y)$ is $N_x + N_y$, we can see that this assumed form includes all possible linear solutions of (2.5). Substituting this assumption into (2.5), they become

$$\Phi_{x_i}\delta\dot{\overline{x}}(t) = \Phi_{x_i}J(t)\delta\overline{x}(t) + \sum_{j=1}^{N_y} A_{ij}\Phi_{y_j}\Gamma_1\delta\overline{y}(t-\tau_y), \quad i = 1,2,\ldots,N_x, \tag{2.6}$$

$$\Phi_{y_j}\delta\dot{\overline{y}}(t) = \Phi_{y_j}W(t)\delta\overline{y}(t) + \sum_{i=1}^{N_x} B_{ji}\Phi_{x_i}\Gamma_2\delta\overline{x}(t-\tau_x), \quad j = 1,2,\ldots,N_y. \tag{2.7}$$

Thus, in order that (2.6) (resp., (2.7)) is satisfied for all i (resp., j), we require that $\Phi_{x_i}^{-1}\sum_j A_{ij}\Phi_{y_j} = \eta_1$, where η_1 is independent of i, and $\Phi_{y_j}^{-1}\sum_i B_{ji}\Phi_{x_i} = \eta_2$, where η_2 is independent of j. Set $\Phi_x = (\Phi_{x_1},\ldots,\Phi_{x_{N_x}})$ and $\Phi_y = (\Phi_{y_1},\ldots,\Phi_{y_{N_y}})$. From this, we get $A\Phi_y = \eta_1\Phi_x$, $B\Phi_x = \eta_2\Phi_y$; that is,

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} \Phi_x \\ \Phi_y \end{pmatrix} = \begin{pmatrix} \eta_1\Phi_x \\ \eta_2\Phi_y \end{pmatrix}. \tag{2.8}$$

Substituting (2.8) in (2.6) and (2.7), we obtain

$$\begin{gathered} \delta\dot{\overline{x}}(t) = J(t)\delta\overline{x}(t) + \eta_1\Gamma_1\delta\overline{y}(t-\tau_y), \\ \delta\dot{\overline{y}}(t) = W(t)\delta\overline{y}(t) + \eta_2\Gamma_2\delta\overline{x}(t-\tau_x). \end{gathered} \tag{2.9}$$

One particular solution of (2.8) is derived when $\eta_1 = \eta_2 = \lambda$, then

$$\Theta\begin{pmatrix} \Phi_x^* \\ \Phi_y^* \end{pmatrix} = \lambda\begin{pmatrix} \Phi_x^* \\ \Phi_y^* \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \tag{2.10}$$

where λ is the (possibly complex) eigenvalues of the matrix Θ.

Set $\eta_1 = \lambda\xi$, $\eta_2 = \lambda/\xi$, $\Phi_x = \Phi_x^*$, $\Phi_y = \xi\Phi_y^*$, where ξ is a free parameter. Equation (2.10) becomes

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} \Phi_x^* \\ \xi\Phi_y^* \end{pmatrix} = \begin{pmatrix} (\lambda\xi)\Phi_x^* \\ \left(\dfrac{\lambda}{\xi}\right)\xi\Phi_y^* \end{pmatrix}, \tag{2.11}$$

which shows that the solution of (2.11) contains all the possible solutions of (2.8). We rewrite (2.9) as

$$\delta\dot{\overline{x}}(t) = J(t)\delta\overline{x}(t) + \lambda\Gamma_1\delta\overline{y}(t-\tau_y),$$
$$\delta\dot{\overline{y}}(t) = W(t)\delta\overline{y}(t) + \lambda\Gamma_2\delta\overline{x}(t-\tau_x), \tag{2.12}$$

where $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_{N_x+N_y}$.

In [14], the authors gave the explicit analysis on the spectrum of Θ, and two forms of the constructed matrix Θ were shown. Here, we adopt one that matrix Θ has real eigenvalues and only study network (2.1). In the sequel, we utilize the LMI to derive a synchronous theorem.

