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Some New Exact Solutions of Jacobian Elliptic Functions in Nonlinear Physics Problem

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Abstract: An extended mapping method with symbolic computation is developed to obtain some new periodic wave solutions in terms of Jacobin elliptic function for nonlinear elastic rod equation arising in physics.As a result,many exact travelling wave solutions are obtained which include Jacobian elliptic functions solutions,combined Jacobian elliptic functions solutions and triangular function solutions.Solutions in the limiting cases have also been studied.It is shown that the mapping method provides a very effective and powerful mathematical tool for solving nonlinear evolution equations in physics.

Keywords: Extended mapping method;Nonlinear physics problem ;Jacobian elliptic functions solutions;Triangular function solutions

1 Introduction

A large variety of physical,chemical,and biological phenomena is governed by nonlinear evolution equations.The analytical study of nonlinear partial differential equations was of great interest during the last decade years.The investigations of the travelling wave solution of nonlinear equations play an important role in the study of nonlinear physical phenomena.The importance of obtaining the exact solutions,if available,of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions.

Exact travelling wave solutions of nonlinear evolution equations is one of the fundamental object of study in mathematical physics.These exact solutions when they exist can be help one to well understand the mechanism of the complicated physical phenomena and dynamical processes modelled by these nonlinear evolution equations.In the past several decades,many significant methods have been established in $[1-20]$

In fact the Jacobian elliptic functions $[3-5]$ degenerate into hyperbolic functions when the modulus approaches 1,has attracted a lot of interest in the investigation of exact solutions.The three basic Jacobian elliptic functions $sn(\xi,m)$,$cn(\xi,m)$ and $dn(\xi,m)$, where m is the modulus of the elliptic function,satisfy the well known type of trigonometric relations such as $sn^2(\xi)+cn^2(\xi)=1$,$dn^2(\xi)+m^2sn^2(\xi)=1$,$(sn(\xi))'=cn(\xi)dn(\xi)$,$(cn(\xi))'=-sn(\xi)dn(\xi)$,$(dn(\xi))'=-m^2sn(\xi)cn(\xi)$.When $m\longrightarrow 0$,the Jacobi elliptic functions degenerate to the triangular functions,i.e.,$sn(\xi)\longrightarrow sin(\xi)$,$cn(\xi)\longrightarrow cos(\xi)$,$dn(\xi)\longrightarrow 1$ and when $m\longrightarrow 1$, the Jacobian elliptic functions degenerate to the hyperbolic functions i.e.,$sn(\xi)\longrightarrow tanh(\xi)$,$cn(\xi)\longrightarrow sech(\xi)$,$dn(\xi)\longrightarrow sech(\xi)$.A mapping method and its extensions have been successfully applied to derive a variety of Jacobian elliptic function solutions for nonlinear evolution equations arising in mathemtical physics.

Solitary and periodic waves solutions of nonlinear evolution equations have been studied intensively.The exact solutions,if these nonlinear equations facilitates the verification of numerical solvers and aids in the

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stability analysis of solutions.The goal of the present work is to document the changes of the traveling wave solutions in the parameter space,from the view point of dynamical systems,for an elastic rod equation.Zhuang et.al.[20] derived the nonlinear wave equation of longitudinal oscillation of a nonlinear elastic rod with lateral inertia [21] as

$$\frac{\partial^2 u}{\partial t^2} - c_0^2[1 + na_n(\frac{\partial u}{\partial x})^{n-1}]\frac{\partial^2 u}{\partial x^2} - \frac{\nu^2 J_p}{s}\frac{\partial^4 u}{\partial t^2 \partial x^2} = 0, \tag{1}$$

where s,J_p and $c_0^2 = \frac{E}{\rho}$,ν,E, and ρ are the cross-section area of the rod,the polar moment of inertia,the square of the linear elastic longitudinal wave velocity,Poisson ratio,the Young's modulus and the density of the rod, respectively.a_n is the material constant,n is an integer.For the soft nonlinear materials $a_n < 0$,for example,majority of the metals.For the hard nonlinear materials such as rubbers and polymers,$a_n > 0$. When $n = 2$,3 [20].

2 Modified Mapping Method

The extended mapping method can be introduced briefly as follows.For a given nonlinear evolution equation,say,in two independent variables,

$$F(u, u_t, u_x, ...) = 0, \tag{2}$$

and its travelling wave solution

$$u(x, t) = u(\xi), \xi = kx - ct, \tag{3}$$

where k,and c are constants to be determined later.

