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FRAGMENTABILITY BY THE DISCRETE METRIC

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Abstract

In a recent paper, topological spaces (X, τ) that are fragmented by a metric that generates the discrete topology were investigated. In the present paper we shall continue this investigation. In particular, we will show, among other things, that such spaces are σ-scattered, that is, a countable union of scattered spaces, and characterise the continuous images of separable metrisable spaces by their fragmentability properties.

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In [7], topological spaces (X, τ) that are fragmented by a metric that generates the discrete topology were investigated. In this paper we show, among other things, that such spaces are σ-scattered. The reason behind the interest in fragmentability lies in the fact that fragmentability (σ-fragmentability) has had numerous applications to many parts of analysis; see [3–6, 8, 17, 23–28, 30, 33, 35–39], to mention but a small selection of them.

Let (X, τ) be a topological space and let ρ be a metric defined on X. Following [12], we shall say that (X, τ) is *fragmented by* ρ if whenever $\varepsilon > 0$ and A is a nonempty subset of X there is a τ-open set U such that $U \cap A \neq \emptyset$ and $\rho - \mathrm{diam}(U \cap A) < \varepsilon$.

A significant generalisation of fragmentability is the following: a topological space (X, τ), endowed with a metric ρ, is *σ-fragmented by* ρ if, for each $\varepsilon > 0$, there exists a cover $\{X_n^\varepsilon : n \in \mathbb{N}\}$ of X (that is, $\bigcup_{n\in\mathbb{N}} X_n^\varepsilon = X$) such that for every $n \in \mathbb{N}$ and every nonempty subset A of X_n^ε there exists a τ-open set U such that $U \cap A \neq \emptyset$ and $\rho - \mathrm{diam}(U \cap A) < \varepsilon$; see [9–11].

Theorem 1. *Let (X, τ) be a Hausdorff regular space. Then the following are equivalent:*

(i) *(X, τ) is fragmented by a metric that generates the discrete topology;*
(ii) *(X, τ) is σ-fragmented by the discrete metric;*
(iii) *(X, τ) is σ-scattered, that is, a countable union of scattered spaces.*

Proof. The proof that (i) $\Rightarrow$ (ii) follows from [21, Proposition 3.1]. To see that (ii) $\Rightarrow$ (iii), we simply apply the definition of σ-fragmentability with

$\varepsilon := 1/2 < 1$. The fact that (iii) $\Rightarrow$ (ii) is obvious. Finally, (ii) $\Rightarrow$ (i) follows from [21, Proposition 3.2]. □

Thus, the study of fragmentability by a metric that generates the discrete topology reduces to the (well studied) study of scattered spaces.

In the presence of Lindelöfness, fragmentability by a metric that generates the discrete topology imposes a severe constraint on the size of the underlying set.

Corollary 2. *Let (X, τ) be a hereditarily Lindelöf Hausdorff regular space. Then (X, τ) is countable provided that (X, τ) is fragmented by a metric that generates the discrete topology. In particular, every subset of a separable metric space that is fragmented by a metric that generates the discrete topology is countable.*

Proof. By Theorem 1, we know that X is a countable union of scattered spaces. Hence, it is sufficient to show that a hereditarily Lindelöf scattered space is countable. Let

$$\mathcal{U} := \{U \in \tau : U \text{ is countable}\} \quad \text{and let} \quad U^* := \bigcup_{U \in \mathcal{U}} U.$$

Since X is hereditarily Lindelöf, it follows that $U^* \in \mathcal{U}$. We claim that $X = U^*$. Indeed, if this were not the case, then $X \backslash U^* \neq \emptyset$ and so there would exist an open set W such that $(X \setminus U^*) \cap W$ is a singleton. Clearly, then, $U^* \cup W \in \mathcal{U}$. However, this is impossible since $U^* \cup W \not\subseteq U^*$. □

At the price of having to introduce several new definitions and several basic results, we can extend Corollary 2 as follows.

