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Volterra Composition Operators from $F(p,q,s)$ Spaces to Bloch-type Spaces

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Abstract. Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. Let φ be a holomorphic self-map of B and $g \in H(B)$. In this paper, we investigate the boundedness and compactness of the Volterra composition operator

$$(V_\varphi^g f)(z) = \int_0^1 f(\varphi(tz))\Re g(tz)\frac{dt}{t},$$

which map from general function space $F(p,q,s)$ to Bloch-type space $\mathcal{B}^\alpha$ in the unit ball.

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1. Introduction

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in the complex vector space $\mathbb{C}^n$ and

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n.$$

Let dv stand for the normalized Lebesgue measure on $\mathbb{C}^n$. For a holomorphic function f we denote

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right).$$

Let $H(B)$ denote the class of all holomorphic functions on the unit ball. Let $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$ stand for the radial derivative of $f \in H(B)$ (see [31]). It is easy to see that, if $f \in H(B)$, $f(z) = \sum_\alpha a_\alpha z^\alpha$, where α is a multi-index, then

$$\Re f(z) = \sum_\alpha |\alpha| a_\alpha z^\alpha.$$

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For $\alpha > 0$, the Bloch-type space (or α-Bloch space) $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$, is the space of all $f \in H(B)$ such that

$$b_\alpha(f) = \sup_{z\in B}(1-|z|^2)^\alpha\,|\Re f(z)| < \infty.$$

On $\mathcal{B}^\alpha$ the norm is introduced by

$$||f||_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f).$$

With this norm $\mathcal{B}^\alpha$ is a Banach space. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha\,|\Re f(z)| = 0.$$

This function space is called little Bloch-type space. If $\alpha = 1$, we denote $\mathcal{B}^\alpha$ simply by $\mathcal{B}$, which is the well-known classical Bloch space.

Let $0 < p, s < \infty$, $-n-1 < q < \infty$. A function $f \in H(B)$ is said to belong to general function space $F(p,q,s) = F(p,q,s)(B)$ (see, e.g. [7, 29, 30]) if

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in B}\int_B |\nabla f(z)|^p(1-|z|^2)^q g^s(z,a)dv(z) < \infty,$$

where $g(z,a) = \log|\varphi_a(z)|^{-1}$ is the Green's function for B with logarithmic singularity at a.

We call $F(p,q,s)$ general function space because we can get many function spaces, such as BMOA space, Q_p space (see [20]), Bergman space, Hardy space, Bloch space, if we take special parameters of p,q,s (see, e.g. [30]). If $q+s \le -1$, then $F(p,q,s)$ is the space of constant functions.

Suppose that $g : B \to \mathbb{C}^1$ is a holomorphic map of the unit ball, for a $f \in H(B)$, define

$$T_g f(z) = \int_0^1 f(tz)\frac{dg(tz)}{dt} = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \quad z \in B. \tag{1.1}$$

This operator is called Riemann-Stieltjes operator (or Extended-Cesàro operator). It was introduced in [4], and studied in [1, 2, 4–7, 9–15, 21, 26, 27, 32, 36].

A product of Riemann-Stieltjes operator T_g and composition operator C_φ is defined as follows:

$$V_\varphi^g f(z) = \int_0^1 f(\varphi(tz))\frac{dg(tz)}{dt} = \int_0^1 f(\varphi(tz))\Re g(tz)\frac{dt}{t},\ f \in H(B), \tag{1.2}$$

which is called Volterra composition operator and studied in [19, 33–35, 37]. See [22–25, 28] for the boundedness and compactness of a related operator on some holomorphic function spaces in the unit ball. In the case of $n = 1$, this operator has form

$$V_\varphi^g f(z) = \int_0^1 f(\varphi(tz))g'(tz)\frac{dt}{t}, \quad f \in H(D),\ z \in D, \tag{1.3}$$

which was introduced in [8]. See [3, 16–18] for the study of composition operators on Bloch spaces.