Theorem 2.1. *Consider network model* (2.1). *If there exist two positive matrices $P, Q > 0$, satisfying*

$$\Xi = \begin{pmatrix} \Psi_1 & 0 & 0 & \lambda P\Gamma_1 \\ 0 & \Psi_2 & \lambda Q\Gamma_2 & 0 \\ 0 & \lambda\Gamma_2^T Q & -I_{n_x} & 0 \\ \lambda\Gamma_1^T P & 0 & 0 & -I_{n_y} \end{pmatrix} < 0, \tag{2.13}$$

where $\Psi_1 = J(t)^T P + PJ(t) + I_{n_x}$, $\Psi_2 = W(t)^T Q + QW(t) + I_{n_y}$, then the network (2.1) *asymptotically synchronizes to x_s, y_s defined by* (2.4) *for the fixed delays $\tau_x, \tau_y > 0$, respectively.*

Proof. Consider (2.12). Choose a Lyapunov-Krasovskii functional as

$$V(t) = \delta\overline{x}^T(t)P\delta\overline{x}(t) + \delta\overline{y}^T(t)Q\delta\overline{y}(t) + \int_{t-\tau_x}^{t} \delta\overline{x}^T(\alpha)\delta\overline{x}(\alpha)d\alpha + \int_{t-\tau_y}^{t} \delta\overline{y}^T(\alpha)\delta\overline{y}(\alpha)d\alpha. \tag{2.14}$$

Therefore,

$$\begin{aligned}
\dot{V}(t) &= \delta\dot{\overline{x}}^T(t)P\delta\overline{x}(t) + \delta\overline{x}^T(t)P\delta\dot{\overline{x}}(t) + \delta\dot{\overline{y}}^T(t)Q\delta\overline{y}(t) + \delta\overline{y}^T(t)Q\delta\dot{\overline{y}}(t) \\
&\quad + \delta\overline{x}^T(t)\delta\overline{x}(t) - \delta\overline{x}^T(t-\tau_x)\delta\overline{x}(t-\tau_x) + \delta\overline{y}^T(t)\delta\overline{y}(t) - \delta\overline{y}^T(t-\tau_y)\delta\overline{y}(t-\tau_y) \\
&= \delta\overline{x}^T(t)\Psi_1\delta\overline{x}(t) + \delta\overline{y}^T(t-\tau_y)\lambda\Gamma_1^T P\delta\overline{x}(t) + \delta\overline{y}^T(t)\Psi_2\delta\overline{y}(t) + \delta\overline{x}^T(t-\tau_x)\lambda\Gamma_2^T Q\delta\overline{y}(t) \\
&\quad + \delta\overline{y}^T(t)\lambda Q\Gamma_2\delta\overline{x}^T(t-\tau_x) + \delta\overline{x}^T(t)\lambda P\Gamma_1\delta\overline{y}^T(t-\tau_y) \\
&\quad - \delta\overline{x}^T(t-\tau_x)\delta\overline{x}(t-\tau_x) - \delta\overline{y}^T(t-\tau_y)\delta\overline{y}(t-\tau_y) \\
&= \begin{pmatrix} \delta\overline{x}(t) \\ \delta\overline{y}(t) \\ \delta\overline{x}(t-\tau_x) \\ \delta\overline{y}(t-\tau_y) \end{pmatrix}^T \Xi \begin{pmatrix} \delta\overline{x}(t) \\ \delta\overline{y}(t) \\ \delta\overline{x}(t-\tau_x) \\ \delta\overline{y}(t-\tau_y) \end{pmatrix}.
\end{aligned} \tag{2.15}$$

According to the known condition $\Xi < 0$ and Lyapunov-Krasovskii stability Theorem [6], the zero solutions of (2.12) are asymptotically stable, which shows that synchronization inside network $\mathbb{X}$ and network $\mathbb{Y}$ happens when t tends to $+\infty$. □

Remark 2.2. Matrix Ξ relies upon t and τ_x, τ_y. The condition can be changed into $\Xi < 0$ when $t > T$ for enough big $T > 0$. The linear matrix equality method is inadequate for assessing the stability of the synchronous solution when both intragroup and extragroup connections are allowed in the network (2.2).

3. Numerical Examples

In this section, we will give two examples to illustrate our obtained results, which includes two cases: $n_x \neq n_y$ and $n_x = n_y$. For the case $n_x \neq n_y$, we only consider the inner synchronization inside each network. On the other hand, $n_x = n_y$, we discuss the inner synchronization within each network and investigate the outer synchronization between two networks.