Let (X, τ) be a topological space. Then we call $\mathcal{P} \subseteq 2^X \setminus \{\emptyset\}$ a *partial exhaustive partition of X* if:

(i) $\bigcup_{P \in \mathcal{P}} P \in \tau$;
(ii) the members of $\mathcal{P}$ are pairwise disjoint;
(iii) for every nonempty subset A of $\bigcup_{P \in \mathcal{P}} P$, there exists a $P \in \mathcal{P}$ such that $A \cap P$ is a nonempty relatively open subset of A.

If $\bigcup_{P \in \mathcal{P}} P = X$, then we simply call $\mathcal{P}$ an *exhaustive partition of X*.

Given partitions $\mathcal{P}$ and $\mathcal{Q}$ of a set X, we shall say that $\mathcal{P}$ is a *refinement* of $\mathcal{Q}$ if for each $P \in \mathcal{P}$ there is a $Q \in \mathcal{Q}$ such that $P \subseteq Q$. Now, if $\mathcal{P}$ and $\mathcal{Q}$ are partitions of X, then

$$\mathcal{P} \vee \mathcal{Q} := \{Y \in 2^X \setminus \{\emptyset\} : Y = P \cap Q \text{ for some } P \in \mathcal{P} \text{ and } Q \in \mathcal{Q}\}$$

is also a partition of X that is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$. Furthermore, if $\mathcal{P}$ and $\mathcal{Q}$ are exhaustive partitions of a topological space (X, τ), then $\mathcal{P} \vee \mathcal{Q}$ is also an exhaustive partition of X.

Proposition 3. *Every exhaustive partition of a hereditarily Lindelöf space is countable.*

Proof. Let (X, τ) be a hereditarily Lindelöf topological space and let $\mathcal{P}$ be an exhaustive partition of X. Let $\mathscr{A}$ be the family of all $\mathcal{Q} \subseteq \mathcal{P}$ such that $\mathcal{Q}$ is a countable partial exhaustive partition of X. Then $(\mathscr{A}, \subseteq)$ is a nonempty partially ordered set.

Furthermore, from Zorn's lemma and the fact that (X, τ) is hereditarily Lindelöf, it follows that $(\mathscr{A}, \subseteq)$ has a maximal element $\mathcal{Q}_{\max}$.

We claim that $\bigcup_{Q \in \mathcal{Q}_{\max}} Q = X$ (which implies that $\mathcal{Q}_{\max} = \mathcal{P}$). Indeed, if $\bigcup_{Q \in \mathcal{Q}_{\max}} Q \neq X$, then $X \setminus (\bigcup_{Q \in \mathcal{Q}_{\max}} Q) \neq \emptyset$. Since $\mathcal{P}$ is exhaustive, there exists a $P \in \mathcal{P}$ such that $\emptyset \neq P \cap (X \setminus (\bigcup_{Q \in \mathcal{Q}_{\max}} Q))$ is relatively open in $X \setminus (\bigcup_{Q \in \mathcal{Q}_{\max}} Q)$. If we let $\mathcal{Q}^* := \mathcal{Q}_{\max} \cup \{P\}$, then $\mathcal{Q}^* \in \mathscr{A}$, $\mathcal{Q}_{\max} \subseteq \mathcal{Q}^*$ and $\mathcal{Q}_{\max} \neq \mathcal{Q}^*$. However, this contradicts the maximality of $\mathcal{Q}_{\max}$. Therefore, $\bigcup_{Q \in \mathcal{Q}_{\max}} Q = X$ and so $\mathcal{P} = \mathcal{Q}_{\max} \in \mathscr{A}$. □

Theorem 4. *Let (X, τ) be a completely regular topological space. Then X is the continuous image of a separable metric space if, and only if, (X, τ) is hereditarily Lindelöf and fragmented by a metric whose topology is at least as strong as τ.*