Inserting Eq.(3) into Eq.(2) yields an ordinary differential equation of $u(\xi)$.Then $u(\xi)$ can be expressed as follows

$$u(\xi) = \sum_{i=0}^{M} g_i f^i(\xi) + \sum_{i=1}^{M} h_i f^{-i}(\xi), \tag{4}$$

where g_i and $h_i(i = 1, ...N)$ are constants to be determined later,M is fixed by balancing the linear term of the highest order derivative with nonlinear term,while $f(\xi)$ satisfy the equation

$$\begin{aligned} f'(\xi) &= \sqrt{bf^2(\xi) + \frac{a}{2}f^4(\xi) + c}, \\ f'' &= bf(\xi) + af^3(\xi), \end{aligned} \tag{5}$$

where the prime denotes derivative with respect to ξ,and b,a,c are constants to be determined.

Substitute anzatz (4) into (3),make use of Eq.(5) with computerized symbolic computation,equating to zero the coefficients of all powers of $f^{\pm i}(\xi)(i = -4, ..., 4)$ yields a set of algebraic equations for g_i and h_i.Solving this system by use of the symbolic computation system MAPLE,we can find the travelling wave solutions of Eq.(1).

3 Exact solutions of nonlinear elastic rod equation

To look for the travelling wave solution of Eq.(1),we use the gauge ransformation

$$u(x, t) = \psi(x - ct) = \psi(\xi), \xi = x - ct, \tag{6}$$

where c is the wave speed.Then Eq.(1) reduces to

$$[c^2 - c_0^2]\psi'' - nc_0^2 a_n(\psi')^{n-1}\psi'' - \frac{\nu^2 c^2 J_p}{s}\psi'''' = 0, \tag{7}$$

where $'$ is the derivative with respect to ξ.Taking $\phi(\xi) = \psi(\xi)'$ and integrating obtained equation once,we have

$$[c^2-c_0^2]\phi - c_0^2 a_n\phi^n - \frac{\nu^2 c^2 J_p}{s}\phi'' + d = 0 \tag{8}$$

where d is an integration constant.Denote that $\alpha=\frac{s(c^2-c_0^2)}{c^2\nu^2 J_p}$,$\beta=\frac{s a_n c_0^2}{c^2\nu^2 J_p}$ for $c^2\nu^2 J_p\neq 0$.Then,for $n=2$ we obtain

$$\phi'' - \alpha\phi + \beta\phi^2 = 0 \tag{9}$$

Then,Eq.(9) can be rewritten as

$$A\phi'' + B\phi + C\phi^2 + d = 0, A=1, B=-\alpha, C=\beta \tag{10}$$

Considering the homogeneous balance between $\phi''(\xi)$ and $\phi^2(\xi)$ in Eq.(9),we have $M=2$.Therefore,we assume that $\phi(\xi)$ can be expressed as

$$\phi(\xi) = g_0 + g_1 f(\xi) + h_1 f^{-1}(\xi) + g_2 f^2(\xi) + h_2 f^{-2}(\xi), \tag{11}$$

where g_0,g_1,g_2,h_1 and h_2 are constants to be determined,and $f(\xi)$ satisfy equation (5).We substitute anzatz (11) into (10),make use of Eq.(5) with computerized symbolic computation,equating to zero the coefficients of all powers of $f(\xi)$ yields a set of algebraic equations for g_0,g_1,g_2,h_1 and h_2.Solving the system of algebraic equations with the aid of Maple,we have three different cases as

Case[i]

$$g_1=0, h_1=0, g_0=-\frac{B+4Ab}{2C}, h_2=-\frac{6Ac}{C}, g_2=\frac{-3Aa}{C}, d=-\frac{B^2+96A^2aca+16A^2b^2}{4C} \tag{12}$$

Case[ii]

$$g_1=0, h_1=0, g_0=-\frac{B+4Ab}{2C}, h_2=0, g_2=\frac{-3Aa}{C}, d=\frac{B^2+24A^2ca-16A^2d^2}{4C} \tag{13}$$

Case[iii]

$$g_1=0, h_1=0, g_0=-\frac{B+4Ab}{2C}, h_2=\frac{-6Ac}{C}, g_2=0, d=\frac{B^2+24A^2ca-16A^2d^2}{4C} \tag{14}$$

Taking example case[i],we obtain the following exact solution of Eq.(1) as

$$\phi(\xi) = -\frac{B+4Ab}{2C} - \frac{3Aa}{C}F^2(\xi) - \frac{6Ac}{C}F^{-2}(xi) \tag{15}$$

Some of new exact Jacobin elliptic function solutions can be obtained according to the different choice of the function $f(\xi)$ and (a,b and c) (See Appendices A,B and C).For simplicity cases (ii) and (iii) should be omitted here.