Example 3.1. Consider the coupled network dynamical equations [14] below, which are in the form (2.1):

$$\begin{gathered}
\dot{x}_i(t) = -a[x_i(t) + s(x_i(t))] + \sum_{j=1}^{N_y} A_{ij} y_{j1}(t - \tau_y), \quad i = 1, \ldots, N_x, \\
\dot{y}_{j1}(t) = -y_{j1}(t) + y_{j2}(t) + \sum_{i=1}^{N_x} B_{ji} x_i(t - \tau_x), \\
\dot{y}_{j2}(t) = -b y_{j1}(t), \quad j = 1, 2, \ldots, N_y,
\end{gathered} \tag{3.1}$$

where $s(x) = m_1 x + 1/2(m_0 - m_1)(|x+1| - |x-1|)$. Here, we take $a = 4.6$, $b = 6.02$, $m_0 = -8/7$, $m_1 = -5/7$, for numerical simulation. We introduce two quantities $E_x = \max_{i=1,\ldots,N_x} \|x_i - x_s\|$ and $E_y = \max_{j=1,\ldots,N_y} \|y_j - y_s\|$ to measure the extent to which synchronization is achieved. The initial values are randomly chosen in $(0, 1)$.

The intercoupling matrices are

$$A = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{3.2}$$

When $a = 4.6$, $b = 6.02$, $m_0 = -8/7$, $m_1 = -5/7$, $\tau_x = \tau_y = 0.8$ and A, B; by using the Matlab LMI Toolbox, we solve the LMI (2.13) for $P > 0$, $Q > 0$, and obtain

$$P = (2.5874); \qquad Q = \begin{pmatrix} 10.8749 & -0.9036 \\ -0.9036 & 1.8461 \end{pmatrix}, \tag{3.3}$$

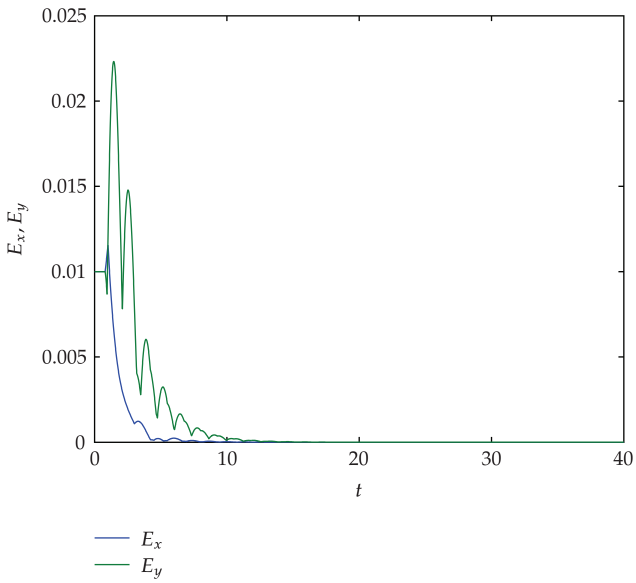

Figure 2: Inner synchronization evolution curves with $\tau_x = \tau_y = 0.8$.

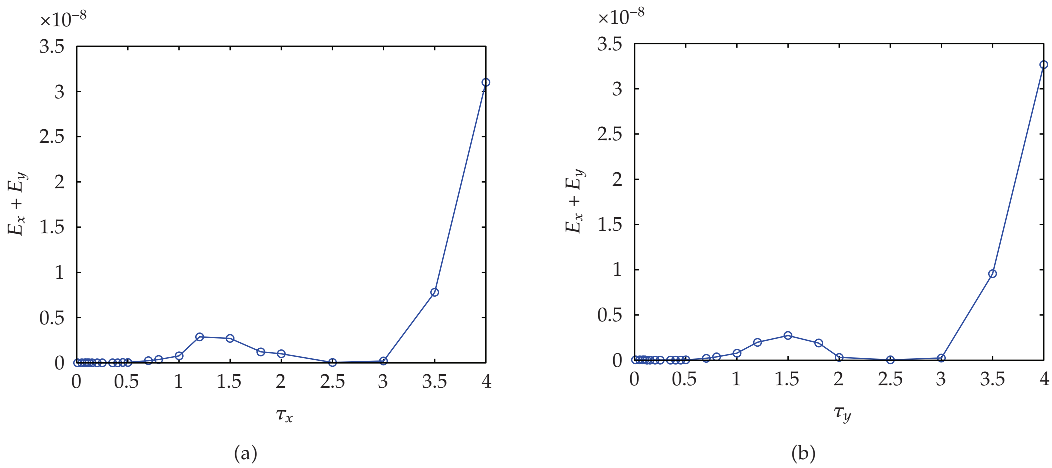

Figure 3: The plot shows $E_x + E_y$ at $t = 40$ with regard to τ_x or τ_y, (a) $\tau_y = 0.8$; (b) $\tau_x = 0.8$.