The purpose of this paper is to study the boundedness and compactness of the Volterra composition operators V_φ^g from $F(p,q,s)$ to the Bloch-type space.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary results

In order to prove our results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 2.1. *Let φ be a holomorphic self-map of B. For every $f, g \in H(B)$, it holds*

$$\Re[V_\varphi^g(f)](z) = f(\varphi(z))\Re g(z) \tag{2.1}$$

Proof. We use the method of [4]. Assume that the holomorphic function $f \circ \varphi \Re g$ has the expansion $\sum_\alpha a_\alpha z^\alpha$. Then

$$\Re[V_\varphi^g(f)](z) = \Re[T_g(f\circ\varphi)](z) = \Re\int_0^1 \sum_\alpha a_\alpha (tz)^\alpha \frac{dt}{t} = \Re\sum_\alpha \frac{a_\alpha}{|\alpha|} z^\alpha = \sum_\alpha a_\alpha z^\alpha,$$

which is what we wanted. ∎

Lemma 2.2. [29] *For $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, if $f \in F(p,q,s)$, then $f \in \mathcal{B}^{\frac{n+1+q}{p}}$ and*

$$\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \le C\|f\|_{F(p,q,s)}. \tag{2.2}$$

The following lemma can be found in [21].

Lemma 2.3. *If $f \in \mathcal{B}^\alpha$, then*

$$|f(z)| \le C \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^\alpha} & : \quad 0 < \alpha < 1; \\ |f(0)| + \|f\|_{\mathcal{B}^\alpha} \ln \frac{e}{1-|z|^2} & : \quad \alpha = 1, \\ |f(0)| + \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & : \quad \alpha > 1, \end{cases}$$

for some C independent of f.

The following criterion for compactness follows from standard arguments similar to those outlined in Proposition 3.11 of [3] or in Lemma 3 of [10]. We omit the details.

Lemma 2.4. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$. Then $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $F(p,q,s)$ which converges to zero uniformly on compact subsets of B as $k \to \infty$, we have $\|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$.*

The next lemma was proved in [16] in the case of $\alpha = 1$ in the unit disk. For the general case the proof is similar, thus we omit the details (see, e.g. [7]).

Lemma 2.5. *A closed set K in $\mathcal{B}_0^\alpha(B)$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)^\alpha |\Re f(z)| = 0.$$

3. Main results and proofs

Theorem 3.1. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p < n+1+q$. Then $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z \in B} \frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} < \infty. \tag{3.1}$$

Moreover, the following relationship holds

$$\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha} \asymp \sup_{z \in B} \frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}}. \tag{3.2}$$

Proof. For $f \in H(B)$, note that $V_\varphi^g f(0) = 0$. By Lemmas 2.1, 2.2 and 2.3,

$$\begin{aligned}
\|V_\varphi^g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha |\Re(V_\varphi^g f)(z)| \\
&= \sup_{z\in B}(1-|z|^2)^\alpha |f(\varphi(z))||\Re g(z)| \\
&\le C\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \sup_{z\in B} \frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} \\
&\le C\|f\|_{F(p,q,s)} \sup_{z\in B} \frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}}.
\end{aligned}$$

Therefore (3.1) implies that $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. For $w \in B$, let

$$f_w(z) = \frac{1-|\varphi(w)|^2}{(1-\langle z, \varphi(w)\rangle)^{\frac{n+1+q}{p}}}. \tag{3.3}$$

It is easy to see that

$$f_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\frac{n+1+q-p}{p}}}, \quad |\Re f_w(\varphi(w))| \asymp \frac{|\varphi(w)|^2}{(1-|\varphi(w)|^2)^{\frac{n+1+q}{p}}}. \tag{3.4}$$