Proof. Suppose that X is the continuous image of a separable metric space. Then, clearly, (X, τ) is hereditarily Lindelöf and, by [23, Proposition 2.1], (X, τ) is fragmented by a metric whose topology is at least as strong as τ. Conversely, suppose that (X, τ) is hereditarily Lindelöf and fragmented by a metric d whose topology on X is at least as strong as τ. For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a maximal partial exhaustive partition of X such that $d-\operatorname{diam}(P) < 1/n$ for each $P \in \mathcal{P}$. Since (X, τ) is fragmented by d, each $\mathcal{P}_n$ is in fact an exhaustive partition of X. By passing to a refinement, we may assume that for each $n \in \mathbb{N}$, $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$. Furthermore, by Proposition 3, we can write, for each $n \in \mathbb{N}$, $\mathcal{P}_n := \{P_n^k : k \in \Omega_n\}$, where $\emptyset \neq \Omega_n \subseteq \mathbb{N}$. Let

$$\Sigma := \Big\{\sigma \in \prod_{n \in \mathbb{N}} \Omega_n : \bigcap_{n \in \mathbb{N}} P_n^{\sigma(n)} \neq \emptyset\Big\}.$$

Endow Σ with the Baire metric d, that is, if $\sigma \neq \sigma'$, then $d(\sigma, \sigma') := 1/n$, where $n := \min\{k \in \mathbb{N} : \sigma(k) \neq \sigma'(k)\}$. Next define $f : (\Sigma, d) \to (X, \tau)$ by $f(\sigma) \in \bigcap_{n \in \mathbb{N}} P_n^{\sigma(n)}$. Note that f is well defined, since $|\bigcap_{n \in \mathbb{N}} P_n^{\sigma(n)}| = 1$ for all $\sigma \in \Sigma$. Clearly, f is a bijection from Σ onto X and, since $f(B(\sigma, 1/n)) \subseteq P_n^{\sigma(n)}$ (where $B(\sigma, 1/n) := \{\sigma' \in \Sigma : d(\sigma, \sigma') < 1/n\}$) and $d-\operatorname{diam}(P_n^{\sigma(n)}) < 1/n$, we see that f is continuous on Σ. □

It is known that fragmentability of a topological space is characterised by the existence of a winning strategy for one of the players (usually called B) in a certain topological game [20, 21]. It is also known that the lack of a winning strategy for the other player (usually called A) in the same game characterises a property that is close to the Namioka property [18, 19]. To be more precise about this, we need the following definition.

Let X be a set with two (not necessarily distinct) topologies τ_1 and τ_2. On X we will consider the $\mathscr{G}(X, \tau_1, \tau_2)$*-game* played between two players A and B. Player A goes first (every time—life is not always fair) and chooses a nonempty subset A_1 of X. Player B must then respond by choosing a nonempty relatively τ_1-open subset B_1 of A_1. Following this, player A must select another nonempty set $A_2 \subseteq B_1 \subseteq A_1$ and in turn player B must again respond by selecting a nonempty relatively τ_1-open subset $B_2 \subseteq A_2 \subseteq B_1 \subseteq A_1$. Continuing this process indefinitely, the players A and B produce a sequence $((A_n, B_n) : n \in \mathbb{N})$ of pairs of nonempty subsets (with B_n relatively τ_1-open in A_n) called a *play* of the $\mathscr{G}(X, \tau_1, \tau_2)$-game. We shall declare that the player B *wins* a play $((A_n, B_n) : n \in \mathbb{N})$ if either (i) $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ or else (ii) $\bigcap_{n\in\mathbb{N}} A_n = \{x\}$ for some $x \in X$ and for every τ_2-open neighbourhood U of x there exists an $n \in \mathbb{N}$ such that $A_n \subseteq U$. Otherwise, the player A is said to have won. By a *strategy* σ for the player B we mean a 'rule' that specifies each move of the player B in every possible situation that can occur. Since in general the moves of the player B may depend upon the previous moves of the player A, we shall denote by $\sigma(A_1, A_2, \ldots, A_n)$ the nth move of the player B under the strategy σ. We shall call a strategy σ, for the player B, a *winning strategy* if he/she wins every play of the $\mathscr{G}(X, \tau_1, \tau_2)$-game, in which they play according to the strategy σ. For a more precise definition of a strategy, see [2].