which indicates that inner synchronization inside each network appears. The inner synchronization evolution curves are depicted in Figure 2. We plots $E_x + E_y$ with different values of τ_x or τ_y in Figure 3, which shows that the delays τ_x, τ_y have little influence on inner synchronization with such intercoupling matrices A, B. Note that the order of magnitude for $E_x + E_y$ is 10^{-8}.

Next, we change the connection topology of A, B and consider a special case—globally connected, the elements of $A_{ij} = 1/N_y$, $B_{ji} = 1/N_x$, $i = 1, \ldots, N_x$, and $j = 1, \ldots, N_y$ with $N_x = 50$, $N_y = 100$. The other parameters are the same as in Figure 2. The numerical results are given in Figures 4 and 5. It is shown that the node numbers and connection topology influence the inner synchronization less.

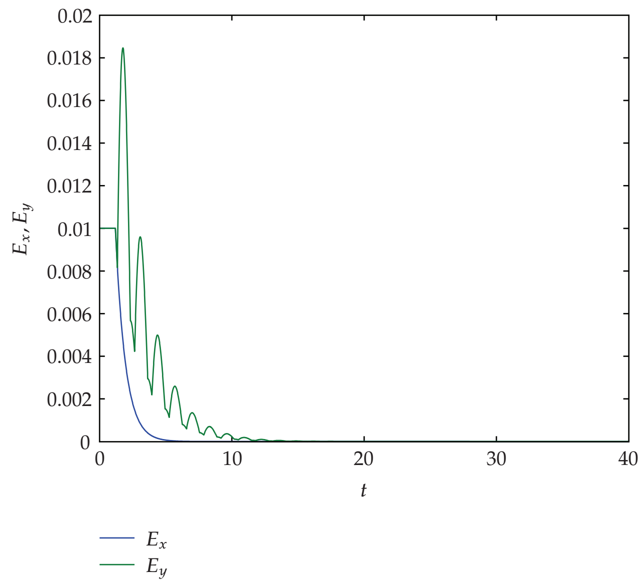

Figure 4: Inner synchronization evolution curves with globally connected topology, where $N_x = 50$, $N_y = 100$, and $\tau_x = \tau_y = 0.8$.

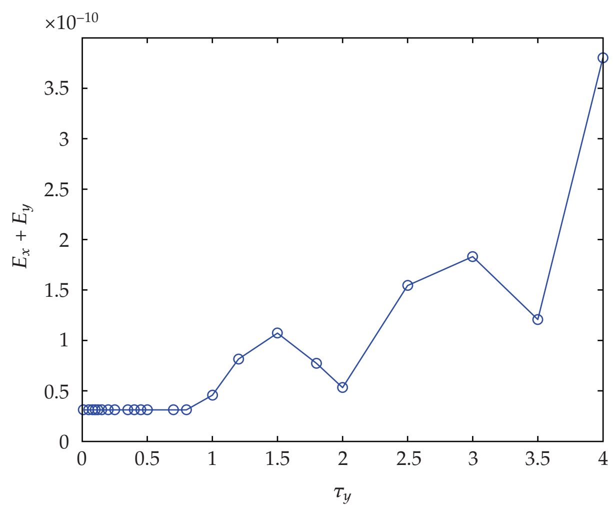

Figure 5: The curves of $E_x + E_y$ at $t = 40$ versus τ_y, with $\tau_x = 0.8$, $N_x = 50$, and $N_y = 100$.