4 New periodic wave solutions

Case (1).When $a=2m^2$,$b=-1-m^2$,$c=1$,$f(\xi)=sn(\xi)$.Thus the Jacobian elliptic function solution of Eq.(1) is

$$\phi_{1,1}(\xi) = -\frac{B+4A(-1-m^2)}{2C} - \frac{6Am^2 sn^2(\xi)}{C} - \frac{6A}{Csn^2(\xi)} \tag{16}$$

For $m\longrightarrow 1$,Eq.(16) admits to new solitary wave solution

$$\phi_{1,2}(\xi) = -\frac{B-8A}{2C} - \frac{6Atanh^2(\xi)}{C} - \frac{6A}{Ctanh^2(\xi)}$$

when m tends to 0,Eq.(16) admits to a new triangular function solution

$$\phi_{1,3}(\xi) = -\frac{B-4A}{2C} - \frac{6A}{Csin^2(\xi)}$$

Case (2).When $a = 2$,$b = -(1+m^2)$,$c = m^2$,$f(\xi) = ns(\xi)$,we have a new Jacobian elliptic function solution of Eq.(1)

$$\phi_{2,1}(\xi) = -\frac{B+4A(-1-m^2)}{2C} - \frac{6Ans^2(\xi)}{C} - \frac{6Am^2}{Cns^2(\xi)} \tag{17}$$

as long as $m \longrightarrow 1$,Eq.(17) gives the solitary wave solution as follows

$$\phi_{2,2}(\xi) = -\frac{B-8A}{2C} - \frac{6Acoth^2(\xi)}{C} - \frac{6A}{Ccoth^2(\xi)}$$

when $m \longrightarrow 0$ in Eq.(17),we have a new triangular function solution

$$\phi_{2,3}(\xi) = -\frac{B-4A}{2C} - \frac{6Acsc^2(\xi)}{C}$$

Case (3).When $a = 2$,$b = 2-m^2$,$c = 1-m^2$,$f(\xi) = cs(\xi)$,we have the Jacobian elliptic function solution of Eq.(1) is

$$\phi_{3,1}(\xi) = -\frac{B+4A(2-m^2)}{2C} - \frac{6Acs^2(\xi)}{C} - \frac{6A(1-m^2)}{Ccs^2(\xi)} \tag{18}$$

For $m \longrightarrow 1$,Eq.(18) admits to solitary wave solution

$$\phi_{3,2}(\xi) = -\frac{B+4A}{2C} - \frac{6Acsch^2(\xi)}{C}$$

when $m \longrightarrow 0$,Eq.(18) reduces to triangular function solution

$$\phi_{3,3}(\xi) = -\frac{B+8A}{2C} - \frac{6Acot^2(\xi)}{C} - \frac{6A}{Ccot^2(\xi)}$$

Case (4).For $a = 2$,$b = 2m^2-1$,$c = m^2(m^2-1)$.In this case,we have $f(\xi) = ds(\xi)$,we obtain a new Jacobian elliptic function solution as

$$\phi_{4,1}(\xi) = -\frac{B+4A(2m^2-1)}{2C} - \frac{6Ads^2(\xi)}{C} - \frac{6Am^2(m^2-1)}{Cds^2(\xi)} \tag{19}$$