The main result connecting the $\mathscr{G}(X, \tau_1, \tau_2)$-game to fragmentability is the following theorem.

Theorem 5 [21, Theorem 1.2]. *Let τ_1, τ_2 be two (not necessarily distinct) topologies on a set X. The space (X, τ_1) is fragmentable by a metric whose topology is at least as strong as τ_2 if, and only if, the player B has a winning strategy in the $\mathscr{G}(X, \tau_1, \tau_2)$-game played on X.*

Throughout the remainder of this paper we will be interested in the case when τ_2 is the discrete topology—which we will denote by τ_d. We have seen in Theorem 1 that fragmentability by a metric that generates the discrete topology (or, equivalently, the existence of a winning strategy for the player B in the $\mathscr{G}(X, \tau_1, \tau_d)$-game) reduces to the study of σ-scattered spaces. However, it might be interesting to see whether the lack of a winning strategy for the player A in the $\mathscr{G}(X, \tau_1, \tau_d)$-game leads to anything more interesting.

Our next result requires two more auxiliary notions. The first is the notion of quasi-continuity. Suppose that $f : (X, \tau) \to (Y, \tau')$ is a function acting between topological spaces (X, τ) and (Y, τ'). Then we say that f is *quasi-continuous* if for each open set W in Y, $f^{-1}(W) \subseteq \overline{\mathrm{int}(f^{-1}(W))}$ [16]. The second notion that is needed is that of an α-favourable space, whose precise definition can be found in [19].

Theorem 6 [19, Theorem 1]. *Let (X, τ) be a Hausdorff regular space. Then the following are equivalent:*

(i) *the $G(X, \tau, \tau_d)$-game is A-unfavourable;*
(ii) *for every quasi-continuous mapping $f : Z \to (X, \tau)$ from a complete metric space Z there is a nonempty open subset U such that f is constant on U;*
(iii) *for every quasi-continuous mapping $f : Z \to (X, \tau)$ from an α-favourable space Z there is a nonempty open subset U such that f is constant on U;*
(iv) *for every continuous mapping $f : Z \to (X, \tau)$ from an α-favourable space Z there is a nonempty open subset U such that f is constant on U.*

If the topology τ is metrisable, then we have the following theorem.

THEOREM 7. *Let (X,τ) be a metrisable space. Then the following are equivalent:*

(i) *the $G(X,\tau,\tau_d)$-game is A-unfavourable;*
(ii) *for every continuous mapping $f : Z \to (X,\tau)$ from a complete metric space Z there is a nonempty open subset U such that f is constant on U;*
(iii) *for every quasi-continuous mapping $f : Z \to (X,\tau)$ from a complete metric space Z there is a nonempty open subset U such that f is constant on U;*
(iv) *for every quasi-continuous mapping $f : Z \to (X,\tau)$ from an α-favourable space Z there is a nonempty open subset U such that f is constant on U;*
(v) *for every continuous mapping $f : Z \to (X,\tau)$ from an α-favourable space Z there is a nonempty open subset U such that f is constant on U.*

PROOF. Clearly, (iii) ⇒ (ii) and so by Theorem 6 it is sufficient to show that (ii) ⇒ (iii). Suppose that Z is a complete metric space and $f : Z \to (X,\tau)$ is quasi-continuous. Since (X,τ) is metrisable, we have from [1] that there exists a dense G_δ subset G of Z such that f is continuous at each point of G. Now, by [15, page 208] or [34, page 164], there exists a complete metric d on G that generates the relative topology on G. Next, by our assumption, $f|_G : G \to X$ has a nonempty open subset U of G such that $f|_G(U) =: \{x\}$ is a singleton. Let U^* be any open subset of Z such that $U^* \cap G = U$. Since f is quasi-continuous on Z, it follows (see for example [28]) that $f(U^*) \subseteq \overline{f|_G(U)} = \overline{\{x\}} = \{x\}$. Hence, f is constant on U^*, which completes the proof. □

We may now apply this result along with the definition of a perfect set to obtain the following useful characterisation. Recall that a subset of a topological space (X,τ) is called *perfect* if it is closed and does not have any isolated points.