If $\varphi(w) = 0$ then $f_w \equiv 1$ obviously belongs to $F(p,q,s)$. From [29] we know that $f_w \in F(p,q,s)$, moreover there is a positive constant K such that $\sup_{w\in B} \|f_w\|_{F(p,q,s)} \le K$. Therefore, for every $z \in B$,

$$\begin{aligned}
(1-|z|^2)^\alpha |f_w(\varphi(z))\Re g(z)| &= (1-|z|^2)^\alpha |\Re(V_\varphi^g f_w)(z)| \\
&\le \|V_\varphi^g f_w\|_{\mathcal{B}^\alpha} \le K\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha}.
\end{aligned} \tag{3.5}$$

From this and (3.3), we get

$$\frac{(1-|w|^2)^\alpha |\Re g(w)|}{(1-|\varphi(w)|^2)^{\frac{n+1+q-p}{p}}} = (1-|w|^2)^\alpha |f_w(\varphi(w))\Re g(w) \le K\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha}.$$

from which (3.1) follows. From the above proof, we see that (3.2) holds. The proof is completed. ∎

Theorem 3.2. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p < n+1+q$. Then $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if $g \in \mathcal{B}^\alpha$ and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} = 0. \tag{3.6}$$

Proof. Assume $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact. Then V_φ^g is bounded. Taking $f \equiv 1$, we get $g \in \mathcal{B}^\alpha$.

Let $\{\varphi(z_k)\}_{k\in\mathbb{N}}$ be a sequence in B such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$. Define

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\langle z, \varphi(z_k)\rangle)^{\frac{n+1+q}{p}}}.$$

Then $f_k \in F(p,q,s)$, and f_k uniformly converges to zero on any compact subset of B. By Lemma 2.4, we have $\lim_{k\to\infty} \|V_\varphi^g(f_k)\|_{\mathcal{B}^\alpha} = 0$.

On the other hand, we have

$$\begin{aligned}
\|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha |\Re(V_\varphi^g f_k)(z)| \\
&= \sup_{z\in B}(1-|z|^2)^\alpha |f_k(\varphi(z))||\Re g(z)| \\
&\geq (1-|z_k|^2)^\alpha |f_k(\varphi(z_k))||\Re g(z_k)| \\
&= \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{\frac{n+1+q}{p}}} |\Re g(z_k)|.
\end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi(z_k)|^2)^{\frac{n+1+q}{p}}} |\Re g(z_k)| = 0,$$

which implies that (3.6) holds.

Conversely, if $g \in \mathcal{B}^\alpha$ and (3.6) holds. From $g \in \mathcal{B}^\alpha$ and (3.6), we see that (3.1) holds. Hence $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded.

Let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $F(p,q,s)$ with

$$\|f_k\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \leq \|f_k\|_{F(p,q,s)} \leq M, \ k \in \mathbb{N},$$

and $f_k \to 0$ uniformly on any compact subset of B as $k \to \infty$. By Lemma 2.4, to show that the operator V_φ^g is compact, we only need to show

$$\lim_{k\to\infty} \|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} = 0.$$

In fact, for any positive number ε, (3.6) implies that there is positive number $\delta < 1$, such that when $\delta < |\varphi(z)| < 1$, we have

$$\frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} < \varepsilon. \tag{3.7}$$

Let $B_\delta = \{w \in B : |w| \leq \delta\}$. (3.7) together with the fact that $g \in \mathcal{B}^\alpha$ show that

$$\begin{aligned}
\|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha |\Re(V_\varphi^g f_k)(z)| \\
&= \sup_{z\in B}(1-|z|^2)^\alpha |f_k(\varphi(z))||\Re g(z)|
\end{aligned}$$

$$\le \Big(\sup_{\{z\in B:|\varphi(z)|\le\delta\}} + \sup_{\{z\in B:\delta<|\varphi(z)|<1\}} \Big)(1-|z|^2)^\alpha |f_k(\varphi(z))||\Re g(z)|$$
$$\le \|g\|_{\mathcal{B}^\alpha} \sup_{w\in B_\delta} |f_k(w)| + \sup_{\{z\in B:\delta<|\varphi(z)|<1\}} (1-|z|^2)^\alpha |f_k(\varphi(z))||\Re g(z)|$$
$$\le \|g\|_{\mathcal{B}^\alpha} \sup_{w\in B_\delta} |f_k(w)| + C\|f_k\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \sup_{\{z\in B:\delta<|\varphi(z)|<1\}} \frac{(1-|z|^2)^\alpha|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}}$$
$$\le \|g\|_{\mathcal{B}^\alpha} \sup_{w\in B_\delta} |f_k(w)| + CM\varepsilon.$$