In the limiting case $m \longrightarrow 1$,Eq.(19) admits to solitary wave solution

$$\phi_{4,2}(\xi) = -\frac{B+4A}{2C} - \frac{6Acsch^2(\xi)}{C}$$

when $m \longrightarrow 0$,Eq.(19) becomes

$$\phi_{4,3}(\xi) = -\frac{B-4A}{2C} - \frac{6Acsc^2(\xi)}{C}$$

Case (5).If $a = 2(1-m^2)$,$b = 2-m^2$,$c = 1$.In this case,we have $f(\xi) = sc(\xi)$ and thus the corresponding a new Jacobian elliptic function solution as

$$\phi_{5,1}(\xi) = -\frac{B+4A(2-m^2)}{2C} - \frac{6A(1-m^2)sc^2(\xi)}{C} - \frac{6A}{Csc^2(\xi)} \tag{20}$$

when $m \longrightarrow 1$,Eq.(20) admits to solitary wave solution as

$$\phi_{5,2}(\xi) = -\frac{B+4A}{2C} - \frac{6A}{Csinh^2(\xi)}$$

For $m \longrightarrow 0$,Eq.(20) admits to a new triangular function solution as follows

$$\phi_{5,3}(\xi) = -\frac{B+8A}{2C} - \frac{6Atan^2(\xi)}{C} - \frac{6A}{Ctan^2(\xi)}$$

Case (6).If we select $a = -1/2$,$b = (1+m^2)/2$,$c = -\frac{(1-m^2)^2}{4}$.We have $f(\xi) = mcn(\xi) \pm dn(\xi)$, we obtain the following new Jacobian elliptic function solution

$$\phi_{6,1}(\xi) = -\frac{B+2A(1+m^2)}{2C} + \frac{\frac{3}{2}A[mcn(\xi)+dn(\xi)]^2}{2C} + \frac{\frac{3}{2}A(1-m^2)62}{C[mcn(\xi) \pm dn(\xi)]^2} \quad (21)$$

For $m \longrightarrow 1$,Eq.(21) admits to

$$\phi_{6,2}(\xi) = -\frac{B+4A}{2C} + \frac{6Asech^2(\xi)}{C}$$

When $m \longrightarrow 0$ in Eq.(21),we have

$$\phi_{6,3}(\xi) = -\frac{B+2A}{2C} - \frac{3A}{C}$$

Case (7).For $a = 1/2$,$b = \frac{1-2m^2}{2}$,$c = 1/4$,$f(\xi) = ns(\xi) \pm cs(\xi)$,we obtain a new Jacobian elliptic function solution as

$$\phi_{7,1}(\xi) = -\frac{B+4A(1/2-m^2)}{2C} - \frac{3A[ns(\xi) \pm cs(\xi)]^2}{2C} - \frac{3A}{2C[ns(\xi) \pm cs(\xi)]^2} \quad (22)$$

As long as $m \longrightarrow 1$ and 0 in Eq.(22),we have new exact solitary and triangular solutions

$$\phi_{7,2}(\xi) = -\frac{B-2A}{2C} - \frac{3A[coth(\xi) \pm csch(\xi)]^2}{2C} - \frac{3A}{2C[coth(\xi) \pm csch(\xi)]^2}$$

$$\phi_{7,3}(\xi) = -\frac{B+2A}{2C} - \frac{3A[csc(\xi) \pm cot(\xi)]^2}{2C} - \frac{3A}{2C[csc(\xi) \pm cot(\xi)]^2}$$

Case (8).For $a = 2$,$b = 2m^2-1$,$c = m^2(m^2-1)$,$f(\xi) = ds(\xi)$,we obtain a new Jacobian elliptic function solution

$$\phi_{8,1}(\xi) = -\frac{B+4A(2m^2-1)}{2C} - \frac{3Ads^2(\xi)}{2C} - \frac{6Am^2(m^2-1)}{Cds^2(\xi)} \quad (23)$$

When $m \longrightarrow 1$ and 0,Eq.(24),reduces to new solitary and triangular solutions as follows

$$\phi_{8,2}(\xi) = -\frac{B+4A}{2C} - \frac{6Acsch^2(\xi)}{2C},$$
$$\phi_{8,3}(\xi) = -\frac{B-4A}{2C} - \frac{6Acsc^2(\xi)}{C}$$

5 Conclusion

Based on the modified mapping method and the use of symbolic computation system Maple,we have constructed explicit some new solutions of the nonlinear elastic rod with lateral inertia.

As a result,many new solutions are obtained which include Jacobian elliptic function solutions,triangular and hyperbolic functions.Solutions in the limiting cases when the modulus m of the elliptic function approach 0 or 1 have also been studied.

It can be easliy seen that the method used in this paper must futher be improved to solve more nonlinear partial differential equations arising in mathematical physics.This is our task in the future.