COROLLARY 8. *Let (X,τ) be a metrisable space. Then the $G(X,\tau,\tau_d)$-game is A-unfavourable if, and only if, X does not contain any perfect compact subsets.*

PROOF. Suppose that the $G(X,\tau,\tau_d)$-game is A-unfavourable. In order to obtain a contradiction, let us suppose that X contains a perfect compact set Z. We shall consider the identity mapping $f : Z \to Z$ defined by $f(z) := z$ for all $z \in Z$. Now, since Z is a perfect set, it does not have any isolated points and so we have a continuous nowhere-constant function defined on a complete metric space. This contradicts part (ii) of Theorem 7.

For the converse, let us start by assuming that X does not contain any perfect compact subsets. From Theorem 7, it is sufficient to show that for any complete metric space M and any continuous function $f : M \to X$ there is a nonempty open subset U of M such that f is constant on U. Let (M,ρ) be a complete metric space. In order to obtain a contradiction, let us suppose that $f : M \to X$ is not constant on any nonempty open subset of M. Let D be the set of all finite sequences of zeros and ones. We shall inductively (on the length $|d|$ of $d \in D$) define a family $\{C_d : d \in D\}$ of nonempty open subsets of M such that:

(i) $\rho - \operatorname{diam}(C_d) < 1/2^{|d|}$;
(ii) $\emptyset \neq \overline{C_{d0}} \cap \overline{C_{d1}} \subseteq \overline{C_{d0}} \cup \overline{C_{d1}} \subseteq C_d$;
(iii) $f(\overline{C_{d0}}) \cap f(\overline{C_{d1}}) = \emptyset$.

Base step: let $C_\emptyset$ be a nonempty open subset of M with $\rho - \operatorname{diam}(C_\emptyset) < 1/2^0$, where the sequence of length zero is denoted by $\emptyset$.

Assuming that we have already defined the nonempty open sets C_d satisfying (i), (ii) and (iii) for all $d \in D$ with $|d| \leq n$, we proceed to the inductive step.

Inductive step: fix $d \in D$ of length n. Therefore, there exist points c_0 and c_1 in C_d such that $f(c_0) \neq f(c_1)$. From the continuity of f, we can choose open neighbourhoods C_{d0} of c_0 and C_{d1} of c_1 such that conditions (i), (ii) and (iii) are satisfied. This completes the induction.

Now, for each $n \in \mathbb{N}$, let $K_n := \bigcup\{\overline{C_d} : d \in D \text{ and } |d| = n\}$ and $K := \bigcap_{n\in\mathbb{N}} K_n$. Then K is closed and totally bounded and hence compact. Furthermore, K is perfect and f is one-to-one on K. Therefore, $f(K)$ is a perfect compact subset of X, which contradicts our assumption concerning X. Therefore, f must be constant on some nonempty open subset U of M. □

In order to state our last result, we need to recall the definition of a Bernstein set. A subset B of $\mathbb{R}$ is called a *Bernstein set* if neither B nor its complement contains a perfect compact subset [32, page 23]. In [32], the construction of a Bernstein set is given. It is also easy to check that every Bernstein set is uncountable.

COROLLARY 9. *Let B be a Bernstein subset of $\mathbb{R}$ endowed with the relative topology τ inherited from $\mathbb{R}$ with its usual topology. Then neither player (A nor B) has a winning strategy in the $G(B, \tau, \tau_d)$-game played on B.*

Interestingly, uncountable subsets of $\mathbb{R}$ that do not contain any perfect compact subsets played an important role in the construction of (i) a Gâteaux differentiability space that is not weak Asplund [13, 14, 27, 29, 30], (ii) a dual differentiation space that does not admit an equivalent locally uniformly rotund norm [22] and (iii) a Namioka space without an equivalent Kadeč norm [31].

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