Note the compactness of the B_δ, we have

$$\lim_{k\to\infty} \sup_{w\in B_\delta} |f_k(w)| = 0.$$

Hence $\lim_{k\to\infty} \|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} \le CM\varepsilon$, i.e. we obtain

$$\lim_{k\to\infty} \|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} = 0.$$

Therefore $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact. The proof is completed. ∎

Theorem 3.3. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p < n+1+q$. Then the following statements are equivalent:*

(i) $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$ *is compact.*

(ii) $g \in \mathcal{B}_0^\alpha$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} = 0. \tag{3.8}$$

(iii)

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} = 0. \tag{3.9}$$

Proof. (i)⇒(ii). Assume $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact. Taking $f \equiv 1$, we get $g \in \mathcal{B}_0^\alpha$. By the compactness of $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$, we see that $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact and hence is bounded. Theorem 3.2 implies that (3.8) holds.

(ii)⇒(iii). Assume that $g \in \mathcal{B}_0^\alpha$ and (3.8) holds. For every $\varepsilon > 0$, there exists a $t \in (0,1)$ such that

$$\frac{(1-|z|^2)^\alpha|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} < \varepsilon$$

when $t < |\varphi(z)| < 1$. Moreover there exists a $r \in (0,1)$ such that when $r < |z| < 1$,

$$(1-|z|^2)^\alpha|\Re g(z)| < \frac{\varepsilon}{(1-t^2)^{\frac{n+1+q-p}{p}}}.$$

Therefore, when $r < |z| < 1$ and $t < |\varphi(z)| < 1$, we have

$$\frac{(1-|z|^2)^\alpha|\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} < \varepsilon.$$

When $r < |z| < 1$ and $|\varphi(z)| \le t$, we obtain

$$\frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} < \frac{\varepsilon}{(1-t^2)^{\frac{n+1+q-p}{p}}}(1-t^2)^{\frac{n+1+q-p}{p}} = \varepsilon.$$

In a word, for every $\varepsilon > 0$, there exists a $r \in (0,1)$, when $r < |z| < 1$ we have

$$\frac{(1-|z|^2)^\alpha |\Re g(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+q-p}{p}}} < \varepsilon,$$

which implies that (3.9) holds.

(iii)⇒(i). If (3.9) holds. From Lemma 2.5, we know that $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{F(p,q,s)}\le 1} (1-|z|^2)^\alpha |\Re(V_\varphi^g f)(z)| = 0. \tag{3.10}$$

On the other hand, by Lemmas 2.1, 2.2 and 2.3, we have that

$$(1-|z|^2)^\alpha |\Re(V_\varphi^g f)(z)| \le \frac{C(1-|z|^2)^\alpha |\Re g(z)|\, \|f\|_{F(p,q,s)}}{(1-|\varphi(z)|^2)^{\frac{q+n+1-p}{p}}}. \tag{3.11}$$

Taking the supremum (sup) in (3.11) over the the unit ball in the space $F(p,q,s)$, and letting $|z| \to 1$, by (3.9) we see that (3.10) holds and hence $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is compact. The proof is completed. ∎

Theorem 3.4. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p > n+1+q$. Then $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if $g \in \mathcal{B}^\alpha$.*

Moreover, the following relationship holds

$$\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha} \asymp \|g\|_{\mathcal{B}^\alpha}. \tag{3.12}$$