Example 3.2. In this example, we let $n_x = n_y$ and $N_x = N_y = N$. Therefore, we discuss inner synchronization inside network $\mathbb{X}$ or $\mathbb{Y}$ and study synchronization between two networks, that is, outer synchronization proposed in [11], denoting the quantity $E_{\text{outer}} = \max_{i=1,\dots,N} \|x_i - y_i\|$, to demonstrate that outer synchronization happens. Consider a one-order dynamical system [23] as the dynamical nodes of the complex networks which is described by

$$\dot{x}_i(t) = -x_i^2(t) + \rho_1 x_i(t) + r_x \sum_{j=1}^{N} A_{ij} y_j(t - \tau_y), \quad i = 1, 2, \dots, N,$$

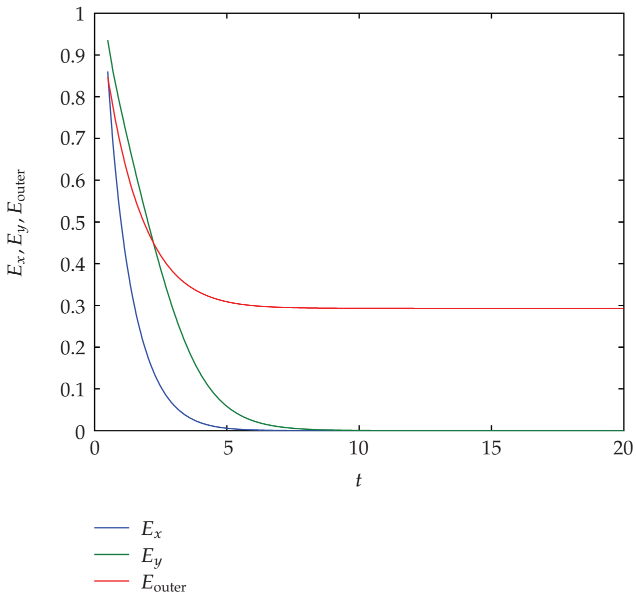

Figure 6: Inner and outer synchronization evolution curves with $r_x = 0.5$, $r_y = 0.2$, $\tau_x = 0.3$, $\tau_y = 0.5$, $\rho_1 = -0.3$, $\rho_2 = 0.6$, and $N = 50$.

$$\dot{y}_j(t) = -y_j^2(t) + \rho_2 y_j(t) + r_y \sum_{i=1}^{N} B_{ji} x_i(t - \tau_x), \quad j = 1, 2, \ldots, N. \tag{3.4}$$

The elements of $A_{ij} = 1/N$, $B_{ji} = 1/N$, $i, j = 1, \ldots, N$, and $N = 50$, when $\rho_1 = -0.3$, $\rho_2 = 0.6$, $\tau_x = 0.3$, and $\tau_y = 0.5$, by solving the LMI (2.13) for $P > 0$, $Q > 0$, we obtain

$$P = (2.0946), \qquad Q = (1.9328), \tag{3.5}$$

satisfying the conditions of a synchronous criterion, then the individual network achieves its own inner synchronized state. Figure 6 plots the inner and outer synchronization evolution curves. In addition, we plot the curves of quantities $E_x, E_y, E_{\text{outer}}$ with regard to r_x for $r_y = 0.2$ in Figure 7. We find that the individual network easily achieves inner synchronization for the arbitrary values of r_x, while only when $r_x = 1.2$, outer synchronization is realized.

In the following, we numerically discuss the delay effect on inner and outer synchronization. Fix the values of $\rho_1 = -0.3$, $\rho_2 = 0.6$, $r_y = 0.2$, $r_x = 1.2$, and $\tau_y = 0.5$, and let the value of τ_x vary. We observe that the delay does not influence inner synchronization, and when $\tau_x > 1.5$, outer synchronization disappears. Figure 8 plots the curves of E_x, E_y, and E_{outer} with regard to τ_x.

Now, we add the individual connections in network $\mathbb{X}$ and network $\mathbb{Y}$, and the following network systems are written as:

$$\begin{aligned} \dot{x}_i(t) &= -x_i^2(t) + \rho_1 x_i(t) + r_x \sum_{j=1}^{N} A_{ij} y_j(t - \tau_y) + \sum_{m=1}^{N} C_{im} x_m(t), \quad i = 1, 2, \ldots, N, \\ \dot{y}_j(t) &= -y_j^2(t) + \rho_2 y_j(t) + r_y \sum_{i=1}^{N} B_{ji} x_i(t - \tau_x) + \sum_{k=1}^{N} D_{jk} y_k(t), \quad j = 1, 2, \ldots, N, \end{aligned} \tag{3.6}$$