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Appendix A

Table 1: Relations between values of (a, b, c) and corresponding $F(\xi)$ in $F^{'2}(\xi) = (a/2)F^4(\xi)+bF^2(\xi)+c$

a	b	c	$F(\xi)$
m^2	$-(1+m^2)$	1	$F(\xi) = sn(\xi)$,$cd(\xi) = \frac{cn(\xi)}{dn(\xi)}$
$-m^2$	$2m^2-1$	$1-m^2$	$F(\xi) = cn(\xi)$
-1	$2-m^2$	m^2-1	$F(\xi) = dn(\xi)$
1	$-(1+m^2)$	m^2	$F(\xi) = ns(\xi) = (sn(\xi))^{-1}$,$dc(\xi) = \frac{dn(\xi)}{cn(\xi)}$
$1-m^2$	$2m^2-1$	$-m^2$	$F(\xi) = nc(\xi) = (cn(\xi))^{-1}$
m^2-1	$2-m^2$	-1	$F(\xi) = nd(\xi) = (dn(\xi))^{-1}$
$1-m^2$	$2-m^2$	1	$F(\xi) = sc(\xi) = \frac{sn(\xi)}{cn(\xi)}$
$-m^2(1-m^2)$	$2m^2-1$	1	$F(\xi) = sd(\xi) = \frac{sn(\xi)}{dn(\xi)}$
1	$2-m^2$	$1-m^2$	$F(\xi) = cs(\xi) = \frac{cn(\xi)}{sn(\xi)}$
1	$2m^2-1$	$-m^2(1-m^2)$	$F(\xi) = ds(\xi) = \frac{dn(\xi)}{sn(\xi)}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$F(\xi) = ns(\xi) \pm cs(\xi)$
$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{2}$	$F(\xi) = nc(\xi) \pm sc(\xi)$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$F(\xi) = ns(\xi) \pm ds(\xi)$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$F(\xi) = sn(\xi) \pm ics(\xi)$

Appendix B

Derivatives of Jacobi elliptic functions

$$\begin{aligned} sn^{'}(\xi) = cn(\xi)dn(\xi), cd^{'}(\xi) = -(1-m^2)sd(\xi)nd(\xi), \\ cn^{'}(\xi) = -sn(\xi)dn(\xi), dn^{'}(\xi) = -m^2sn(\xi)cn(\xi), \\ ns^{'}(\xi) = -cs(\xi)ds(\xi), dc^{'}(\xi) = (1-m^2)nc(\xi)sc(\xi), \\ nc^{'}(\xi) = sc(\xi)dc(\xi), nd^{'}(\xi) = m^2cd(\xi)sd(\xi), \\ sc^{'}(\xi) = dc(\xi)nc(\xi), cs^{'}(\xi) = -ns(\xi)ds(\xi), \\ ds^{'}(\xi) = -cs(\xi)ns(\xi), sd^{'}(\xi) = nd(\xi)cd(\xi) \end{aligned}$$

Appendix C

Table 2: Jacobi elliptic functions degenerate as hyperbolic functions when $m \longrightarrow 1$

$sn(\xi)$	$cn(\xi)$	$dn(\xi)$	$sc(\xi)$	$sd(\xi)$	$cd(\xi)$	$ns(\xi)$	$nc(\xi)$	$nd(\xi)$	$cs(\xi)$	$ds(\xi)$	$dc(\xi)$
$tanh(\xi)$	$sech(\xi)$	$sech(\xi)$	$sinh(\xi)$	$sinh(\xi)$	1	$coth(\xi)$	$cosh(\xi)$	$cosh(\xi)$	$csch(\xi)$	$csch(\xi)$	1

Table 3: Jacobi elliptic functions degenerate as hyperbolic functions when $m \longrightarrow 0$

$sn(\xi)$	$cn(\xi)$	$dn(\xi)$	$sc(\xi)$	$sd(\xi)$	$cd(\xi)$	$ns(\xi)$	$nc(\xi)$	$nd(\xi)$	$cs(\xi)$	$ds(\xi)$	$dc(\xi)$
$sin(\xi)$	$cos(\xi)$	1	$tan(\xi)$	$sin(\xi)$	$cos(\xi)$	$csc(\xi)$	$sec(\xi)$	1	$cot(\xi)$	$csc(\xi)$	$sec(\xi)$