Proof. For $f \in H(B)$, note that $V_\varphi^g f(0) = 0$. By Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned}
\|V_\varphi^g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha |\Re(V_\varphi^g f)(z)| \\
&= \sup_{z\in B}(1-|z|^2)^\alpha |f(\varphi(z))||\Re g(z)| \\
&\le C\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \sup_{z\in B}(1-|z|^2)^\alpha |\Re g(z)| \\
&\le C\|f\|_{F(p,q,s)} \sup_{z\in B}(1-|z|^2)^\alpha |\Re g(z)|,
\end{aligned}$$

By $g \in \mathcal{B}^\alpha$, we have that $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. Taking $f(z) = 1$, then

$$\begin{aligned}
(1-|z|^2)^\alpha |f_w(\varphi(z))\Re g(z)| &= (1-|z|^2)^\alpha |\Re(V_\varphi^g f_w)(z)| \\
&\le \|V_\varphi^g f_w\|_{\mathcal{B}^\alpha} \le K\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha},
\end{aligned}$$

which implies $g \in \mathcal{B}^\alpha$. From the above proof, we see that (3.12) holds. The proof is completed. ∎

Theorem 3.5. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p > n+1+q$. Then $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact if and only if $g \in \mathcal{B}^\alpha$.*

Proof. Assume $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is compact, and then V_φ^g is bounded. By Theorem 3.4, we get $g \in \mathcal{B}^\alpha$.

Conversely, assume that $g \in \mathcal{B}^\alpha$. By Theorem 3.4 we see that $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in $F(p,q,s)$ and $f_k \to 0$ uniformly on B as $k \to \infty$. Similarly to the proof of Lemma 4 and Theorem 12 of [35], we have

$$\|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} = \sup_{z\in B}(1-|z|^2)^\alpha |f_k(\varphi(z))\Re g(z)| \le \|g\|_{\mathcal{B}^\alpha} \sup_{z\in B}|f_k(\varphi(z))| \to 0,$$

as $k \to \infty$, which implies the desired result. ∎

Theorem 3.6. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p > n+1+q$. Then the following statements are equivalent:*

(i) $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$ *is compact.*
(ii) $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$ *is bounded.*
(iii) $g \in \mathcal{B}_0^\alpha$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). If $V_\varphi^g : F(p,q,s) \to \mathcal{B}_0^\alpha$ is bounded. Taking $f \equiv 1$, we get $g \in \mathcal{B}_0^\alpha$.

(iii)⇒(i). With little modifying the proof of (iii)⇒(i) in Theorem 3.3, we can get the desired result. The proof is completed. ∎

Theorem 3.7. *Let $g \in H(B)$ and φ be a holomorphic self-map of B, $0 < \alpha, p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p = n+1+q$. Then $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^\alpha |\Re g(z)| \ln\frac{e}{1-|\varphi(z)|^2} < \infty. \tag{3.13}$$

Moreover, the following relationship holds

$$\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha} \asymp \sup_{z\in B}(1-|z|^2)^\alpha |\Re g(z)| \ln\frac{e}{1-|\varphi(z)|^2}. \tag{3.14}$$

Proof. For $f \in H(B)$, by Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned}\|V_\varphi^g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha |f(\varphi(z))||\Re g(z)| \\ &\le C\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} \sup_{z\in B}(1-|z|^2)^\alpha |\Re g(z)| \ln\frac{e}{1-|\varphi(z)|^2} \\ &\le C\|f\|_{F(p,q,s)} \sup_{z\in B}(1-|z|^2)^\alpha |\Re g(z)| \ln\frac{e}{1-|\varphi(z)|^2}.\end{aligned}$$