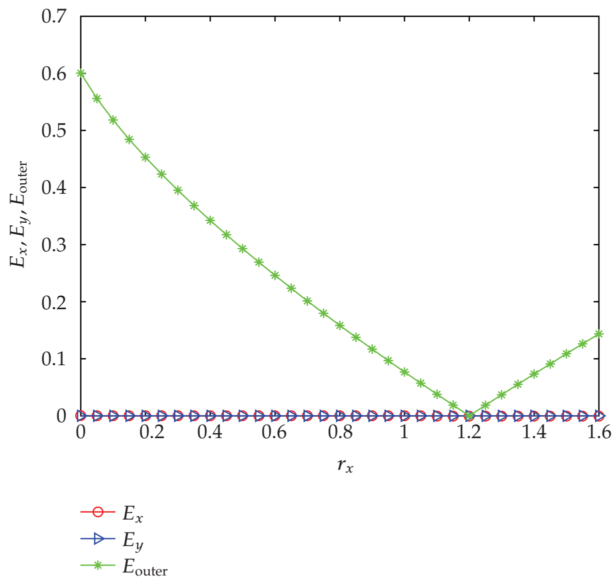

Figure 7: The curves of E_x, E_y, and E_{outer} versus r_x for $r_y = 0.2$. The other parameters are $\tau_x = 0.3$, $\tau_y = 0.5$, $\rho_1 = -0.3$, and $\rho_2 = 0.6$.

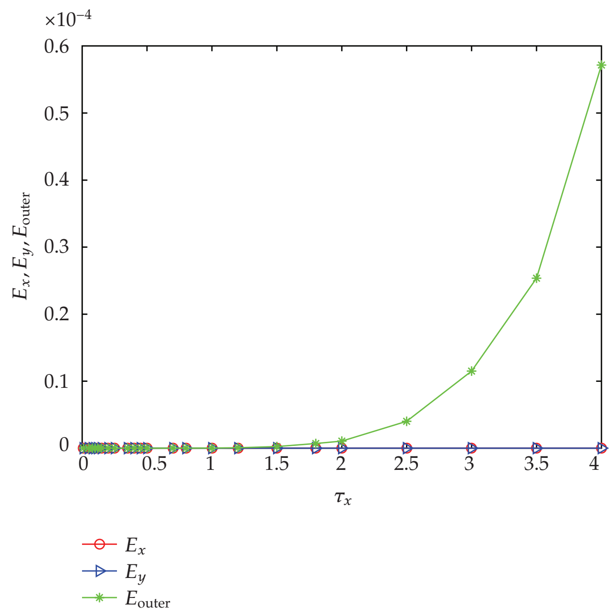

Figure 8: The plot show E_x, E_y, and E_{outer} versus τ_x with $\tau_y = 0.5$, $r_y = 0.3$, $r_x = 1.2$, $\rho_1 = -0.3$, and $\rho_2 = 0.6$.

where $(-C) = -\{C_{im}\}$, $(-D) = -\{D_{jk}\}$ are the Laplacian matrices: $\sum_{m=1}^{N} C_{im} = 0$ for all i and $\sum_{k=1}^{N} D_{jk} = 0$ for all j; obviously, the diffusive couplings $\sum_{m=1}^{N} C_{im}$, $\sum_{k=1}^{N} D_{jk}$ are null in the synchronized manifold. The parameters are chosen as the same as those in Figure 8, and individual couplings C, D are taken as two random matrices. Numerical results show that the adding individual connections have little effect on inner and outer synchronization.

4. Conclusions

In conclusion, we have studied inner synchronization inside networks $\mathbb{X}$ and $\mathbb{Y}$ and outer synchronization between them with time delays. A synchronous criterion for the occurrence of inner synchronization has been derived. Numerical examples show that the delays have little effect on inner synchronization, how to estimate the domain of delays τ_x and τ_y is an interesting work. It is noted that in Example 3.2, outer synchronization only happens in the coupling strength $r_x = 1.2$, and perhaps the node dynamics in networks $\mathbb{X}$ and $\mathbb{Y}$ plays a central role, or the bidirectional delayed coupling is weak. Because of the various kinds of actions between two networks, how to derive the criteria on inner and outer synchronization simultaneously is an technical challenging. The theoretical understanding is helpful to study the synchronization between two or more neural networks [24] with appropriate couplings. We hope that such work will appear elsewhere.

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