By (3.13) we see that $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose $V_\varphi^g : F(p,q,s) \to \mathcal{B}^\alpha$ is bounded. For $w \in B$, let

$$f_w(z) = \ln\frac{e}{1-\langle z, \varphi(w)\rangle}. \tag{3.15}$$

From [29] we know that $f_w \in F(p,q,s)$, moreover there is a positive constant K such that $\sup_{w\in B}\|f_w\|_{F(p,q,s)} \le K$. Therefore, for every $z \in B$

$$\begin{aligned}(1-|z|^2)^\alpha |f_w(\varphi(z))\Re g(z)| &= (1-|z|^2)^\alpha |\Re(V_\varphi^g f_w)(z)| \\ &\le \|V_\varphi^g f_w\|_{\mathcal{B}^\alpha} \le K\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha}.\end{aligned} \tag{3.16}$$

Therefore

$$(1-|w|^2)^\alpha|\Re g(w)|\ln\frac{e}{1-|\varphi(w)|^2} = (1-|w|^2)^\alpha|f_w(\varphi(w))\Re g(w)| \tag{3.17}$$
$$\le K\|V_\varphi^g\|_{F(p,q,s)\to\mathcal{B}^\alpha},$$

which implies (3.13). From the above proof, we see that (3.14) holds. The proof is completed. ∎

Theorem 3.8. *Let $g\in H(B)$ and φ be a holomorphic self-map of B, $0<\alpha,p,s<\infty$, $-n-1<q<\infty$, $q+s>-1$, $p=n+1+q$. Then $V_\varphi^g : F(p,q,s)\to\mathcal{B}^\alpha$ is compact if and only if $g\in\mathcal{B}^\alpha$ and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|\Re g(z)|\ln\frac{e}{1-|\varphi(z)|^2}=0. \tag{3.18}$$

Proof. Since the sufficiency part is similar to the proof of Theorem 3.2, we omit the details.

Now we prove the necessity part. Assume that $V_\varphi^g : F(p,q,s)\to\mathcal{B}^\alpha$ is compact. Then V_φ^g is bounded. Taking $f\equiv 1$, by the boundedness of V_φ^g, we get that $g\in\mathcal{B}^\alpha$.

Let $\{\varphi(z_k)\}_{k\in N}$ be a sequence in B such that $\lim_{k\to\infty}|\varphi(z_k)|=1$. Define

$$f_k(z)=\Big(\ln\frac{e}{1-\langle z,\varphi(z_k)\rangle}\Big)^2\Big(\ln\frac{e}{1-|\varphi(z_k)|^2}\Big)^{-1}.$$

Then $f_k\in F(p,q,s)$ and f_k uniformly converges to zero on any compact subset of B. By Lemma 2.4, we have $\lim_{k\to\infty}\|V_\varphi^g(f_k)\|_{\mathcal{B}^\alpha}=0$.

On the other hand, we have

$$\begin{aligned}\|V_\varphi^g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\Re(V_\varphi^g f_k)(z)|\\ &= \sup_{z\in B}(1-|z|^2)^\alpha|f_k(\varphi(z))||\Re g(z)|\\ &\ge (1-|z_k|^2)^\alpha|f_k(\varphi(z_k))||\Re g(z_k)|\\ &= (1-|z_k|^2)^\alpha|\Re g(z_k)|\ln\frac{e}{1-|\varphi(z_k)|^2}.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}(1-|z_k|^2)^\alpha|\Re g(z_k)|\ln\frac{e}{1-|\varphi(z_k)|^2}=0,$$

which implies that (3.18) holds. The proof is completed. ∎

Theorem 3.9. *Let $g\in H(B)$ and φ be a holomorphic self-map of B, $0<\alpha,p,s<\infty$, $-n-1<q<\infty$, $q+s>-1$, $p=n+1+q$. Then the following statements are equivalent:*

(i) $V_\varphi^g : F(p,q,s)\to\mathcal{B}_0^\alpha$ *is compact.*

(ii) $g\in\mathcal{B}_0^\alpha$ *and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|\Re g(z)|\ln\frac{e}{1-|\varphi(z)|^2}=0. \tag{3.19}$$

(iii)

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|\Re g(z)|\ln\frac{e}{1-|\varphi(z)|^2}=0. \tag{3.20}$$

Proof. The proof is similar to the proof of Theorem 3.3. We omit the details. ∎

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